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Adversarial Fragmentation of Robotic Teams Operating Under Reynolds' Rules with Bounded Communication Radius

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***Abstract*— In this paper, we examine a class of counter-swarm problems featuring small teams of antagonists. The objective of the counter-swarm algorithm is to minimize the connectivity of the team using a single adversarial pursuer. We devise a novel criterion for the connectivity of an undirected graph, based on an augmentation of its Laplacian. We prove theoretically how the criterion depends on the size and the connectivity of the graph. Next, we pose optimal control problems which use this criterion as well as the eigenvalues of the Laplacian. We show how the properties of the team dynamics and their interaction with the pursuer define an envelope within which the pursuer achieves the desired objectives.**

I. INTRODUCTION

Robotic swarms are a promising technology for carrying out complex missions using numerous simple, low-cost robots. The development of methods that enable robust swarming has been accompanied by the development of methods that *disrupt* swarms, primarily through some form of herding or combat. In this paper, we consider the problem where the objective is to fragment the swarm into disconnected pieces using a single adversarial agent as shown in Figure 1. This objective can be viewed in two ways. First, the fragmentation of the swarm into disconnected pieces can be seen as a prelude to functional failure. This is relevant in applications where a quorum of agents is needed to complete the mission. Second, the act of breaking the swarm can be seen as the prelude to effective herding using multiple pursuers, since herding a single cohesive swarm can be challenging due to its sheer inertia [6].

A. Overview of the literature

The most common counter-swarm technique explored in the literature is that of herding. The objective of herding is to the preserve the swarm as a cohesive unit and divert its flight path. That way, the swarm can be prevented from entering sensitive airspace such as those around airports [6]. This task can be accomplished using either a single pursuer nudging an inherently cohesive swarm [6] or a group of pursuers working to contain and herd the swarm simultaneously, even when it is splintered into one or more components [2], [1]. An alternative to herding is to engage the swarm using a group of pursuers that seek to destroy individual agents of the swarm [9], [8]. While it can be challenging to herd swarms using one or more pursuers, additional difficulties arise from the need to estimate the dynamics of the swarms [3] and dealing with instances where the swarm might prefer to split in order to maximize effectiveness [1]. The problem of swarm engagement can be posed as an optimal control problem, such as in [9] where the survival probability was maximized during a combative engagement.

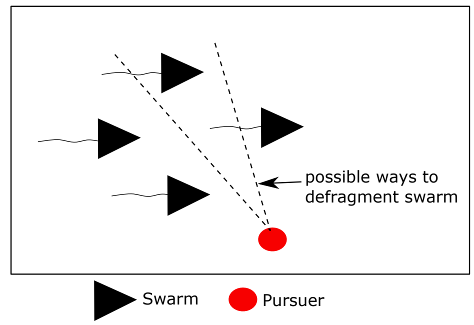

Fig. 1: A pursuer engaging a swarm has two options to fragment the swarm, as shown in dashed lines. The dashed line on the right separates the leading agent from the rest of the swarm, while the one on the left breaks the swarm into two equal, mutually disconnected halves.

B. Contribution

We consider the problem of breaking the swarm into disconnected components and pose it as an optimal control problem (OCP). This is a novel problem formulation, to the best of our knowledge. As an example, consider a swarm consisting of agents with specialized capabilities, all of which are essential for completing its mission. For each capability, such a swarm may have redundancies. However, if the agents can be separated from each other into smaller groups that are functionally incomplete, it follows that the swarm's mission can be successfully obstructed. This may be viewed as a middle ground between herding and combat.

In this paper, we pose the problem of minimizing the connectivity of a homogeneous swarm as an OCP which we proceed to solve numerically. We investigate several candidate objective functions that capture the connectivity of the graph and are differentiable with respect to the separation between the agents in the swarm. This enables us to use these objective functions in a standard OCP solver such as ICLOCS2 [5]. The numerical solutions help identify, in terms of the system parameters, the envelope within which a successful counter-swarm operation is feasible.

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Of possible theoretical interest, we introduce a novel metric to capture connectivity, based on the determinant of an augmented Laplacian matrix. While the traditional Laplacian matrix is rank-deficient by default, the augmented Laplacian introduced in the paper loses rank only when the graph is disconnected.

The rest of the paper is organized as follows. We present the optimal control problem formulation in Sec. II, and the augmented Laplacian matrix in Sec. III. We present numerical solutions to the optimal control problem in Sec. IV, followed by the concluding discussion in Sec. V.

II. Problem Formulation

We consider a group of n agents (antagonists in our setting) with second order dynamics flying in a formation under Reynolds' rules [7]. We assume that each agent i maintains a fixed address-book $\mathcal{A}_i$ of the other agents in the graph.

Assumption 1: The address book $\mathcal{A}_i$ of each agent i is set once and for all at time $t = 0$ as follows: $\mathcal{A}_i = \{j \mid r_{ij}(t = 0) < R_{\text{com}}\}$.

At each time instant, we define the set

$$\mathcal{N}_i = \{j \mid j \in \mathcal{A}_i \text{ and } r_{ij} < R_{\text{com}}\}$$

where R_{com} is communication threshold and r_{ij} is the distance between agents i and j. Notice that the set $\mathcal{A}_i$ is fixed for all time, while $\mathcal{N}_i$ can be time-varying, and even empty if agent i is disconnected from the swarm.

The communication network can be modeled as an undirected graph $G = (V, E)$, where V is the set of n agents and $E \subseteq (V, V)$ is the set of edges with $(i, j) \in E$ if and only if $j \in \mathcal{N}_i$.

Assumption 2: The graph G is connected at time $t = 0$.

We consider the motion of the swarm in 2-D space and denote the position and the velocity of each agent by $\boldsymbol{x}_{\{\cdot\}}, \boldsymbol{v}_{\{\cdot\}} \in \mathbb{R}^2$. The swarm is engaged by a single pursuer whose position and velocity are denoted by $\boldsymbol{x}_p$, and $\boldsymbol{v}_p \in \mathbb{R}^2$, respectively. The dynamics of the i^{th} agent are described by

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= \boldsymbol{v}_i \\ \dot{\boldsymbol{v}}_i &= \sum_{j\in\mathcal{N}_i} k_r \left(1 - \left(\frac{R_{\text{safe}}}{\|\boldsymbol{r}_{ij}\|}\right)^3\right) \boldsymbol{r}_{ij} + k_v \sum_{j\in\mathcal{N}_i} (\boldsymbol{v}_j - \boldsymbol{v}_i) \\ &\quad + k_p H(\boldsymbol{r}_{pi}) + k_d(\boldsymbol{v}_d - \boldsymbol{v}_i) \end{aligned} \tag{1}$$

where $\boldsymbol{r}_{ij} = \boldsymbol{x}_j - \boldsymbol{x}_i$, the subscript p denotes a pursuer, $\boldsymbol{v}_d$ denotes a reference velocity for the swarm, and $H(\boldsymbol{r}_{pi})$ denotes the evasive response to the pursuer. The gains $k_{\{\cdot\}} > 0$ are typically unknown and need to be estimated. We assume in this paper that the gains are known. We model $H(\cdot)$ as follows

$$H(\boldsymbol{r}_{pi}) = \begin{cases} \frac{\boldsymbol{r}_{ip}}{\|\boldsymbol{r}_{ip}\|^2}, & \text{if } \|\boldsymbol{r}_{ip}\| < R_{\text{fear}} \\ 0, & \text{otherwise} \end{cases}, \tag{2}$$

where R_{fear} denotes the distance from an agent within which a pursuer can effectively influence its motion. We assume that the motion of the pursuer p is governed by the first order equation

$$\dot{\boldsymbol{r}}_p = \boldsymbol{v}_p, \quad \|\boldsymbol{v}_p\| \leq v_{p,\max} \tag{3}$$

Let $C \in \mathbb{R}_{\geq 0}$ denote the instantaneous connectivity metric under consideration. Given a metric C, we wish to solve the following optimal control problem (OCP) for a known horizon T:

$$\min_{\boldsymbol{v}_p[0:T]} \int_0^T C(t)\, dt \tag{4}$$

subject to the dynamics in (1) and (3), where $\boldsymbol{v}_p[0 : T]$ denotes the values of the pursuer velocity over the time interval $[0, T]$.

Remark 1: An alternate way to formulate the OCP is to solve for the minimum time T such that $C(T) < C_{\min}$. We choose the fixed horizon problem for its relative simplicity, and because it suffices to illustrate our general approach to formulating and solving the swarm fragmentation problem.

III. Connectivity Metrics and the Augmented Laplacian

Recall that the Laplacian matrix $L \in \mathbb{R}^{n\times n}$ of an undirected graph consisting of n vertices is defined as

$$L_{ij} = \begin{cases} 1 & j \in \mathcal{N}_i \\ 0, & j \neq i, j \notin \mathcal{N}_i \\ -\sum_{j\neq i} L_{ij} & \text{otherwise} \end{cases}$$

We recall the following properties of the Laplacian.

Lemma 1: Let $L \in \mathbb{R}^{n\times n}$ denote the Laplacian matrix of the undirected graph G. Clearly, $L = L^\top$. Let $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ denote the ordered list of eigenvalues of L. Then, we have that

1) $\lambda_1 = 0$, with $\mathbf{1_n} \in \mathbb{R}^{\mathbf{n}}$ as the corresponding eigenvector. We will drop the subscript n where the dimensionality is unambiguous.
2) G is connected if and only if $\lambda_2 > 0$.
3) The edge set is empty, and the graph is fully disconnected, if and only $\lambda_n = 0$.

It is clear that a combination of λ_2 and λ_n could serve as a connectivity metric for the OCP. However, L is not a continuous function of the distance between two agents. Moreover, the eigenvalues may not be computable as part of off-the-shelf OCP solvers and, therefore, we need a simpler way to get information about the connectivity.

A continuously differentiable, weighted (distance-based) Laplacian $L_w \equiv L_w(\boldsymbol{x})$ is constructed as follows.

Definition 1 (Smooth step): For $\delta \in \mathbb{R}$, let $\sigma_\alpha(\cdot; \delta) : \mathbb{R} \to [0, 1]$ denote the smooth step function given by

$$\sigma_\alpha(x, \delta) = \frac{1}{2} \left(1 - \tanh(\alpha(x - \delta))\right)$$

where $\alpha > 0$ governs the steepness of the step.

Definition 2: Let $i = (j, k) \in E$ denote the edge with label i, and connecting nodes $j, k \in V$. The set of edge weights is written as $W_E = \{w_1, \dots, w_m\}$ denote the set of edge weights, where

$$w_i = w_{j,k} \triangleq \sigma_\alpha(\|\boldsymbol{r}_i\|; r_{\text{thr}})$$

for a sufficiently large $\alpha \gg 1$.

Definition 3: We define the weighted Laplacian L_w as

$$L_{w,ij} = \begin{cases} -w_{i,j} & i \neq j, \quad (i,j) \in E \\ 0 & i \neq j, \ (i,j) \notin E \\ \sum_k w_{i,k} & i = j \end{cases} \quad (5)$$

and the augmented (unweighted and weighted) Laplacians as

$$\Lambda = L + \mathbf{1}\mathbf{1}^\top, \ \Lambda_w = L_w + \mathbf{1}\mathbf{1}^\top \quad (6)$$

The following lemma allows us to obtain an analytical expression for the determinant of the augmented Laplacian.

Lemma 2: Let $\mathcal{S}_T$ denote the set of all spanning trees of G; i.e., $\mathcal{S}_\mathcal{T} = \{(i_1, \ldots, i_k) \colon i_1, \ldots, i_k \in E \text{ and } (i_1, \ldots, i_k) \text{ is a spanning tree of } G\}$ Let $\mathcal{S}$ denote the set of products of edge weights containing all the spanning trees of G; i.e., $\mathcal{S} = \{\mathbf{S} \colon \mathbf{S} = w_{i_1} \cdots w_{i_k}; (i_1, \ldots, i_k) \in \mathcal{S}_T\}$. Then,

$$\det(L_w + \mathbf{1}\mathbf{1}^\top) = n^2 \left(\sum_{\mathbf{S} \in \mathcal{S}} \mathbf{S} \right). \quad (7)$$

Proof: Since L_w is a weighted graph Laplacian, it follows that $\det(L_w) = 0$ and all the cofactors of the matrix are equal (from the weighted matrix-tree theorem) . Let $L_w(i,j)$ denote the $(i,j)^{th}$ cofactor of L_w; i.e., the cofactor obtained by removing the i^{th} row and the j^{th} column. According to the weighted matrix-tree theorem,

$$\sum_{\mathcal{S}} w_{i_1} \cdots w_{i_k} = \left(\sum_{\mathbf{S} \in \mathcal{S}} \mathbf{S} \right) = (-1)^{i+j} L_w(i,j) \quad (8)$$

for any pair $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ [4]. Let $\mathrm{adj}(L_w)$ denote the adjugate (adjoint) of L_w. Since, the cofactors L_w are all equal (8), one can then deduce that $\mathrm{adj}(L_w) = \left(\sum_{\mathbf{S} \in \mathcal{S}} \mathbf{S} \right) \mathbf{1}\mathbf{1}^\top$. Using matrix-determinant lemma, one can then obtain

$$\begin{aligned} \det(L_w + \mathbf{1}\mathbf{1}^\mathrm{T}) &= \det(L_w) + \mathbf{1}^\top \mathrm{adj}(L_w) \mathbf{1} \\ &= 0 + \mathbf{1}^\top \left(\sum_{\mathbf{S} \in \mathcal{S}} \mathbf{S} \right) \mathbf{1}\mathbf{1}^\top \mathbf{1} \\ &= n^2 \left(\sum_{\mathbf{S} \in \mathcal{S}} \mathbf{S} \right). \end{aligned}$$

We state the main result of this section.

Theorem 1: Let Λ_w denoted the weighted augmented Laplacian of an undirected graph G. Then, $\mathrm{Rank}(\Lambda_w) = n$ iff G is connected.

Proof: We start by proving the sufficiency of the result. Consider the case where the graph G is connected and $\mathrm{Rank}(L_w) = n - 1$. It follows from Lemma 2 that Λ_w is also full ranked (since its determinant is non-zero).

To prove the necessity, suppose that Λ_w is full-ranked but that G is not connected. Since G is not connected, there exists a vector $\boldsymbol{p} \in \mathbb{R}^n$ satisfying

$$L_w \boldsymbol{p} = 0, \quad \boldsymbol{p}^\top \mathbf{1} = 0$$

where the second equation follows from $L_w = L_w^\top$. It follows that $\mathbf{p}^\top \Lambda_w = 0$, which contradicts our supposition that Λ_w is full-ranked. Thus, it follows that if Λ_w is full-ranked, then G must be connected. ■

Example 1: Consider the 3-complete graph G with edge weights w_{12}, w_{23}, and w_{31}. Then, it is easy to verify that

$$\det(\Lambda_w) = 9(w_{12}w_{23} + w_{31}w_{23} + w_{12}w_{31}) \ ■$$

Lemma 3: Let $0 \leq \lambda_1' \leq \ddots \leq \lambda_n'$ denote the ordered set of eigenvalues of Λ_w for an undirected graph G with n vertices. Then G is fully disconnected if and only $\lambda_{n-1}' = 0$.

Proof: We first prove the necessity. For a fully disconnected graph, $L_w = 0$, so that $\Lambda_w = \mathbf{1}\mathbf{1}^\top$. It is easy to check that $\mathrm{Rank}(\mathbf{1}\mathbf{1}^\top) = 1$. Thus, $\lambda_{n-1}' = 0$. In order to prove the sufficiency, suppose that $\lambda_{n-1}' = 0$. Then, there exist $n - 1$ mutually orthogonal eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ satisfying $\boldsymbol{v}_i^\top \mathbf{1} = 0$ (since $\mathbf{1}$ is an eigenvector of Λ_w with eigenvalue n) and $L_w \boldsymbol{v}_i = 0$ for all $i = 1, \ldots n-1$. Since $L_w \mathbf{1} = 0$, it follows that all eigenvalues of L_w are zero. Hence L_w is a zero matrix and G is fully disconnected. ■

The analysis presented above suggests that $\det(\Lambda_w)$ can serve, entirely by itself or in combination with other terms, as the objective function for the OCP. Notice that it is a positive semi-definite function of the graph connectivity, takes a value of zero if and only if the graph is disconnected, and is continuously differentiable with respect to the inter-agent distances.

There are two significant limitations of $\det(\Lambda_w)$ as an objective function. First, its value is proportional to n^2 and the number of spanning trees. Second, while $\det(\Lambda_w) = 0$ indicates a loss of connectivity, it does not provide any information about the connectivity of the result components of the swarm. As a result, loss of connectivity needs to be interpreted conservatively, as possibly just one agent is being pulled away from the remaining swarm.

To circumvent this limitation of $\det(\Lambda_w)$, we need additional metrics which capture the size of the largest component of the fragmented swarm. Thus, for the optimal control problems described in the previous section, we use an additional cost function $C(t) = \lambda_{n-1}'(1 + \lambda_1')$.

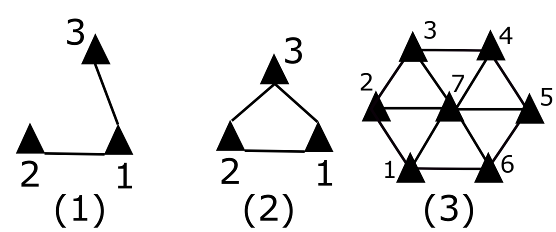


Fig. 2: Different simulation scenarios. Lines between the agents show the initial communication link. Case (1) Agent 1 has communication link with Agents 2 and 3 only. Case (2) All three agents can communicate with each other. Case (3) Six agents are arranged in a hexagon around agent 7, where agent 7 can communicate with all agents while other agents can communicate with their geometric neighbors and agent 7.

IV. Numerical Results

A. Problem formulation

In this section, we solve a modified version of the OCP (4)

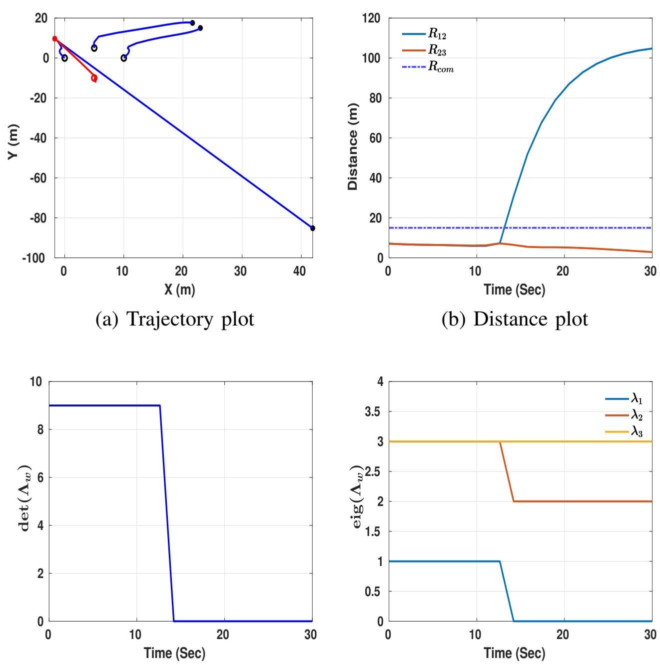
$$\min_{u_p[0:t_f]} \int_0^{t_f} \left(C + W_1 u_p^T u_p + W_2 \left(\frac{1}{\|\boldsymbol{r}_{ip}\|} \right)^2 \right) dt \qquad (9)$$

subject to the dynamics (1) and (3). We consider two cost functions for a comparative study:

$$C_1(t) = \det(\Lambda_w), C_2(t) = \lambda'_{n-1}(1 + \lambda'_1) \qquad (10)$$

where $0 \leq \lambda'_1 \leq \cdots \leq \lambda'_n$ is the ordered set of eigenvalues of Λ_w. The parameters of the flocking model used for the simulation are as follows: $k_r = 0.02$, $k_v = 0.2$, $k_d = 0.2$, $k_p = 10$, $R_{\text{safe}} = 5m$, $R_{\text{com}} = 15m, R_{\text{fear}} = 15$, $\alpha = 4$, $\delta = 14.5, W_1 = W_2 = 0.1$ to perform the simulations. We consider three starting scenarios as shown in Figure 2. We use the ICLOCS2 toolbox [5] in MATLAB to solve the OCP.

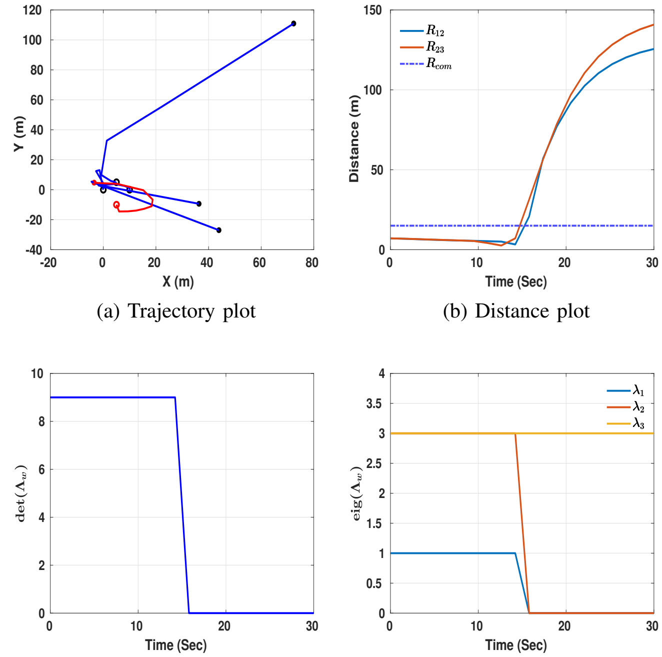
We use the following metric to analyze the results (i) the distance between swarm agents (ii) the determinant of the augmented Laplacian det(Λ_w), and (iii) the eigenvalues of the augmented Laplacian eig(Λ_w). For each case, we evaluate the performance with the proposed two connectivity metrics in (10).



(a) Trajectory plot (b) Distance plot

(c) det(Λ_w) (d) eig(Λ_w)

Fig. 3: Simulation Results for Case 1 with $C_1(t)$. In the trajectory subplot, the pursuer's trajectory is shown in red, while the agents' trajectory is shown in blue.



(a) Trajectory plot (b) Distance plot

(c) det(Λ_w) (d) eig(Λ_w)

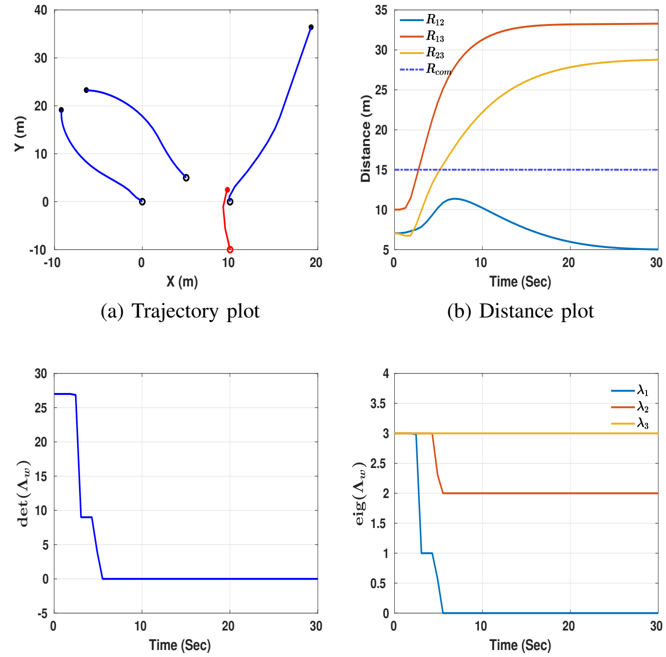
Fig. 4: Simulation Results for Case 1 with $C_2(t)$

B. Simulation results for the three cases

The results for Case 1 using different connectivity metrics are shown in Figures 3 and 4, respectively. We briefly clarify the notation used in the plots. In the plot showing the trajectory, the red curve indicates the pursuer's trajectory while the blue curves show the antagonistic agents' trajectory. The plot showing the inter-agent distance shows the inter-agent communication threshold using a dotted line. A necessary condition for the swarm to be fragmented is the $\det(\Lambda_w) = 0$ (see the subplots (c)). Finally, the eigenvalues of Λ_w are shown in subplot (d).

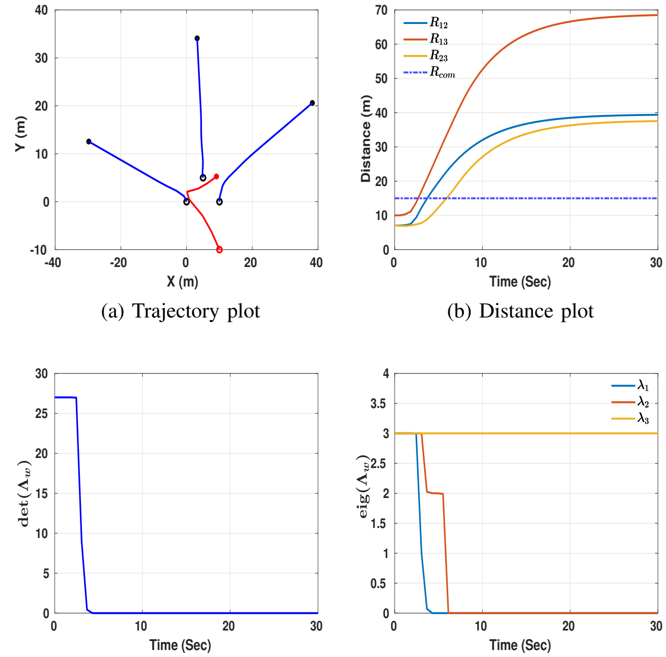
The results show that when the connectivity metric C_1 is used as the cost function, the optimal control law ensures that one agent is pushed away from the the other two agents which continue to remain connected. In contrast, when the metric C_2 is used, we observe that the pursuer is able to execute a trajectory which fragments the swarm entirely; i.e., $|E| = 0$ which is equivalent to $\lambda_2' = 0$ (see Lemma 3). The results for Case 2, presented in Figures 5 and 6 for connectivity metrics C_1 and C_2 respectively, are similar to those for Case 1.

Finally, the results for Case 3, with connectivity metrics C_1 and C_2, respectively, are presented in Figs. 7 and 8. Like Case 1 and Case 2, the use of the connectivity metric C_2 leads to a higher degree of fragmentation compared to C_1. The trajectories of the individual swarm agents shows that they are pushed to distances far greater than those seen in Cases 1 and 2. This appears to be due to the repeller term in (1), which creates a rebound effect when the distance between two agents becomes considerably smaller than the



(a) Trajectory plot (b) Distance plot (c) $\det(\Lambda_w)$ (d) $\mathrm{eig}(\Lambda_w)$

Fig. 5: Simulation Results for Case 2 with $C_1(t)$



(a) Trajectory plot (b) Distance plot (c) $\det(\Lambda_w)$ (d) $\mathrm{eig}(\Lambda_w)$

Fig. 6: Simulation Results for Case 2 with $C_2(t)$

safe inter-agent distance R_{safe}.

C. Effect of variations in k_p, k_v on the feasibility of optimization for Case 2

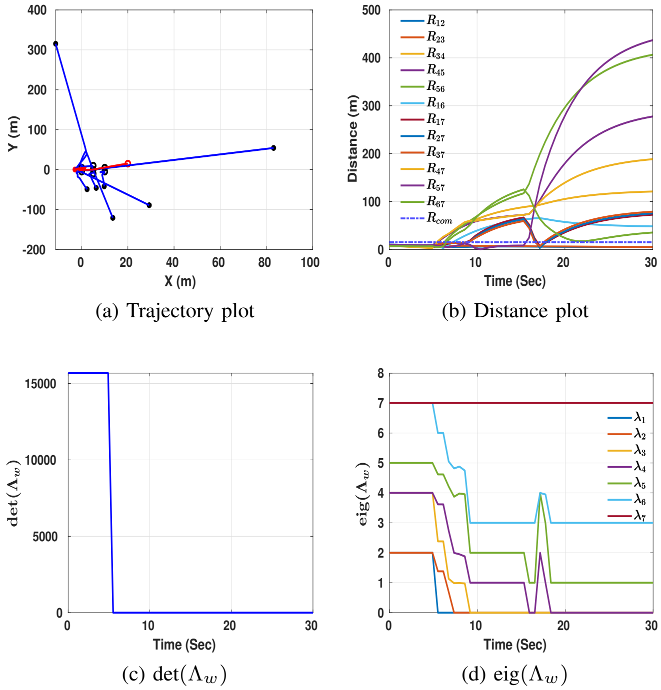
The optimal control problem illustrated earlier may lead to fragmentation or loss of connectivity for all values of the system parameters (k_p, k_v, etc. in (1)). As an illustration, we identify the values of k_p and k_v for which the solution of the optimal control problem leads to a fragmented or disconnected swarm. We use the metric C_2 from (10) with $W_1 = W_2 = 0.01$ in (9). Figure 9 shows the effect of k_p vs k_v on the fragmention of the swarm. The plot marks the tested values of (k_v, k_p), with the color of the marker indicating the outcome. The broad trend appears to be that k_p and k_v need to lie inside a bounded region of the quadrant. When k_p is small, the penalty imposed by C_2 on the pursuer's approach distance from individual agents limits the ability of the pursuer to fragment the swarm. An unexpected observation is that the ability of the pursuer to break the swarm may also be jeopardized by high values of k_p. We would have expected a high value of k_p ought to generate a high degree of instability in the swarm; however, for the three-agent team in Case 2, the high value of k_p simply causes *all* of the agents to move away rapidly from the pursuer without losing connectivity. This explanation suggests that the success of the optimal control technique may be sensitive to k_v, k_p as well as the size of the swarm. Further analysis along these lines is left as a subject for future work.

V. Conclusions

In this paper, we addressed the problem of fragmenting a swarm of agents using a single pursuer. We posed the fragmentation problem as an optimal control problem (OCP) and solved it numerically. We introduced novel connectivity metrics which enabled the formulation of the OCP as well as its numerical solution. We illustrated our techniques for a few canonical examples and identified the bounds on the modeling parameters for which fragmentation is feasible.

The goal of fragmentation was motivated by swarms which rely on heterogeneous agents with non-overlapping capabilities to complete their mission. Fragmentation can be viewed as a tool to break the swarm into functionally inadequate subunits, thereby providing an intermediate solution between herding the swarm and engaging in a combat of attrition.

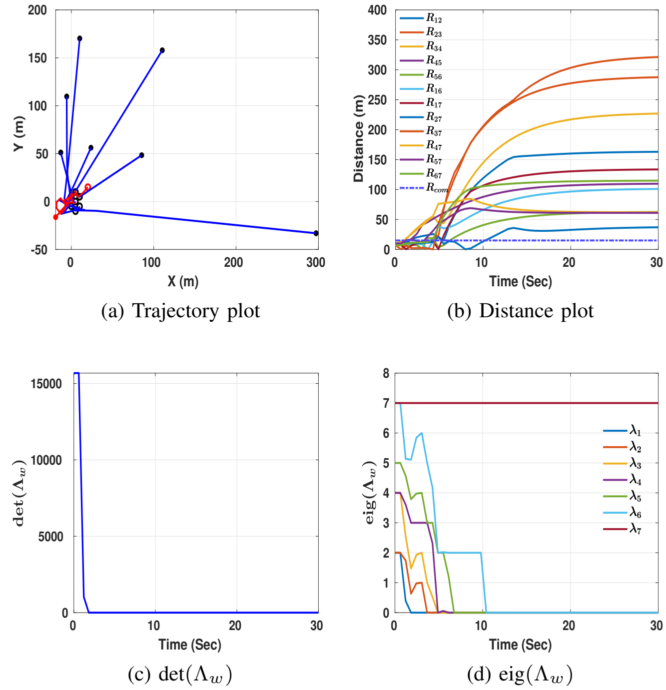
The work presented in this paper can be extended along a number of lines. First, it is open problem to determine how the results presented in the paper would scale to larger swarms. Second, the OCP presented here relies on the knowledge of the underlying graph and it is essential to extend it using a combination of estimation and methods robust to uncertainty. Finally, as seen in Sec. IV, fragmentation works for a range of system parameters, and herding or attrition-driven engagement might be essential when the parameters fall outside this range. It remains to be seen whether a general OCP can be formulated such that modes (herding, functional incapacitation via fragmentation, and attrition) actually arise as a solution to the problem.



(a) Trajectory plot (b) Distance plot

(c) $\det(\Lambda_w)$ (d) $\mathrm{eig}(\Lambda_w)$

Fig. 7: Simulation Results for Case 3 with $C_1(t)$



(a) Trajectory plot (b) Distance plot

(c) $\det(\Lambda_w)$ (d) $\mathrm{eig}(\Lambda_w)$

Fig. 8: Simulation Results for Case 3 with $C_2(t)$

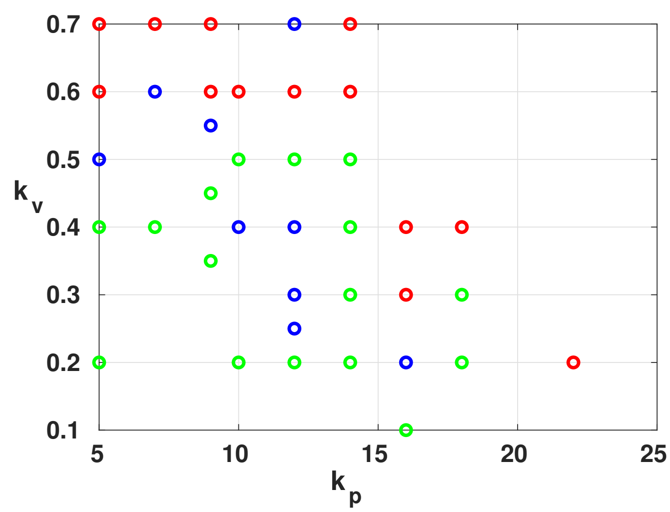


Fig. 9: Optimal solutions to a 3-agent, fully connected problem with $C_2(t)$ over varying k_p, k_v. ○ shows fully fragmented swarm i.e. $|E| = 0$, ○ shows only one agent is disconnected from the swarm i.e. $\lambda_1' = 0$, and ○ shows a fully connected swarm.

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