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A Novel Free-Matrix-Based Summation Inequality for Stability Analysis of Discrete-Time Delayed System

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Abstract— **This paper introduces an improved stability criterion for discrete-time systems with time-varying delay. A novel summation inequality based on the free-matrix is suggested which considers the augmented vector of the state and its forward difference. Additionally, the proposed summation inequality is employed to derive an improved stability criterion for the discrete-time system with time-varying delay. A new Lyapunov-Krasovskii functional is established for applying the summation lemma to reduce the conservatism of the stability analysis. To verify the effectiveness of the proposed approach, the maximum admissible upper bounds of the proposed method is presented in comparison to existing methods with two numerical examples.**

I. INTRODUCTION

In the real world, signals and information require communication time to be transmitted. Due to this physical limitation, there are various dynamic system models with time-varying delays, whose subsequent outputs depend on the previous state of their system [1][2]. However, such time-varying delays may cause the performance degradation of the systems and even become a source of system instabilities. Therefore, stability analysis of the time-delay systems garnered great attention from academic and industrial researchers in recent decades[3]-[4].

Since there is no analytic method available for stability analysis with the time-varying delayed system, the Lyapunov-Krasovskii Functional(LKF) approach is one of the primary methods used for stability of the system with time-varying delay. The two main steps of the LKF approach involve constructing appropriate LKFs and applying precise bounding techniques [5]. Therefore several techniques have been proposed to derive stability criterion with reduced conservativeness for this purpose. For example, the novel delay-square-dependent LKF was suggested for stability analysis of discrete-time delayed systems [6].

A common method used for stability analysis is the application of summation inequalities. There are various types of integral inequalities and summation inequalities that have been proposed for stability analysis [7][8]. By estimating these terms, the stability criterion can be represented based on the linear matrix inequalities (LMIs) approach. For example, Jensen's inequality [9][10] showed acceptable performance behavior with fewer decision variables. Moreover, several other improved summation inequalities were developed to figure out the more concise bounds such as the Wirtinger-based integral inequality[11][12], Bessel inequality[13], auxiliary function-based integral inequality[14]. These inequalities enable the estimation of the upper bound of the summation terms that arise in the forward difference of LKFs.

Recently, some free-matrix-based integral inequalities and summation inequalities have been proposed which allows handling with the use of multiple integral terms for the stability analysis of a continuous-time and discrete-time system with time-varying delay[15][16]. Nevertheless, the free-matrix-based inequality for discrete-time such as in [17] only gives the estimation of energy of the state or its forward difference without consideration of the cross information of them. In continuous-time, the novel free-matrix-based integral inequalities are suggested that involve the augmented vectors to enlarge freedom for reducing the conservatism of the inequality[18][19]. However, there is still room for discrete-time stability analysis by reducing the conservativeness of the stability criterion.

In this paper, inspired by the preceding discussion, we develop a new summation inequality based on the free-matrix-based method. By using the augmented vector of the state and its forward difference, the summation inequality reduces conservativeness. Also, a novel improved stability criterion for the discrete-time system is derived by using the proposed summation inequality. Two numerical examples show the effectiveness of the proposed method compared to the existing methods.

Notation: Throughout this paper, the superscripts '-1' and 'T' indicate the inverse matrix and the transpose matrix of a given matrix. $\mathbb{R}^n$ represents the n-dimensional Euclidean space and '$*$' stands for the symmetric terms in a symmetric matrix. $P > 0$ implies that P is the positive definite matrix. **sym**$\{X\}$ denotes $X + X^T$ for square matrix X and $diag\{\cdot\}$ signifies the a block-diagonal matrix. The matrix I_n stands for the $n \times n$ identity matrix and the matrix 0 represents the zero matrix with appropriate dimension. Define $x_a(i) = x(a+i)$, and $y_a(i) = x_a(i+1) - x_a(i)$.

II. PROBLEM STATEMENT

Consider the following linear system that contains a time-varying delay term for $h(k)$:

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$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-h(k)), & k \geq 0, \\ x(k) = \psi(k), & k \in [-h_2, 0], \end{cases} \tag{1}$$

where the $x(k)$ represent the state vector, $\psi(k)$ is the initial condition with the time instant $k \in [-h_2, 0]$. A, $A_d \in \mathbb{R}^{n\times n}$ are the constant system matrices.

The time-varying delay $h(k)$ with time instant k satisfies the condition

$$h_1 \leq h(k) \leq h_2,$$

for the positive constant integer h_1 and h_2.

In this section, motivated from [18], a new free-matrix-based summation inequality lemma are suggested.

Lemma 2.1 Let $x(k) \in \mathbb{R}^n$ be a vector function and m are constant integer with $m > 1$. The positive definite matrix $R \in \mathbb{R}^{2n\times 2n}$ and the free matrices $Y_i \in \mathbb{R}^{mn\times n} (i = 1, 2, 3)$, the following summation inequality holds:

$$-\sum_{i=0}^{m-1} \begin{bmatrix} x_a(i) \\ y_a(i) \end{bmatrix}^T R \begin{bmatrix} x_a(i) \\ y_a(i) \end{bmatrix} \leq \zeta^T(k)\Omega(m)\zeta(k) \tag{2}$$

where

$$\Omega(h) = h\,[Y_1 \quad Y_2]\,R^{-1}\,[Y_1 \quad Y_2]^T + \frac{h}{3}\,[Y_3 \quad 0]\,R^{-1}\,[Y_3 \quad 0]^T$$
$$+ sym\left\{Y_1 N_1^T + Y_2 N_2^T + Y_3 N_3^T\right\},$$
$$N_1^T \zeta(k) = x_a(m) - x_a(0),$$
$$N_2^T \zeta(k) = \sum_{i=0}^{m} x_a(i) - x_a(m),$$
$$N_3^T \zeta(k) = x_a(m) + x_a(0) - \frac{2}{m+1}\sum_{i=0}^{m} x_a(i).$$

Proof 2.1: Before the prove process, define the following structured matrix $Y \in \mathbb{R}^{2mn\times 2n}$ and orthogonal scalar functions $p_i(s) (i = 1, 2)$ such that

$$Y = \begin{bmatrix} Y_3 & 0 \\ Y_1 & Y_2 \end{bmatrix}, p_1(i) = 1,\ p_2(i) = \frac{2i - (m-1)}{m+1},$$
$$\sum_{i=0}^{m-1} p_1^2(i) = m,\ \sum_{i=0}^{m-1} p_2^2(i) = \frac{m(m-1)}{3(m+1)},$$
$$\sum_{i=0}^{m-1} p_1(i)p_2(i) = 0,$$

Since $R > 0$, the following summation inequality can be derived as follows:

$$0 \leq \sum_{i=0}^{m-1} \bar{\zeta}^T(k) \begin{bmatrix} YR^{-1}Y^T & Y \\ Y^T & R \end{bmatrix} \bar{\zeta}(k)$$
$$= \sum_{i=0}^{m-1} \begin{bmatrix} x_a(i) \\ y_a(i) \end{bmatrix}^T R \begin{bmatrix} x_a(i) \\ y_a(i) \end{bmatrix}$$
$$+ \zeta^T(k) \sum_{i=0}^{m-1} \begin{bmatrix} p_2(i)I \\ p_1(i)I \end{bmatrix}^T YR^{-1}Y^T \begin{bmatrix} p_2(i)I \\ p_1(i)I \end{bmatrix} \zeta(k)$$
$$+2\zeta^T(k) \sum_{i=0}^{m-1} \begin{bmatrix} p_2(i)I \\ p_1(i)I \end{bmatrix}^T Y \begin{bmatrix} I & 0 \\ 0 & (m+1)I \end{bmatrix} \begin{bmatrix} y_a(i) \\ \frac{1}{(m+1)} x_a(i) \end{bmatrix}$$
$$= \sum_{i=0}^{m-1} \begin{bmatrix} x_a(i) \\ y_a(i) \end{bmatrix}^T R \begin{bmatrix} x_a(i) \\ y_a(i) \end{bmatrix}$$
$$+ \zeta(k)^T \{m\,[Y_1\ Y_2]\,R^{-1}\,[Y_1\ Y_2]^T$$
$$+ \frac{m(m-1)}{3(m+1)}\,[Y_3\ 0]\,R^{-1}\,[Y_3\ 0]^T\}\zeta(k)$$
$$+ \mathbf{sym}\left\{\zeta^T(k)\left(Y_1(x_a(m) - x_a(0))\right.\right.$$
$$\left.\left. + Y_2\big(\sum_{i=0}^{m} x_a(i) - x_a(m)\big) + Y_3 \sum_{i=0}^{m-1} p_2(i) y_a(i)\right)\right\},$$

where

$$\bar{\zeta}(k) = \left[(p_2(s)\zeta(k))^T \quad (p_1(s)\zeta(k))^T \mid x_a^T(i) \quad y_a^T(i)\right]^T.$$

Since

$$\sum_{i=0}^{m-1} p_2(i) y_a(i) = x_a(m) + x_a(0) - \frac{2}{(m+1)} \sum_{i=0}^{m} x_a(i),$$

and $\frac{m-1}{m+1} \leq 1$, the (2) can be derived. This completes the proof. ■

Remark 1: Note that the new free-matrix-based summation inequality proposed in Lemma 2.1 is the generalized version of the summation inequality in [20]. By letting $Y_2 = 0$, the summation inequality can be driven easily. Since Y_2 is related to both $x_a(i)$ and $y_a(i)$, the proposed lemma in Lemma 2.1 includes the additional information of them for reducing the conservatism.

III. MAIN RESULTS

In this section, the improved stability analysis are introduced with using the developed lemma. For brevity, the following notations are introduced to simplify the representation:

$$\zeta(k) = [\zeta_1^T(k), \zeta_2^T(k)]^T$$
$$\zeta_1(k) = [x^T(k), x^T(k-h_1), x^T(k-h_k), x^T(k-h_2)]^T,$$
$$\zeta_2(k) = [s_1^T(k), s_2^T(k), s_3^T(k)]^T,$$
$$h_k = h(k),\ h_{12} = h_2 - h_1,\ h_{k1} = h_k - h_1,$$
$$h_{2k} = h_2 - h_k,\ g_1(h) = h + 1,$$
$$s_1(k) = \sum_{i=k-h_1}^{k} \frac{x(i)}{g_1(h_1)}, \quad s_2(k) = \sum_{i=k-h_k}^{k-h_1} \frac{x(i)}{g_1(h_{k1})},$$
$$s_3(k) = \sum_{i=k-h_2}^{k-h_k} \frac{x(i)}{g_1(h_{2k})},$$
$$\chi_1(k) = [x^T(k), \sum_{i=k-h_1}^{k-1} x^T(i), \sum_{i=k-h_2}^{k-h_1-1} x^T(i)]^T,$$
$$\chi_{2,k}(i) = [x^T(k), x^T(i)]^T$$
$$e_i = \left[0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(7-i)n}\right], \quad (i = 1, 2, \cdots, 7)$$
$$e_s = Ae_1 + A_d e_3,$$

The following stability criterion for the discrete-time linear system (1) can be derived using the summation inequality lemma (2).

Theorem 3.1 For the time-varying delay h_k with given nonnegative constant h_1 and h_2 satisfying (2), linear discrete-time varying delay system (1) is asymptotically stable if there exists symmetric positive definite matrices $P > 0$, $Q_1 > 0 (i = 1, 2)$, $R_i > 0 (i = 1, 2)$ and free matrices Y_1, Y_2, Y_3 such that

$$\begin{bmatrix} \Gamma(h_1) & h_1\overline{Y_1} & h_{12}\overline{Y}_3 \\ * & -h_1\overline{R_1} & 0 \\ * & * & -h_{12}\overline{R_2} \end{bmatrix} < 0, \quad (3)$$

$$\begin{bmatrix} \Gamma(h_2) & h_1\overline{Y_1} & h_{12}\overline{Y}_2 \\ * & -h_1\overline{R_1} & 0 \\ * & * & -h_{12}\overline{R_2} \end{bmatrix} < 0, \quad (4)$$

where

$$\begin{aligned}
\Gamma(h_k) &= \Phi_1(h_k) + \Phi_2(h_k) + \Phi_3(h_k), \\
\Phi_1(h_k) &= \Lambda_2^T\Lambda_2 - \Lambda_1^T P\Lambda_1 + \textbf{sym}\{(\Lambda_2 - \Lambda_1)^T P\Lambda_0(h_k)\}, \\
\Phi_2(h_k) &= \omega_1^T Q_1\omega_1 - \omega_2^T Q_1\omega_2 + h_1\omega_3^T Q_1\omega_3 \\
&\quad + \omega_2^T Q_2\omega_2 - \omega_4^T Q_2\omega_4 + h_{12}\omega_3^T Q_2\omega_3, \\
&\quad + \textbf{sym}\{\omega_3^T Q_1\omega_5 + \omega_3^T Q_2\omega_6(h_k)\}, \\
\Phi_3(h_k) &= \omega_7^T(h_1R_1 + h_{21}R_2)\omega_7 \\
&\quad + \textbf{sym}\{Y_{11}N_{11}^T + Y_{12}N_{12}^T + Y_{13}N_{13}^T + Y_{21}N_{21}^T \\
&\quad + Y_{22}N_{22}^T + Y_{23}N_{23}^T + Y_{31}N_{31}^T + Y_{32}N_{32}^T + Y_{33}N_{33}^T\} \\
\Lambda_1 &= [e_1^T,\ -e_1^T,\ -e_2^T - e_3^T]^T, \\
\Lambda_2 &= [e_s^T,\ -e_2^T,\ -e_3^T - e_4^T]^T, \\
\Lambda_0(h_k) &= [0,\ g_1(h_1)e_5^T,\ g_1(h_{k1})e_6^T + g_1(h_{2k})e_7^T]^T, \\
\omega_1 &= [e_1^T,\ e_1^T]^T\zeta(k), \qquad \omega_2 = [e_1^T,\ e_2^T]^T\zeta(k), \\
\omega_3 &= [e_s^T - e_1^T,\ 0]^T\zeta(k),\ \omega_4 = [e_1^T,\ e_4^T]^T\zeta(k), \\
\omega_5 &= [h_1e_1^T,\ g_1(h_1)e_5^T - e_2^T]^T\zeta(k), \\
\omega_6(h_k) &= [h_{21}e_1^T,\ g_1(h_{k1})e_6^T + g_1(h_{2k})e_7^T - e_3^T - e_4^T]^T\zeta(k), \\
\omega_7 &= [e_s^T - e_1^T, e_1^T]^T\zeta(k), \\
\Omega_i(h) &= h\,[Y_{i1}\ Y_{i2}]\,R_1^{-1}\,[Y_{i1}\ Y_{i2}]^T + \frac{h}{3}\,[Y_{i3}\ 0]\,R_1^{-1}\,[Y_{i3}\ 0]^T \\
&\quad + \textbf{sym}\{Y_{i1}N_{i1}^T + Y_{i2}N_{i2}^T + Y_{i3}N_{i3}^T\} \\
&\quad (i = 1, 2, 3) \\
\overline{Y_i} &= [Y_{i1}\ Y_{i2}\ Y_{i3}\ 0],\ Y_{ij} \in \mathbb{R}^{7n\times n} \qquad (i = 1, 2, 3) \\
\overline{R}_i &= diag\{R_i, 3R_i\},\ R_i \in \mathbb{R}^{2n\times 2n} \qquad (i = 1, 2)
\end{aligned}$$

Proof) Consider a LKF $V(k) = \sum_{i=1}^{3} V_i(k)$, where

$$V_1(k) = \begin{bmatrix} x(k) \\ \sum_{i=k-h_1}^{k-1} x(i) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix}^T P \begin{bmatrix} x(k) \\ \sum_{i=k-h_1}^{k-1} x(i) \\ \sum_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix}, \quad (5)$$

$$\begin{aligned}
V_2(k) &= \sum_{i=k-h_1}^{k-1} \begin{bmatrix} x(k) \\ x(i) \end{bmatrix}^T Q_1 \begin{bmatrix} x(k) \\ x(i) \end{bmatrix} \\
&\quad + \sum_{i=k-h_2}^{k-h_1-1} \begin{bmatrix} x(k) \\ x(i) \end{bmatrix}^T Q_2 \begin{bmatrix} x(k) \\ x(i) \end{bmatrix}, \quad (6)
\end{aligned}$$

$$\begin{aligned}
V_3(k) &= \sum_{i=k-h_1}^{k-1}\sum_{j=i}^{k-1} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}^T R_1 \begin{bmatrix} x(j) \\ y(j) \end{bmatrix} \\
&\quad + \sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-1} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}^T R_2 \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}, \quad (7)
\end{aligned}$$

where $y(k) = x(k+1) - x(k)$.

Throughout the trajectory of the system (1), the forward difference of $V_i(k)$ is defined as $\Delta V_i(k) = V_i(k+1) - V_i(k)$ $(i = 1, 2, 3)$. Note that $\chi_1(k) = (\Lambda_0(h_k) + \Lambda_1)\zeta(k)$ and $\chi_1(k+1) = (\Lambda_0(h_k) + \Lambda_2)\zeta(k)$, so the forward difference of the V_1 yields:

$$\begin{aligned}
\Delta V_1(k) &= \chi_1^T(k+1)P\chi_1(k+1) - \chi_1(k)^T P\chi_1(k) \\
&= \xi(k)^T\Phi_1(h_k)\xi(k). \quad (8)
\end{aligned}$$

Then, we compute the forward difference of the V_2, throughout the trajectory of the system (1).

$$\begin{aligned}
&\Delta V_2(k) \\
&= \chi_{2,k}^T(k)Q_1\chi_{2,k}(k) - \chi_{2,k}^T(k-h_1)Q_1\chi_{2,k}(k-h_1) \\
&\quad + \sum_{i=k-h_1+1}^{k} \Delta(\chi_{2,k}(i)^T Q_1\chi_{2,k}(i)) \\
&\quad + \chi_{2,k}(k-h_1)^T Q_2\chi_{2,k}(k-h_1) - \chi_{2,k}(k-h_2)^T Q_2 \\
&\quad\quad \chi_{2,k}(k-h_2) + \sum_{i=k-h_2+1}^{k-h_1} \Delta(\chi_{2,k}(i)^T Q_2\chi_{2,k}(i)) \\
&= \zeta^T(k)\Phi_2(h_k)\zeta(k) \quad (9)
\end{aligned}$$

where $\Delta\big(\chi_{2,k}^T(i)Q_i\chi_{2,k}(i)\big) = \chi_{2,k+1}^T(i)Q_1\chi_{2,k+1}(i) - \chi_{2,k}^T(i)Q_1\chi_{2,k}(i)$.

Throughout the trajectory of the system (1), calculating the forward difference of V_3 leads to:

$$\begin{aligned}
\Delta V_3(k) &= \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T (h_1R_1 + h_{21}R_2) \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \\
&\quad - \sum_{i=k-h_1}^{k-1} \begin{bmatrix} y(i) \\ x(i) \end{bmatrix}^T R_1 \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} - \sum_{i=k-h_k}^{k-h_1-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T R_2 \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
&\quad - \sum_{i=k-h_2}^{k-h_k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T R_2 \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \quad (10)
\end{aligned}$$

By applying Lemma 1 to estimate the summation terms in (10), we can get

$$\begin{aligned}
\Delta V_3(k) &\leq \zeta^T(k)\Big\{\omega_7^T(h_1R_1 + h_{21}R_2)\omega_7 + \Omega_1(h_1) \\
&\quad + \Omega_2(h_{k1}) + \Omega_3(h_{2k})\Big\}\zeta(k) \quad (11)
\end{aligned}$$

Therefore, combining the (8) with (9), (11), we get:

$$\begin{aligned}
\Delta V(k) &\leq \zeta^T(k)\Big\{\Phi_1(h_k) + \Phi_2(h_k) + \Phi_3(h_k) + h_1\bar{Y_1}\bar{R_1}\bar{Y_1}^T \\
&\quad + h_{k1}\bar{Y_2}\bar{R_2}\bar{Y_2}^T + h_{2k}\bar{Y_3}\bar{R_3}\bar{Y_3}^T\Big\}\zeta(k)
\end{aligned}$$

$$= \zeta^T(k)\,\Xi(h_k)\,\zeta(k) \tag{12}$$

According to Schur's complement, inequality (12) can be represented as LMI forms which are equivalent to the LMI condition (3) and (4). Since $\Xi(h_k) \leq 0$ is affine with $h_k \in [h_1, h_2]$, the negativity condition of the LKF (5) is equivalent to (3) and (4). This completes the proof. ■

Remark 2: Compared to Theorem 1 proposed in [21], it is shown that Theorem 1 can provide the less conservative stability criterion. The double summation terms using augmented vectors with $x(j)$ and $y(j)$ are used for the LKF term $V_3(k)$ to utilize more information about the given system. Also, comparing to [15], the cross information between state $x(k)$ and its forward difference $y(k)$ are utilized by using a summation lemma.

IV. NUMERICAL EXAMPLES

This section presents a comparison of the performance of the proposed method with that of existing methods through the use of the following two numerical examples.

Example 1: Consider the discrete linear time-delayed system (1) with the following matrices

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix},\ A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}.$$

Example (1) is widely used for stability analysis of discrete-time systems. Table I lists the maximum admissible upper bounds (MAUBs) h_2 for difference h_1 obtained by the proposed method and the other existing methods. Compared with result by obtained with existing method [22]-[23], the enlarged or at least the same MAUBs obtained by the proposed method. Furthermore, Theorem 1 produces comparable results to those of the existing method in [20][21] with fewer the number of variables(NVs). Thus, this indicates that Theorem 1 gives a less conservative result while also reducing the burden of computational complexity.

Example 2: Consider the discrete linear time-delayed system (1) with the following matrices

$$A = \begin{bmatrix} 0.68 & -0.4 \\ 0.40 & 0.52 \end{bmatrix},\ A_d = \begin{bmatrix} -0.1 & -0.2 \\ -0.2 & -0.1 \end{bmatrix}.$$

Table II lists the MAUBs of $h(k)$ for the different h_1. It is shown that Theorem 1 gives the larger or at least the same upper bound than those of obtained by existing method[27]-[29]. Considering the given system in *Example 2*, utilizing the cross information of state $x(k)$ and its forward difference $y(k)$ leads Theorem 1 to ensure the larger stability region. Thus, it is shown that Theorem 1 gives less conservatism of the stability criterion.

TABLE I
THE MAXIMUM ADMISSIBLE UPPER BOUNDS h_2 FOR DIFFERENT h_1 IN EXAMPLE 1

h_1	5	7	9	11	13	NVs
Thm. 5 [22]	21	22	23	22	23	$10.5n^2 + 3.5n$
Thm. 1 [24]	20	21	21	22	23	$29.5n^2 + 12.5n$
Thm. 1 [25]	20	21	21	22	23	$32.5n^2 + 6.5n$
Thm. 1 [26]	21	22	22	23	23	$78.5n^2 + 12.5n$
Thm. 1 [23]	21	22	22	23	24	$10.5n^2 + 3.5n$
Thm. 2 [20]	22	22	22	23	24	$97n^2 + 4n$
Thm. 1 [21]	22	22	22	23	24	$160.5n^2 + 5.5n$
Thm. 1 [proposed]	22	22	23	24	24	$75.5n^2 + 4.5n$

TABLE II
THE MAXIMUM ADMISSIBLE UPPER BOUNDS h_2 FOR DIFFERENT h_1 IN EXAMPLE 2

h_1	7	9	10	15	20	24	26	28
Thm.1 [27]	8	10	11	16	21	14	27	29
Thm.1 [28]	8	10	11	16	21	14	27	29
Thm.1 [29]	8	10	11	16	21	14	27	29
Thm.1 [30]	8	10	11	16	21	14	27	29
Thm.1 [13] (m=2)	9	10	11	16	21	25	27	29
Thm.1 [13] (m=8)	10	11	12	17	22	26	28	29
Thm.1 [proposed]	10	12	13	18	23	27	29	31

V. CONCLUSIONS

This paper presented the improved stability analysis for the discrete-time system that has a time-varying delay. The new novel summation inequality lemma based on free-matrix-based is introduced to provide the less conservative stability criterion of the system. By considering the cross terms between the state and the forward difference, the proposed lemma provides additional information to reduce the conservativeness of stability analysis. Furthermore, the novel LKFs with an augmented vector containing $x(t)$ and $y(t)$ are constructed to apply the lemma. The results of two numerical examples indicate that the proposed stability criterion guarantees the expanded stability region with lower computation complexity. Therefore, the proposed stability criterion could provide a less conservative result for stability analysis.

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