2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Large-Population Risk-Sensitive Linear-Quadratic Optimal Control

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Abstract— **We study a risk-sensitive linear-quadratic optimal control problem where a large number of N agents have mean-field interactions. We derive the centralized optimal control law and the resulting decentralized individual control law by passing to the mean-field limit. This procedure is similar to the so-called direct approach in mean-field control. We further compare the asymptotic performances of the above two control laws. The performance difference between the two sets of control laws does not vanish, and instead has an upper bound depending on the risk sensitivity parameter and the noise intensity. This phenomenon is very different from both risk-neural social optimization and risk-sensitive mean-field games, and is inherently due to the exponential functional structure of the cost, which is closely related to large deviations theory.**

I. INTRODUCTION

There has been a long history of research on risk-sensitive control problems. In 1972, Howard and Matheson [1] investigated a risk-sensitive control problem with Markov decision processes. Whittle [2] studied risk-sensitive control for discrete-time linear-quadratic Gaussian systems. Fleming and McEneaney [3] derived the optimal control for the nonlinear risk-sensitive control problem on an infinite time horizon by a large deviations method. Meanwhile, they demonstrated that the original problem leads to a robust control problem of a deterministic system as the noise tends to zero. Nagai [4] investigated a class of nonlinear risk-sensitive control problems by dynamic programming and identified the relationship between asymptotic solutions and the large deviation principle. Lim and Zhou [5] established a maximum principle for risk-sensitive optimal control problems, where control enters the diffusion term. For the application of risk-sensitive control problems in finance, one can refer to Fleming and Sheu [6]-[7], Bielecki and Pliska [8].

With the introduction of mean-field game theory [9]-[10], related risk-sensitive mean-field game problems have attracted significant attention; see [11]-[14]. In parallel to mean-field games studying noncooperative decision-makers, a different optimization paradigm has been successfully developed for cooperative mean-field decision models in which agents coordinate for social optimality. Huang et al. [15] introduce the social certainty equivalence approach in large-population linear-quadratic optimal control problems (or called mean-field social optimization problems), and derive decentralized feedback control laws to asymptotically achieve the social optima. Nourian et al. [16] study the relationship among Nash, social and centralized solutions of mean-field control models. One can refer to [17]-[19] for further details on mean-field social optima.

It is worth noting that research on large-population risk-sensitive optimal control problems is still limited. Existing studies have primarily focused on seeking Nash equilibria, with little literature addressing large-population risk-sensitive control seeking the social optimum. This paper considers a class of large-population risk-sensitive linear-quadratic optimal control problems, and some innovative findings have been obtained. The contributions are as follows. (i) By the direct approach via dynamic programming, we obtain the centralized control law and the limit decentralized control law of the large-population risk-sensitive optimal control problem; (ii) using the value function method, we demonstrate that the difference between the costs of centralized and (limit) decentralized control laws depends on the risk sensitivity parameter and the diffusion coefficient. Such a persistent cost gap is in dramatic contrast to mean-field games [20] and mean-field social optimization with risk neutral cost [19]; (iii) for comparison, we also derive the centralized strategy and the decentralized strategy of the risk-sensitive linear-quadratic mean-field game. Again using the value function method, we prove that the difference between the costs of centralized and decentralized strategies tends 0 as the population size $N \to \infty$, i.e., the set of decentralized strategies is an $O(1/N)$-Nash equilibrium of the risk-sensitive mean-field game. Our method for the risk-sensitive mean-field game provides much tighter bound on performance loss than in the literature [13]-[14].

The paper is organized as follows. Section II formulates the large-population risk-sensitive optimal control problem, which is compared with the mean-field game formulation. Section III derives the centralized optimal control law, and quantitatively examines the performance difference of the optimal control and the resulting decentralized control law obtained as the former's mean-field limit. For comparison, we further extend our method to risk-sensitive mean-field games in Section IV. Section V provides some numerical computations to illustrate the theoretical results. Section VI concludes the paper.

II. PROBLEM FORMULATION

A. Notation

Let $\otimes$ stand for the Kronecker product, $\mathbf{1}_{m\times n}$ for an $m\times n$ matrix with all entries equal to 1 (in particular, $\mathbf{1}_m = \mathbf{1}_{m\times 1}$), I_n for an $n \times n$ identity matrix, and the column vectors $\{e_1^n, \cdots, e_n^n\}$ for the canonical basis of $\mathbb{R}^n$.

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B. Risk-Sensitive Mean-Field Model

For $1 \le i \le N$, the state equation of player $\mathcal{A}_i$ is

$$dX_i(t) = \big[AX_i(t) + Bu_i(t) + GX^{(N)}(t)\big]dt + DdW_i(t), \quad (1)$$

where $X_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^{n_1}$ are state and control of player $\mathcal{A}_i$, respectively, $X^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N X_i(t)$, and $\{W_i, 1 \le i \le N\}$ are N independent $\mathbb{R}^{n_2}$-valued Brownian motions. The coefficient matrices A, B, G and D are deterministic with suitable dimensions. The initial state $X_i(0) = x_i$ is deterministic.

Denote $u_{-i} = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N)$. The social cost functional of players is

$$J_{\rm soc}^{(N)}(\mathbf{u}) = \frac{1}{\alpha}\ln \mathbb{E}\left[e^{\alpha\sum_{i=1}^N L_i(u_i,u_{-i})}\right], \quad (2)$$

where $\alpha > 0$ is a risk-sensitive parameter and

$$L_i(u_i, u_{-i}) = \int_0^T \left(|X_i(t) - \Gamma X^{(N)}(t)|_Q^2 + |u_i(t)|_R^2\right) dt + |X_i(T) - \Gamma_f X^{(N)}(T)|_{Q_f}^2. \quad (3)$$

Here $R > 0, Q \ge 0, Q_f \ge 0, \Gamma, \Gamma_f$ are constant matrices.

Denote $\phi : [0,T]\times\mathbb{R}^n\times\cdots\times\mathbb{R}^n \to \mathbb{R}^{n_1}$ and $\varphi : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n_1}$. For player $\mathcal{A}_i$, we introduce

$\mathcal{U}_i^c = \{u_i | u_i = \phi(t, X_1, \cdots, X_N)\}$ and $\mathcal{U}_i^d = \{u_i | u_i = \varphi(t, X_i)\}$

as sets for the centralized control laws and decentralized control laws ensuring a well defined closed-loop state process, respectively. Let $\mathbf{u} = (u_1^\top, \cdots, u_N^\top)^\top$, $\mathcal{U}^c = \mathcal{U}_1^c \times\cdots\times \mathcal{U}_N^c$ and $\mathcal{U}^d = \mathcal{U}_1^d \times\cdots\times\mathcal{U}_N^d$.

Problem (RS-OC) Find a centralized optimal control law $\mathbf{u}^*(\cdot) \in \mathcal{U}^c$ such that

$$J_{\rm soc}^{(N)}(\mathbf{u}^*) = \inf_{\mathbf{u}\in\mathcal{U}^c} J_{\rm soc}^{(N)}(\mathbf{u}). \quad (4)$$

Subsequently, by taking the mean-field limit of the centralized optimal control law $\mathbf{u}^*$, we obtain the corresponding decentralized control law $\check{\mathbf{u}}$. We are interested in the asymptotic performance of the decentralized control law, and will analyze the cost gap between $\check{\mathbf{u}}$ and $\mathbf{u}^*$.

C. Risk-Sensitive Mean-Field Game

We will further extend our analysis to treat a risk-sensitive mean-field game, where each player $\mathcal{A}_i, 1 \le i \le N$ has individual cost

$$J_i^{(N)}(u_i, u_{-i}) = \frac{1}{\alpha}\ln\mathbb{E}\left[e^{\alpha L_i(u_i,u_{-i})}\right]. \quad (5)$$

Definition 2.1: The set of decentralized strategies $\hat{u}_i(\cdot) \in \mathcal{U}_i^d, 1 \le i \le N$ is an ε_N-Nash equilibrium if for each i,

$$J_i^{(N)}(\hat{u}_i, \hat{u}_{-i}) \le J_i^{(N)}(u_i, \hat{u}_{-i}) + \varepsilon_N, \ \forall u_i \in \mathcal{U}_i^c. \quad (6)$$

Problem (RS-Nash) For the mean-field game with dynamics (1) and cost (5), find a set of decentralized strategies that possesses an ε-Nash equilibrium property.

Existing works [13]-[14] rely heavily on the fixed-point approach. By estimating the state processes, they can only show a bound of $O(1/\sqrt{N})$ for the cost gap between centralized strategies and the decentralized strategies. We employ the direct approach and use the Feynman-Kac formula to directly estimate the cost difference, leading to a bound of $O(1/N)$. The reader is referred to [20] for an overview of the fixed point approach and the direct approach, as two fundamental methodologies in mean field games. Compared with the existing works, our approach needs weaker conditions and yields tighter estimates.

III. RISK-SENSITIVE OPTIMAL CONTROL

Define

$$\begin{aligned}
&X = (X_1^\top, \cdots, X_N^\top)^\top, \ W = (W_1^\top, \cdots, W_N^\top)^\top,\\
&\mathbf{x}_0 = (x_1^\top, \cdots, x_N^\top)^\top, \ \mathbf{B} = \mathrm{diag}[B, \cdots, B],\\
&\mathbf{A} = \mathrm{diag}[A, \cdots, A] + \mathbf{1}_{N\times N}\otimes(G/N),\\
&\mathbf{D} = \mathrm{diag}[D, \cdots, D], \ \mathbf{R} = \mathrm{diag}[R, \cdots, R]\\
&Q^\Gamma = |I - \Gamma|_Q^2 - Q, \ Q_f^\Gamma = |I - \Gamma_f|_{Q_f}^2 - Q_f,\\
&\mathbf{Q}_1 = \mathrm{diag}[Q, \cdots, Q], \ \mathbf{Q}_{1f} = \mathrm{diag}[Q_f, \cdots, Q_f],\\
&\mathbf{Q}_2 = \mathbf{1}_{N\times N}\otimes(Q^\Gamma/N), \ \mathbf{Q}_{2f} = \mathbf{1}_{N\times N}\otimes(Q_f^\Gamma/N),\\
&\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2, \ \mathbf{Q}_f = \mathbf{Q}_{1f} + \mathbf{Q}_{2f}.
\end{aligned} \quad (7)$$

The system dynamics (1) and cost (2) can be rewritten as

$$dX(t) = \left(\mathbf{A}X(t) + \mathbf{B}\mathbf{u}(t)\right)dt + \mathbf{D}dW(t), \quad (8)$$

$$J_{\rm soc}^{(N)}(\mathbf{u}) = \frac{1}{\alpha}\ln\mathbb{E}\left[e^{\alpha\left[\int_0^T\left(|X(t)|_{\mathbf{Q}}^2 + |\mathbf{u}(t)|_{\mathbf{R}}^2\right)dt + |X(T)|_{\mathbf{Q}_f}^2\right]}\right]. \quad (9)$$

A. Centralized Control Law for Social Optima

For given time $t \in [0,T]$, we set the initial condition $(X_1^\top(t), \cdots, X_N^\top(t))^\top = \mathbf{x} \in \mathbb{R}^{nN}$ and define

$$W(t, \mathbf{x}) = \inf_{\mathbf{u}\in\mathcal{U}^c}\mathbb{E}\left[e^{\alpha\left[\int_t^T\left(|X(s)|_{\mathbf{Q}}^2 + |\mathbf{u}(s)|_{\mathbf{R}}^2\right)ds + |X(T)|_{\mathbf{Q}_f}^2\right]}\right].$$

By dynamic programming, we obtain

$$-\partial_t W(t,\mathbf{x}) = \inf_{\mathbf{u}}\Big[\partial_{\mathbf{x}}^\top W\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right) + \frac{1}{2}\mathrm{Tr}\left(\partial_{\mathbf{x}\mathbf{x}}^2 W\mathbf{D}\mathbf{D}^\top\right) + \alpha W(t,\mathbf{x})\left(|\mathbf{x}|_{\mathbf{Q}}^2 + |\mathbf{u}|_{\mathbf{R}}^2\right)\Big]. \quad (10)$$

Setting $V(t,\mathbf{x}) = \frac{1}{\alpha}\ln W(t,\mathbf{x})$, we use (10) to derive

$$\begin{cases}
-\partial_t V(t,\mathbf{x}) = \partial_{\mathbf{x}}^\top V\mathbf{A}\mathbf{x} + \frac{1}{2}\mathrm{Tr}\left(\partial_{\mathbf{x}\mathbf{x}}^2 V\mathbf{D}\mathbf{D}^\top\right) + \frac{\alpha}{2}|\mathbf{D}^\top\partial_{\mathbf{x}}V|^2 \\
\qquad + |\mathbf{x}|_{\mathbf{Q}}^2 - \frac{1}{4}\partial_{\mathbf{x}}^\top V\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\partial_{\mathbf{x}}V,\\
V(T,\mathbf{x}) = |\mathbf{x}|_{\mathbf{Q}_f}^2.
\end{cases} \quad (11)$$

The optimal control law is given by

$$\mathbf{u}^* = -\frac{1}{2}\mathbf{R}^{-1}\mathbf{B}^\top\partial_{\mathbf{x}}V. \quad (12)$$

Taking the ansatz $V(t,\mathbf{x}) = \mathbf{x}^\top\mathbf{P}(t)\mathbf{x} + \mathbf{r}(t)$ and substituting it into (11), we obtain

$$\begin{cases}
\dot{\mathbf{P}} + \mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\left(\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top - 2\alpha\mathbf{D}\mathbf{D}^\top\right)\mathbf{P} + \mathbf{Q} = 0,\\
\mathbf{P}(T) = \mathbf{Q}_f,
\end{cases} \quad (13)$$

and

$$\dot{\mathbf{r}} + \mathrm{Tr}\left(\mathbf{P}\mathbf{D}\mathbf{D}^\top\right) = 0, \quad \mathbf{r}(T) = 0. \quad (14)$$

We introduce the following assumption.

Assumption 3.1: $BR^{-1}B^\top - 2\alpha DD^\top \geq 0$.

Lemma 3.2: Under Assumption 3.1, equations (13)-(14) have a unique solution.

Proof: $\mathbf{Q}$ in (9) is obtained by adding up all quadratic forms of X in (2)-(3). Consequently, we have the symmetric matrix $\mathbf{Q} \geq 0$ since $Q \geq 0$. Similarly we derive $\mathbf{Q}_f \geq 0$. Therefore, the Riccati equation (13) has a unique solution under Assumption 3.1. Then (14) has a unique solution. ■

Due to the high dimensionality of $\mathbf{P} \in \mathbb{R}^{nN\times nN}$, the centralized optimal control law (12) is not useful for implementation. Next, we show a simple structure for $\mathbf{P}$, which will be used to get a limit form of the optimal control law.

Denote $M_0 = BR^{-1}B^\top$, $N_0 = 2\alpha DD^\top$, and $M = M_0 - N_0$. We introduce the following ODEs:

$$\begin{cases} \dot{\Lambda}_1 + \Lambda_1 A + A^\top \Lambda_1 - \Lambda_1 M \Lambda_1 + Q = 0, \\ \Lambda_1(T) = Q_f, \end{cases} \tag{15}$$

$$\begin{cases} \dot{\Lambda}_2 + \Lambda_2 (A + G - M\Lambda_1) + (A + G - M\Lambda_1)^\top \Lambda_2 \\ \quad + G^\top \Lambda_1 + \Lambda_1 G - \Lambda_2 M \Lambda_2 + Q^\Gamma = 0, \\ \Lambda_2(T) = Q_f^\Gamma, \end{cases} \tag{16}$$

$$\dot{r} + \mathrm{Tr}(\Lambda_1 DD^\top) = 0, \quad r(T) = 0. \tag{17}$$

We have the following lemma.

Lemma 3.3: Under Assumption 3.1, (i) the ODEs (15)-(17) have a unique solution. (ii) $\mathbf{P}(t)$ has the representation

$$\mathbf{P}(t) = \begin{pmatrix} \Pi_1^N(t) & \Pi_2^N(t) & \cdots & \Pi_2^N(t) \\ \Pi_2^N(t) & \Pi_1^N(t) & \cdots & \Pi_2^N(t) \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_2^N(t) & \Pi_2^N(t) & \cdots & \Pi_1^N(t) \end{pmatrix}, \tag{18}$$

and

$$\sup_{0\leq t\leq T} \big(|\Pi_1^N - \Lambda_1| + |N\Pi_2^N - \Lambda_2| \\ + |(1/N)\mathbf{r} - r|\big) = O(1/N). \tag{19}$$

Proof: By Lemma 3.2, $\mathbf{P}(t)$ has a solution for each N. See [21] for detailed proof of the lemma. ■

We introduce the following assumption stating that the initial states in (1) have convergent empirical mean and convergent empirical covariance.

Assumption 3.4: For initial states $x_i, 1 \leq i \leq N$, we have $\frac{1}{N}\sum_{i=1}^N x_i \to \bar{x}_0$ and $\frac{1}{N}\sum_{i=1}^N (x_i - \bar{x}_0)^T(x_i - \bar{x}_0) \to \Sigma_0$ as $N \to \infty$.

Theorem 3.5: Suppose Assumptions 3.1 and 3.4 hold. For $1 \leq i \leq N$, player $\mathcal{A}_i$'s centralized optimal control law is

$$u_i^* = -R^{-1}B^\top \Big(\Pi_1^N X_i + \sum_{j\neq i}^N \Pi_2^N X_j\Big) \tag{20}$$

and the corresponding social cost is

$$\begin{aligned} J_{\mathrm{soc}}^{(N)}(\mathbf{u}^*) &= N\big[\mathrm{Tr}\left(\Lambda_1(0)\Sigma_0\right) + \bar{x}_0^\top \left(\Lambda_1(0)\right. \\ &\quad \left. + \Lambda_2(0)\right)\bar{x}_0 + r(0) + o(1)\big] \\ &= O(N). \end{aligned} \tag{21}$$

Proof: According to (9) and Lemma 3.3, we have

$$\begin{aligned} J_{\mathrm{soc}}^{(N)}(\mathbf{u}^*) =& \sum_{i=1}^N x_i^\top \left(\Pi_1^N(0) - \Pi_2^N(0)\right) x_i \\ &+ x^{(N)\top} N^2 \Pi_2^N(0) x^{(N)} + \mathbf{r}(0) \\ =& \sum_{i=1}^N x_i^\top \Lambda_1(0) x_i + x^{(N)\top} N \Lambda_2(0) x^{(N)} \\ &+ N r(0) + O(1). \end{aligned} \tag{22}$$

By Assumption 3.4, we obtain (21). ■

B. Decentralized Control Law

Taking the centralized control law in (1), we have

$$\begin{aligned} dX_i = \Big[& \left(A - M_0 \Pi_1^N\right) X_i + G X^{(N)} \\ & - M_0 \Big(\sum_{j\neq i}^N \Pi_2^N X_j\Big)\Big] dt + D dW_i, \end{aligned} \tag{23}$$

where $X_i(0) = x_i$, $1 \leq i \leq N$. Let the (limit) decentralized control law be

$$\check{u}_i = -R^{-1}B^\top \left(\Lambda_1 \check{X}_i + \Lambda_2 \bar{X}\right), \qquad 1 \leq i \leq N, \tag{24}$$

where

$$\begin{cases} d\check{X}_i = \Big[(A - M_0\Lambda_1)\check{X}_i + G\check{X}^{(N)} - M_0\Lambda_2 \bar{X}\Big] dt \\ \qquad + D dW_i, \\ d\bar{X} = (A + G - M_0\Lambda_1 - M_0\Lambda_2)\bar{X} dt, \\ \check{X}_i(0) = x_i, \ \bar{X}(0) = \bar{x}_0, \qquad 1 \leq i \leq N. \end{cases} \tag{25}$$

In view of the approximation in (19), equations (24)-(25) can be obtained by taking $N \to \infty$ in (20) and (23), and approximating $\frac{1}{N}\sum_{j\neq i} X_j(t)$ by $\bar{X}(t)$. Here $\bar{X}(t)$ is regarded as the limit of $X^{(N)}(t)$.

Assumption 3.6: The Riccati equation

$$\begin{cases} \dot{\check{\Lambda}} + (A + G - M\Lambda_1)^\top \check{\Lambda} + \check{\Lambda}(A + G - M\Lambda_1) \\ \quad + G^\top \Lambda_1 + \Lambda_1 G + Q^\Gamma + \check{\Lambda} N_0 \check{\Lambda} = 0, \\ \check{\Lambda}(T) = Q_f^\Gamma \end{cases}$$

has a solution $\check{\Lambda}$ on $[0, T]$.

Theorem 3.7: If Assumptions 3.1, 3.4 and 3.6 hold, the difference between the costs of centralized optimal control law (20) and decentralized control law (24) satisfies

$$|J_{\mathrm{soc}}^{(N)}(\check{\mathbf{u}}) - J_{\mathrm{soc}}^{(N)}(\mathbf{u}^*)| = O\left(N\right)\left(\alpha \mathrm{Tr}(DD^\top) + o(1)\right), \tag{26}$$

where $\mathbf{u}^* = (u_1^{*\top}, \cdots, u_N^{*\top})^\top$ and $\check{\mathbf{u}} = (\check{u}_1^\top, \cdots, \check{u}_N^\top)^\top$.

Proof: See Appendix. ■

Remark 3.8: Theorems 3.5 and 3.7 imply the difference between the costs of centralized and decentralized control laws is of the same order of magnitude as the cost of the centralized control law. If $|\alpha|$ or $\mathrm{Tr}(DD^\top)$ is small, the decentralized control law can approximate the centralized control law well.

IV. RISK-SENSITIVE MEAN-FIELD GAME

For $1 \le i \le N$, we define

$$\mathbf{K}_i = e_i^{N\top} - \frac{1}{N}\mathbf{1}_N^\top \otimes \Gamma,\ \mathbf{K}_{if} = e_i^{N\top} - \frac{1}{N}\mathbf{1}_N^\top \otimes \Gamma_f,$$
$$\mathbf{Q}_i = \mathbf{K}_i^\top Q \mathbf{K}_i,\ \mathbf{Q}_{if} = \mathbf{K}_{if}^\top Q_f \mathbf{K}_{if},\ \mathbf{B}_i = e_i^N \otimes B. \quad (27)$$

With the notation in (7) and (27), we rewrite (1) and (5) in the following form:

$$dX(t) = \Big(\mathbf{A}X(t) + \sum_{i=1}^N \mathbf{B}_i u_i(t)\Big)dt + \mathbf{D}dW(t), \quad (28)$$

and

$$J_i^{(N)}(u_i, u_{-i}) = \frac{1}{\alpha}\ln \mathbb{E}\left[e^{\alpha\left[\int_0^T \left(|X(t)|^2_{\mathbf{Q}_i} + |u_i(t)|^2_R\right)dt + |X(T)|^2_{\mathbf{Q}_{if}}\right]}\right]. \quad (29)$$

A. Centralized Nash Equilibrium Strategy

Under closed-loop perfect state information, we may apply dynamic programming to derive a system of HJB equations for the value functions of the N players, which further leads to the following Riccati equation system:

$$\begin{cases} \dot{\mathbf{P}}_i + \mathbf{A}^\top \mathbf{P}_i + \mathbf{P}_i\mathbf{A} + \mathbf{P}_i(\mathbf{B}_i R^{-1}\mathbf{B}_i^\top + 2\alpha \mathbf{D}\mathbf{D}^\top)\mathbf{P}_i + \mathbf{Q}_i \\ -\mathbf{P}_i \sum_{j=1}^N \mathbf{B}_j R^{-1}\mathbf{B}_j^\top \mathbf{P}_j - \sum_{j=1}^N \mathbf{P}_j \mathbf{B}_j R^{-1}\mathbf{B}_j^\top \mathbf{P}_i = 0, \\ \mathbf{P}_i(T) = \mathbf{Q}_{if}, \qquad 1 \le i \le N. \end{cases} \quad (30)$$

If the Riccati ODE system (30) has a solution on $[0, T]$, the set of (centralized) Nash equilibrium strategies is given by

$$v_i^* = -R^{-1}\mathbf{B}_i^\top \mathbf{P}_i X, \qquad 1 \le i \le N, \quad (31)$$

which gives the closed-loop state process

$$dX = \Big(\mathbf{A}X - \sum_{i=1}^N \mathbf{B}_i R^{-1}\mathbf{B}_i^\top \mathbf{P}_i X\Big)dt + \mathbf{D}dW. \quad (32)$$

To analyze the solvability of (30), we introduce

$$\begin{cases} \dot{\Lambda}_1^o + \Lambda_1^o A + A^\top \Lambda_1^o - \Lambda_1^o M \Lambda_1^o + Q = 0, \\ \Lambda_1^o(T) = Q_f, \end{cases} \quad (33)$$

$$\begin{cases} \dot{\Lambda}_2^o + \Lambda_1^o G + A^\top \Lambda_2^o + \Lambda_2^o(A+G) - \Lambda_2^o M_0 \Lambda_1^o \\ -\Lambda_1^o M_0 \Lambda_2^o + \Lambda_1^o N_0 \Lambda_2^o - \Lambda_2^o M_0 \Lambda_2^o - Q\Gamma = 0, \\ \Lambda_2^o(T) = -Q_f \Gamma_f, \end{cases} \quad (34)$$

$$\begin{cases} \dot{\Lambda}_3^o + \Lambda_2^{o\top} G + G^\top \Lambda_2^o + \Lambda_3^o(A + G - M_0\Lambda_{12}^o) \\ +(A + G - M_0 \Lambda_{12}^o)^\top \Lambda_3^o - \Lambda_2^{o\top} M \Lambda_2^o + \Gamma^\top Q \Gamma = 0, \\ \Lambda_3^o(T) = \Gamma_f^\top Q_f \Gamma_f \end{cases} \quad (35)$$

where $\Lambda_{12}^o = \Lambda_1^o + \Lambda_2^o$.

Note that (33) is a standard Riccati equation and has a unique solution on $[0, T]$ if Assumption 3.1 holds. Let us introduce the following assumption.

Assumption 4.1: Riccati equation (34) has a solution on $[0, T]$ (which is then unique).

In analogue to Lemma 3.3 and Theorem 3.5, we have the following theorem.

Theorem 4.2: Suppose Assumptions 3.1 and 4.1 hold. Then we have the following assertions.

(i) Equation (35) has a unique solution on $[0, T]$.

(ii) For all sufficiently large N, the system (30) has a unique solution on $[0, T]$, with the representation

$$\mathbf{P}_1 = \begin{pmatrix} \Pi_1^{oN} & \Pi_2^{oN} & \cdots & \Pi_2^{oN} \\ \Pi_2^{oN\top} & \Pi_3^{oN} & \cdots & \Pi_3^{oN} \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_2^{oN\top} & \Pi_3^{oN} & \cdots & \Pi_3^{oN} \end{pmatrix}, \mathbf{P}_i = J_{1i}^\top \mathbf{P}_1 J_{1i}, \quad (36)$$

where $J_{ij}, 1 \le i \ne j \le N$ represents the matrix obtained by exchanging the i-th row and the j-th row of submatrices in matrix I_{nN}.

(iii) We have

$$\sup_{0\le t\le T}\big(|\Pi_1^{oN} - \Lambda_1^o| + |N\Pi_2^{oN} - \Lambda_2^o| + |N^2\Pi_3^{oN} - \Lambda_3^o|\big) = O(1/N).$$

The centralized Nash equilibrium strategy of player $\mathcal{A}_i$ is

$$v_i^* = -R^{-1}B^\top\Big(\Pi_1^{oN}X_i + \sum_{j\ne i}\Pi_2^{oN}X_j\Big),\ 1 \le i \le N. \quad (37)$$

Proof: The proof is similar to that of Theorems 3, 4, 5 and 6 in [20]. ■

B. Limit Decentralized Strategy

Taking $N \to \infty$ in equation (37), the limit decentralized strategy of player $\mathcal{A}_i$ is

$$\hat{u}_i = -R^{-1}B^\top(\Lambda_1^o \hat{X}_i + \Lambda_2^o \bar{X}), \qquad 1 \le i \le N, \quad (38)$$

where Λ_1^o and Λ_2^o satisfy (33)-(34), and the closed-loop dynamics are

$$\begin{cases} d\hat{X}_i = \Big[(A - M_0\Lambda_1^o)\hat{X}_i + G\hat{X}^{(N)} - M_0\Lambda_2^o\bar{X}\Big]dt + DdW_i, \\ d\bar{X} = (A + G - M_0\Lambda_1^o - M\Lambda_2^o)\bar{X}dt, \\ \hat{X}_i(0) = x_i,\ \bar{X}(0) = \bar{x}_0, \qquad 1 \le i \le N, \end{cases} \quad (39)$$

where $\hat{X}^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{X}_i$. We have the following ε-Nash equilibrium property for the decentralized strategies.

Theorem 4.3: If Assumptions 3.1, 3.4 and 4.1 hold, the set of decentralized strategies $\hat{u}_i, 1 \le i \le N$ in (38) satisfies

$$|J_i^{(N)}(\hat{u}_i, \hat{u}_{-i}) - \inf_{u_i \in \mathcal{U}_i^c} J_i^{(N)}(u_i, \hat{u}_{-i})| = O(1/N), \quad (40)$$

which implies $\{\hat{u}_i, 1 \le i \le N\}$ is an $O(1/N)$-Nash equilibrium of Problem (RS-Nash).

Proof: See [21] for detailed proof. ■

V. NUMERICAL SIMULATION

We illustrate our results in Section III through a numerical example. Let the parameters in (1)-(3) be

$$A = 2,\ G = 0.2,\ B = 1,\ D = 0.7,\ R = 1.2,\ Q = 0.6,$$
$$\Gamma = -0.7,\ Q_f = 2,\ \Gamma_f = 1,\ T = 1,\ \bar{x}_0 = 1,\ \Sigma_0 = 0.5.$$

Here $\bar{x}_0$ and Σ_0 denote the limits of the empirical mean and covariance of the initial states $x_i, 1 \le i \le N$, as $N \to \infty$.

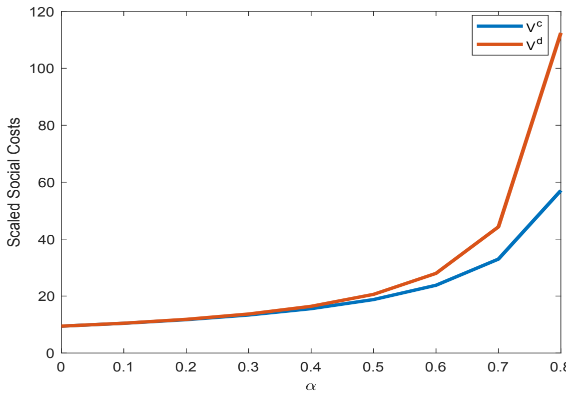

Fig. 1. Scaled social costs for the two control laws, $\bar{x}_0 = 1$

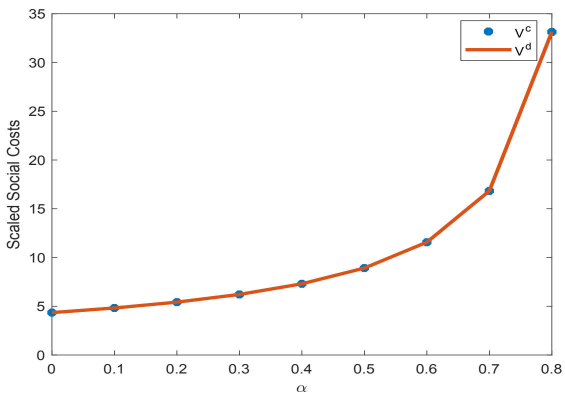

Fig. 2. Scaled social costs for the two control laws, $\bar{x}_0 = 0$

Note that the social costs under the optimal control law $\mathbf{u}^*$ and the decentralized control law $\check{\mathbf{u}}$ are unbounded as $N \to \infty$. We normalize them by $1/N$ and define the limits $V^c = \lim_{N\to\infty} \frac{1}{N} J_{\rm soc}^{(N)}(\mathbf{u}^*)$ and $V^d = \lim_{N\to\infty} \frac{1}{N} J_{\rm soc}^{(N)}(\check{\mathbf{u}})$, which will simply be called the scaled social costs.

Fig. 1 compares the scaled social costs under the control laws $\mathbf{u}^*$ and $\check{\mathbf{u}}$, and shows a persistent gap between the two costs. Furthermore, the cost gap widens as the risk sensitivity parameter α increases. In Fig. 2, we set $\bar{x}_0 = 0$ while keeping all other parameters unchanged; we find that the cost gap disappears in this case.

VI. CONCLUSION

This paper considers a class of large-population risk-sensitive linear-quadratic optimal control problems. We derive the centralized optimal control law and construct a limit decentralized control law. We show that the difference between the costs of the centralized and limit decentralized control laws depends on the risk sensitivity parameter and the diffusion coefficient. This phenomenon of non-vanishing performance gap is very different from the case of social optimization with a risk-neural social cost [19], and may be attributed to the exponential functional structure which has connections with large deviations theory [22].

For comparison, we also consider the risk-sensitive mean-field game and prove that the set of decentralized strategies constructed is an $O(1/N)$-Nash equilibrium.

For future work, we will consider optimizing a decentralized control law directly for solving the risk-sensitive social optimal control problem, and compare its cost with that of the centralized control law.

APPENDIX: PROOF OF THEOREM 3.7

Set $(\check{X}_1^\top, \cdots, \check{X}_N^\top)^\top = \check{X}$. Combining (7) with (24), the limit decentralized control law is

$$\check{\mathbf{u}} = -\check{\Theta}_1 \check{X} - \check{\Theta}_2 \bar{X}, \tag{41}$$

where $\Theta_1 = R^{-1}B^\top \Lambda_1$, $\Theta_2 = R^{-1}B^\top \Lambda_2$, $\check{\Theta}_1 = I_N \otimes \Theta_1$, and $\check{\Theta}_2 = \mathbf{1}_N \otimes \Theta_2$. The closed-loop dynamics are

$$\begin{cases} d\check{X} = \left(\mathbf{A}\check{X} - \mathbf{B}\check{\Theta}_1\check{X} - \mathbf{B}\check{\Theta}_2\bar{X}\right)dt + \mathbf{D}dW, \\ d\bar{X} = (A + G - B\Theta_1 - B\Theta_2)\bar{X}dt, \\ \check{X}(0) = \mathbf{x}_0 \in \mathbb{R}^{nN}, \ \bar{X}(0) = \bar{x}_0. \end{cases} \tag{42}$$

For given time $t \in [0, T]$, we set the initial conditions $\check{X}(t) = \check{\mathbf{x}}$, and $\bar{X}(t) = \bar{x}$ and define the social cost

$$\begin{aligned} &J_{\rm soc}^{(N)}(t, \check{\mathbf{x}}, \bar{x}; \check{\mathbf{u}}) \\ &= \frac{1}{\alpha} \ln \mathbb{E}\left[e^{\alpha\left[\int_t^T \left(|\check{X}|^2_{\mathbf{Q}} + |\check{\Theta}_1\check{X} + \check{\Theta}_2\bar{X}|^2_{\mathbf{R}}\right)ds + |\check{X}(T)|^2_{\mathbf{Q}_f}\right]}\right]. \end{aligned} \tag{43}$$

Set $\check{V}(t, \check{\mathbf{x}}, \bar{x}) = J_{\rm soc}^{(N)}(t, \check{\mathbf{x}}, \bar{x}; \check{\mathbf{u}})$. According to the Feynman-Kac formula, we have

$$\begin{cases} -\partial_t \check{V}(t, \check{\mathbf{x}}, \bar{x}) = \partial_{\check{\mathbf{x}}}^\top \check{V}\left(\mathbf{A}\check{\mathbf{x}} - \mathbf{B}\check{\Theta}_1\check{\mathbf{x}} - \mathbf{B}\check{\Theta}_2\bar{x}\right) \\ \qquad + \partial_{\bar{x}}^\top \check{V}(A + G - B\Theta_1 - B\Theta_2)\bar{x} \\ \qquad + \frac{1}{2}\mathrm{Tr}(\partial^2_{\check{\mathbf{x}}\check{\mathbf{x}}}\check{V}\mathbf{D}\mathbf{D}^\top) + \frac{\alpha}{2}|\mathbf{D}^\top\partial_{\check{\mathbf{x}}}\check{V}|^2 \\ \qquad + |\check{\mathbf{x}}|^2_{\mathbf{Q}} + |\check{\Theta}_1\check{\mathbf{x}} + \check{\Theta}_2\bar{x}|^2_{\mathbf{R}}, \\ \check{V}(T, \check{\mathbf{x}}, \bar{x}) = \check{\mathbf{x}}^\top \mathbf{Q}_f \check{\mathbf{x}}. \end{cases} \tag{44}$$

Suppose $\check{V}$ takes the following form

$$\check{V}(t, \check{\mathbf{x}}, \bar{x}) = \check{\mathbf{x}}^\top \check{\mathbf{P}}_{11}(t)\check{\mathbf{x}} + 2\check{\mathbf{x}}^\top \check{\mathbf{P}}_{12}(t)\bar{x} + \bar{x}^\top \check{\mathbf{P}}_{22}(t)\bar{x} + \check{\mathbf{r}}(t). \tag{45}$$

Substituting (45) into (44), we have

$$\begin{cases} \dot{\check{\mathbf{P}}}_{11} + \check{\mathbf{P}}_{11}(\mathbf{A} - \mathbf{B}\check{\Theta}_1) + (\mathbf{A} - \mathbf{B}\check{\Theta}_1)^\top \check{\mathbf{P}}_{11} \\ \quad + 2\alpha\check{\mathbf{P}}_{11}\mathbf{D}\mathbf{D}^\top\check{\mathbf{P}}_{11} + \mathbf{Q} + \check{\Theta}_1^\top \mathbf{R}\check{\Theta}_1 = 0, \\ \check{\mathbf{P}}_{11}(T) = \mathbf{Q}_f, \end{cases} \tag{46}$$

$$\begin{cases} \dot{\check{\mathbf{P}}}_{12} + \check{\mathbf{P}}_{12}(A + G - B\Theta_1 - B\Theta_2) - \check{\mathbf{P}}_{11}\mathbf{B}\check{\Theta}_2 \\ \quad + (\mathbf{A} - \mathbf{B}\check{\Theta}_1)^\top\check{\mathbf{P}}_{12} + 2\alpha\check{\mathbf{P}}_{11}\mathbf{D}\mathbf{D}^\top\check{\mathbf{P}}_{12} + \check{\Theta}_1^\top\mathbf{R}\check{\Theta}_2 = 0, \\ \check{\mathbf{P}}_{12}(T) = 0, \end{cases} \tag{47}$$

$$\begin{cases} \dot{\check{\mathbf{P}}}_{22} + \check{\mathbf{P}}_{22}(A + G - B\Theta_1 - B\Theta_2) + (A + G - B\Theta_1 \\ \quad - B\Theta_2)^\top\check{\mathbf{P}}_{22} - \check{\mathbf{P}}_{12}^\top\mathbf{B}\check{\Theta}_2 - (\mathbf{B}\check{\Theta}_2)^\top\check{\mathbf{P}}_{12} \\ \quad + 2\alpha\check{\mathbf{P}}_{12}^\top\mathbf{D}\mathbf{D}^\top\check{\mathbf{P}}_{12} + \check{\Theta}_2^\top\mathbf{R}\check{\Theta}_2 = 0, \\ \check{\mathbf{P}}_{22}(T) = 0, \end{cases} \tag{48}$$

and $\dot{\check{\mathbf{r}}} + \mathrm{Tr}(\check{\mathbf{P}}_{11}\mathbf{D}\mathbf{D}^\top) = 0$ with $\check{\mathbf{r}}(T) = 0$.

Note that the solutions of (46)-(47) are in high dimension. In order to characterize the asymptotic behaviour (as $N \to \infty$) of $\check{\mathbf{P}}_{11}$ and $\check{\mathbf{P}}_{12}$, we introduce the following ODEs:

$$\begin{cases} \dot{\check{\Lambda}}_1 + (A - B\Theta_1)^\top \check{\Lambda}_1 + \check{\Lambda}_1(A - B\Theta_1) + \check{\Lambda}_1 N_0 \check{\Lambda}_1 \\ + Q + \Theta_1^\top R \Theta_1 = 0, \\ \check{\Lambda}_1(T) = Q_f, \end{cases} \tag{49}$$

$$\begin{cases} \dot{\check{\Lambda}}_2 + (A + G - B\Theta_1)^\top \check{\Lambda}_2 + \check{\Lambda}_2(A + G - B\Theta_1) + G^\top \check{\Lambda}_1 \\ + \check{\Lambda}_1 G + Q^\Gamma + \check{\Lambda}_2 N_0 \check{\Lambda}_2 + \check{\Lambda}_1 N_0 \check{\Lambda}_2 + \check{\Lambda}_2 N_0 \check{\Lambda}_1 = 0, \\ \check{\Lambda}_2(T) = Q_f^\Gamma, \end{cases} \tag{50}$$

$$\begin{cases} \dot{\check{\Lambda}}_3 + (A + G - B\Theta_1)^\top \check{\Lambda}_3 + \check{\Lambda}_3(A + G - B\Theta_1 - B\Theta_2) \\ + (\check{\Lambda}_1 + \check{\Lambda}_2)(N_0 \check{\Lambda}_3 - B\Theta_2) + \Theta_1^\top R \Theta_2 = 0, \\ \check{\Lambda}_3(T) = 0, \end{cases} \tag{51}$$

$$\begin{cases} \dot{\check{\Lambda}}_4 + (A + G - B\Theta_1 - B\Theta_2)^\top \check{\Lambda}_4 + \check{\Lambda}_4(A + G \\ - B\Theta_1 - B\Theta_2) - \check{\Lambda}_3^\top B\Theta_2 - \Theta_2^\top B^\top \check{\Lambda}_3 \\ + \check{\Lambda}_3^\top N_0 \check{\Lambda}_3 + \Theta_2^\top R \Theta_2 = 0, \\ \check{\Lambda}_4(T) = 0. \end{cases} \tag{52}$$

Under Assumption 3.1, we verify that Λ_1 is the unique solution of (49), which implies $\Lambda_1(t) = \check{\Lambda}_1(t)$. Under Assumption 3.6, (50) has a unique solution. Subsequently, we obtain the solvability of (51) and (52). Then, using Theorems 3 and 4 in [20], we show that (46)-(48) have a solution if N is sufficiently large.

In analogue to Lemma 3.3, we obtain the representation of $\check{\mathbf{P}}_{11}$ and $\check{\mathbf{P}}_{12}$:

$$\check{\mathbf{P}}_{11} = \begin{pmatrix} \check{\Pi}_1^N & \check{\Pi}_2^N & \cdots & \check{\Pi}_2^N \\ \check{\Pi}_2^N & \check{\Pi}_1^N & \cdots & \check{\Pi}_2^N \\ \vdots & \vdots & \ddots & \vdots \\ \check{\Pi}_2^N & \check{\Pi}_2^N & \cdots & \check{\Pi}_1^N \end{pmatrix}, \ \check{\mathbf{P}}_{12} = \begin{pmatrix} \check{\Pi}_3^N \\ \vdots \\ \check{\Pi}_3^N \end{pmatrix}.$$

For the solutions of (46)-(48) and of (49)-(52), we have

$$\sup_{0 \le t \le T} \big(|\check{\Pi}_1^N - \check{\Lambda}_1| + |N\check{\Pi}_2^N - \check{\Lambda}_2| + |\check{\Pi}_3^N - \check{\Lambda}_3| \\ + |(1/N)\check{\mathbf{P}}_{22} - \check{\Lambda}_4| \big) = O(1/N).$$

Then the cost functional becomes

$$J_{\rm soc}^{(N)}(\check{\mathbf{u}}) = \sum_{i=1}^N x_i^\top \left(\check{\Pi}_1^N(0) - \check{\Pi}_2^N(0) \right) x_i \\ + x^{(N)\top} N^2 \check{\Pi}_2^N(0) x^{(N)} + \bar{x}_0^\top \check{\mathbf{P}}_{22}^N(0) \bar{x}_0 \\ + 2x^{(N)\top} N \check{\Pi}_3^N(0) \bar{x}_0 + \check{\mathbf{r}}(0). \tag{53}$$

Combining (21) with (53), we have

$$\left| J_{\rm soc}^{(N)}(\check{\mathbf{u}}) - J_{\rm soc}^{(N)}(\mathbf{u}^*) \right| \\ = N \Big[{\rm Tr} \left((\check{\Lambda}_1(0) - \Lambda_1(0)) \Sigma_0 \right) + \bar{x}_0^\top \big(\check{\Lambda}_1(0) + \check{\Lambda}_2(0) + \check{\Lambda}_3(0) \\ + \check{\Lambda}_3^\top(0) + \check{\Lambda}_4(0) - \Lambda_1(0) - \Lambda_2(0) \big) \bar{x}_0 + o(1) \Big]. \tag{54}$$

Set $\Delta = \check{\Lambda}_1 + \check{\Lambda}_2 + \check{\Lambda}_3 + \check{\Lambda}_3^\top + \check{\Lambda}_4 - \Lambda_1 - \Lambda_2$. Then

$$\begin{cases} \dot{\Delta} + \Delta(A + G - B\Theta_1 - B\Theta_2) + (A + G - B\Theta_1 \\ - B\Theta_2)^\top \Delta + \left(\check{\Lambda}_1 + \check{\Lambda}_2 + \check{\Lambda}_3 \right)^\top N_0 \left(\check{\Lambda}_1 + \check{\Lambda}_2 + \check{\Lambda}_3 \right) \\ - (\Lambda_1 + \Lambda_2)^\top N_0 (\Lambda_1 + \Lambda_2) = 0, \\ \Delta(T) = 0. \end{cases}$$

Note that $N_0 = 2\alpha DD^\top$. If α or D tends to 0, Δ tends to 0. Since $\Lambda_1(t) = \check{\Lambda}_1(t)$, the trace part within (54) is equal to 0. Then (26) holds.

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