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Rejecting an unknown matched disturbance from an infinite-dimensional control system

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***Abstract*— In this paper we address the problem of rejecting an unknown disturbance, which is matched with the input, from an infinite-dimensional plant belonging to the class of regular linear systems. The plant input and output are finite-dimensional and the time-derivative of the disturbance is assumed to be bounded with a known bound. In our solution approach to this problem, we drive a stable ODE using the output of the plant. Via a state transformation obtained by solving a Sylvester equation with possibly unbounded operators, we derive an auxiliary ODE in which the disturbance and the input are matched. We then build a nonlinear disturbance observer for the auxiliary ODE, based on the super-twisting sliding mode algorithm, to generate asymptotically accurate estimates for the unknown disturbance. By letting the input to the plant to be the negative of the disturbance estimate obtained, the matched disturbance in the plant can be rejected. In case the plant is unstable, including a stabilizing feedback signal in the input will ensure that the plant state converges to zero asymptotically. Our approach requires the state of the plant to be known. When only the plant output is known, our approach can be implemented using a state observer for the plant and then modifying the disturbance observer suitably. We demonstrate the efficacy of our approach in simulations by taking the plant to be an anti-stable 1D wave equation and assuming output measurement.**

I. INTRODUCTION

Stabilizing plants while rejecting unknown matched disturbances, i.e. disturbances present in the input channel of the plant, is a problem of practical interest. Several papers in the literature have addressed this problem for finite-dimensional plants, see for instance [1], [17] and the references therein. However, for linear infinite-dimensional plants this problem is harder to solve and fewer works have addressed it. One of the earliest works to study this problem for infinite-dimensional plants in an abstract setting is [12], where a discontinuous control law is proposed based on the sliding mode technique. In that work it is assumed that the input operator is bounded, so plants with boundary control inputs cannot be considered, and that the full-state of the plant can be measured. Under similar assumptions [13] addressed this problem (and a tracking problem) for the 1D heat and wave equations. Subsequently, other works have addressed the above stabilization/disturbance rejection problem for particular boundary controlled infinite-dimensional plants such as the 1D heat and wave equations.

The problem of stabilizing boundary controlled 1D heat equations while rejecting matched disturbances has been addressed in [2], [14] and [15]. While [2] develops a state-feedback controller for solving the problem, [14] and [15] develop output-feedback controllers. The heat equation considered in [15] is more general than those in [2] and [14]. For PDE-ODE cascade systems in which the PDE is a boundary controlled 1D heat equation with matched disturbance whose output drives an unstable ODE system, the stabilization with disturbance rejection problem is addressed in [20] by developing a state-feedback control law and in [8] by developing an output-feedback control law.

In the case of boundary controlled 1D wave equations with unknown matched disturbances, the problem of stabilization with disturbance rejection has been addressed in [4], [5], [7] and [11]. For an anti-stable wave equation [7] develops an output-feedback controller (which can ensure the decay only of the vibrating energy), [5] develops a state-feedback controller and [4] develops an observer-based output feedback controller. The work [11] presents an output-feedback controller for an unstable wave equation. For PDE-ODE cascade systems in which the PDE is a boundary controlled unstable wave equation with matched disturbance whose output drives an unstable ODE system, the stabilization with disturbance rejection problem is addressed in [22] by developing an output-feedback controller. The above problem is solved for an Euler-Bernoulli beam equation in [6] and a linear 2×2 hyperbolic system in [18] using state-feedback controllers.

A natural scheme for addressing the problem of stabilization with matched disturbance rejection is as follows: develop an observer for estimating the disturbance, cancel the disturbance directly using the estimate and then address the stabilization problem by ignoring the cancelled disturbance. This scheme is implemented in [4], [5], [6] and [22], for specific PDE models, by designing a high-gain observer for estimating the disturbance. In this work, we propose an alternate observer for estimating the disturbance which can be used in conjunction with the above scheme. In our observer design approach, we introduce an ODE system in cascade with the infinite-dimensional plant. Using a state transformation obtained by solving a Sylvester equation, we derive an auxiliary ODE in which the disturbance and the input are matched. We then build a nonlinear disturbance observer for the auxiliary ODE to generate asymptotically accurate estimates for the unknown disturbance. Our approach requires the state of the plant to be known, but can be combined with an state observer when only the plant output is known. Our observer design is based on the super-twisting sliding mode algorithm and so does not require high gains. We have developed our approach in an abstract setting and it is applicable to plants belonging to a large class of linear

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systems called regular linear systems.

The rest of the paper is organized as follows. In Section II we introduce our assumptions and state the disturbance estimation problem addressed in this paper. Section III and Section IV present the design of observers using full-state measurement and output measurement, respectively, for estimating the unknown disturbance. In Section V we illustrate the efficacy of our output-based disturbance observer design numerically by taking the plant to be an anti-stable 1D wave equation. We show in simulations that by cancelling the disturbance using the estimate from the observer and implementing the stabilizing output-feedback control law from [5], the state of the plant can be taken to zero asymptotically.

Notations. Let X and Y be two Hilbert spaces. The norm in X is written as $\|\cdot\|_X$. We denote the space of bounded linear operators from X to Y by $\mathcal{L}(X,Y)$ and write $\mathcal{L}(X,X)$ as $\mathcal{L}(X)$. The space of X-valued locally square integrable functions on $[0,\infty)$ is denoted as $L^2_{\rm loc}([0,\infty);X)$. For a linear operator $A:\mathcal{D}(A)\subset X\to X$, where $\mathcal{D}(A)$ is the domain of A, let $\sigma(A)$ be its spectrum and $\rho(A)$ its resolvent set. For $x\in\mathbb{R}$, the set-valued function ${\rm sign}(x)$ is defined as

$$\mathrm{sign}(x)=\begin{cases}1 & x>0,\\ [-1,1] & x=0,\\ -1 & x<0.\end{cases}$$

For a vector $v=[v_1 \ \ v_2 \ \cdots \ v_n]^\top\in\mathbb{R}^n$, we define ${\rm sign}(v)=[{\rm sign}(v_1) \ \ {\rm sign}(v_2) \ \cdots \ {\rm sign}(v_n)]^\top$ and $|v|^{\frac{1}{2}}=[|v_1|^{\frac{1}{2}} \ \ |v_2|^{\frac{1}{2}} \ \cdots \ |v_n|^{\frac{1}{2}}]^\top$.

II. Problem formulation

In this work, we consider an infinite-dimensional plant which can be written as an abstract evolution equation on a Hilbert space Z as follows: for $t>0$,

$$\dot z(t)=Az(t)+B(u(t)+d(t)), \tag{2.1}$$

$$y(t)=C_\Lambda z(t). \tag{2.2}$$

Here $z(t)\in Z$ is the plant state, $u(t)\in\mathbb{R}^m$ is the control input, $d(t)\in\mathbb{R}^m$ is the unknown disturbance and $y(t)\in\mathbb{R}^p$ is the plant output. The state operator A is the generator of a strongly continuous semigroup $\mathbb{T}$ on Z. For some $\beta\in\rho(A)$, let Z_{-1} be the Hilbert space obtained by completing Z with respect to the norm $\|z\|_{Z_{-1}}=\|(\beta I-A)^{-1}z\|_Z$ and let Z_1 be the domain of A with the norm $\|z\|_{Z_1}=\|(\beta I-A)z\|_Z$. We assume that $B\in\mathcal{L}(\mathbb{R}^m,Z_{-1})$ is an admissible control operator for $\mathbb{T}$ and $C\in\mathcal{L}(Z_1,\mathbb{R}^p)$ is an admissible observation operator for $\mathbb{T}$. The Λ-extension of C, denoted as C_Λ, is defined on Z as follows:

$$C_\Lambda z=\lim_{\lambda\to+\infty}C\lambda(\lambda I-A)^{-1}z \quad \text{whenever the limit exists.}$$

The domain of C_Λ, written as $\mathcal{D}(C_\Lambda)$, is the set of all $z\in Z$ for which the above limit exists. We have $\mathcal{D}(A)\subset\mathcal{D}(C_\Lambda)$ and the restriction of C_Λ to $\mathcal{D}(A)$ is C. We suppose that the triple (A,B,C) is regular, so that the plant (2.1)-(2.2) is a regular linear system. For more details regarding the notions of admissibility and regularity, see [19] and [21].

For any initial state $z(0)\in Z$, input $u\in L^2_{\rm loc}([0,\infty);\mathbb{R}^m)$ and disturbance $d\in L^2_{\rm loc}([0,\infty);\mathbb{R}^m)$, the state trajectory of (2.1) is the continuous function from $[0,\infty)$ to Z defined as follows:

$$z(t)=\mathbb{T}_t z(0)+\int_0^t\mathbb{T}_{t-\tau}B[u(\tau)+d(\tau)]{\rm d}\tau \qquad \forall\, t\geq 0.$$

For almost all $t\geq 0$, the state trajectory z satisfies the state equation (2.1) in Z_{-1} and $z(t)\in\mathcal{D}(C_\Lambda)$. The output equation (2.2) determines the output $y\in L^2_{\rm loc}([0,\infty);\mathbb{R}^p)$. The regularity of the triple (A,B,C) implies that $C_\Lambda(sI-A)^{-1}B$ exists for each $s\in\rho(A)$.

We suppose that the unknown disturbance d satisfies the following assumption.

Assumption 2.1: The disturbance $d:[0,\infty)\to\mathbb{R}^m$ is a continuous function such that any finite time interval can be partitioned into finitely many disjoint subintervals on each of which d is continuously differentiable. Furthermore, there exists a known constant $R>0$ such that $\|\dot d(t)\|_{\mathbb{R}^m}\leq R$ for all t at which d is differentiable.

We now state the disturbance estimation problem addressed in this paper.

Problem 2.2: Design a disturbance observer for the regular linear system (2.1)-(2.2) such that the disturbance estimate $\hat d$ generated by the observer satisfies

$$\lim_{t\to\infty}\|d(t)-\hat d(t)\|_{\mathbb{R}^m}=0.$$

We first address this problem in Section III by assuming that the full-state of the plant can be measured. Then, under an additional assumption regarding the existence of an observer for the plant state, we address this problem in Section IV by supposing that only the plant output can be measured. Taking $u=v+\hat d$ in (2.1), where $\hat d$ is the estimate generated by the disturbance observer and v is a stabilizing state/output feedback control law designed for (2.1) by assuming that $d=0$, we can cancel the effect of the disturbance d and ensure that the state trajectory converges to zero asymptotically, see Remarks 3.5 and 4.4.

III. Disturbance observer using full-state measurement

In this section, we address Problem 2.2 by designing a disturbance observer for the plant (2.1)-(2.2) using full-state measurement, see Theorem 3.3. In our design approach, we introduce the following exponentially stable ODE system driven by the plant output y in (2.2):

$$\dot w(t)=Ew(t)+FC_\Lambda z(t). \tag{3.3}$$

Here $w(t)\in\mathbb{R}^m$ is the state of the ODE system, $E\in\mathbb{R}^{m\times m}$ is a diagonal matrix whose eigenvalues have a negative real part and $F\in\mathbb{R}^{m\times p}$. Using the solution to a Sylvester equation, we diagonalize the state operator of the cascade interconnection of the plant (2.1)-(2.2) and the ODE system (3.3) to obtain an auxiliary ODE system in which the input u and the disturbance d are matched. We then obtain estimates $\hat d$ for d by building a disturbance observer for the auxiliary ODE system using the super-twisting sliding mode algorithm, which has been widely employed in the finite-dimensional literature.

We first recall a lemma from our paper [10] regarding the existence of solutions to Sylvester equations.

Lemma 3.1: Let the operators A, B and C be as introduced in (2.1)-(2.2) and let the matrices E and F be as defined in (3.3). Suppose that $\sigma(A) \cap \sigma(E) = \emptyset$. Then there exists a linear map $\Pi : A\mathcal{D}(C_\Lambda) \to \mathbb{R}^m$ with $\Pi \in \mathcal{L}(Z, \mathbb{R}^m)$ such that

$$E\Pi z = \Pi A z + F C_\Lambda z \qquad \forall\, z \in \mathcal{D}(C_\Lambda). \tag{3.4}$$

Furthermore $\Pi B \in \mathbb{R}^{m\times m}$.

Proof: Recall that E is an exponentially stable diagonal matrix. So e^{-Et} can be written as follows:

$$e^{-Et} = \sum_{k=1}^{m} e^{-\lambda_k t} E_k,$$

where $\lambda_k < 0$ is the $k^{\rm th}$ diagonal entry of E and $E_k \in \mathbb{R}^{m\times m}$ is a matrix whose $k^{\rm th}$ diagonal entry is 1 and all other entries are 0. Then

$$\Pi = \sum_{k=1}^{m} E_k F C_\Lambda (\lambda_k I - A)^{-1} \tag{3.5}$$

is the required linear map, see the proof of Lemma 3.6 in [10] for details. ■

Assumption 3.2: There exists matrices $E \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times p}$ such that

(i) E is a diagonal matrix whose eigenvalues have a negative real part and $\sigma(A) \cap \sigma(E) = \emptyset$,
(ii) for Π given in (3.5), the matrix $\Pi B \in \mathbb{R}^{m\times m}$ is invertible.

Note that Item (i) in the assumption can be satisfied easily and Item (ii) is the non-trivial requirement. The above assumption is a sufficient condition for designing a disturbance observer for the regular linear system (2.1)-(2.2) (see Theorem 3.3). It also appears to be a necessary condition. Indeed, when $m = 1$ the above assumption is equivalent to the requirement that the transfer function of (2.1)-(2.2), given by $C_\Lambda(sI - A)^{-1}B$, exists and does not vanish at at least one point on the negative real-axis which, for most operators A, is equivalent to the requirement that $B \neq 0$ and $C \neq 0$.

The following theorem presents a disturbance observer which solves Problem 2.2. Recall the notation $\mathrm{sign}(v)$ and $|v|^{\frac{1}{2}}$ for $v \in \mathbb{R}^n$ introduced at the end of Section I.

Theorem 3.3: Consider the infinite-dimensional plant (2.1)-(2.2) and the ODE system (3.3) driven by the plant output. Let Assumptions 2.1 and 3.2 hold. Define

$$p = \Pi z + w, \tag{3.6}$$

where Π is the linear operator in (3.5), z is the state trajectory of (2.1) for some initial state $z(0) \in Z$, input $u \in L^2_{\rm loc}([0,\infty); \mathbb{R}^m)$ and disturbance $d \in L^2_{\rm loc}([0,\infty); \mathbb{R}^m)$ and w is the state trajectory of (3.3) for some initial state $w(0) \in \mathbb{R}^m$. Let $k_1 = 14\sqrt{R\|\Pi B\|_{\mathbb{R}^{m\times m}}}$ and $k_2 = 9R\|\Pi B\|_{\mathbb{R}^{m\times m}}$ with R being the constant in Assumption 2.1. Then the estimate $\hat{d}$ provided by the disturbance observer

$$\dot{\hat{p}}(t) = Ep(t) + k_1|p(t) - \hat{p}(t)|^{\frac{1}{2}}\mathrm{sign}(p(t) - \hat{p}(t)) + \Pi B u(t) + k_2 \int_0^t \mathrm{sign}(p(\tau) - \hat{p}(\tau))\mathrm{d}\tau, \tag{3.7}$$

$$\hat{d}(t) = k_2 (\Pi B)^{-1} \int_0^t \mathrm{sign}(p(\tau) - \hat{p}(\tau))\mathrm{d}\tau \tag{3.8}$$

satisfies

$$\|d(t) - \hat{d}(t)\|_{\mathbb{R}^m} = 0 \qquad \forall\, t > T \tag{3.9}$$

and some finite time $T > 0$.

Remark 3.4: The solution to (3.7), and other differential equations with non-smooth right-hand sides encountered in this paper, are to be understood in the sense of Filippov [3]. The differential equations with discontinuous right-hand sides studied in this paper are very similar to those considered in [9].

Proof: Taking the time derivative of p in (3.6) and using (2.1)-(2.2) and (3.3), it follows after a simple calculation based on the properties of Π mentioned in Lemma 3.1 that

$$\dot{p}(t) = Ep(t) + \Pi B(u(t) + d(t)). \tag{3.10}$$

(An alternate derivation for the above expression using Laplace transforms is shown above Eq. (3.19) in [10].) Unlike the ODE in (3.3), the above ODE is decoupled from the infinite-dimensional dynamics (2.1)-(2.2). Define

$$e_p(t) = p(t) - \hat{p}(t),$$
$$q_p(t) = \Pi B d(t) - k_2 \int_0^t \mathrm{sign}(p(\tau) - \hat{p}(\tau))\mathrm{d}\tau.$$

Using (3.7) and (3.10) we get

$$\dot{e}_p(t) = -k_1|e_p(t)|^{\frac{1}{2}}\mathrm{sign}(e_p(t)) + q_p(t), \tag{3.11}$$
$$\dot{q}_p(t) = -k_2\mathrm{sign}(e_p(t)) + \Pi B\dot{d}(t). \tag{3.12}$$

Applying [9, Theorem 2] to the above equation we get that there exists a finite time $T > 0$ such that $q_p(t) = 0$ for all $t > T$. The claim in (3.9) now follows directly using the definitions of q_p and $\hat{d}$ in (3.8). ■

We remark that [9, Theorem 2] has been derived for scalar equations of the form (3.11)-(3.12), i.e. $e_p(t)$ and $q_p(t)$ are scalars in [9, Theorem 2]. We could use this result in the above proof since (3.11)-(3.12) with $e_p(t) \in \mathbb{R}^m$ and $q_p(t) \in \mathbb{R}^m$ can be regarded as m scalar equations of the form (3.11)-(3.12) which are not coupled to one another.

Remark 3.5: Suppose that the pair (A, B) is stabilizable and there exists a state-feedback control law $u = Kz$ such that $A + BK$ is the generator of an exponentially stable semigroup (for a formal definition of the stabilizability of the pair (A, B), see [21]). Then taking $u = Kz - \hat{d}$, where $\hat{d}$ is given in (3.8), ensures that the state trajectory of (2.1) converges to zero exponentially for any initial state $z(0)$.

IV. Disturbance observer using output measurement

For implementing the disturbance observer (3.7)-(3.8) proposed in Section III, the full-state $z(t)$ of the infinite-dimensional plant (2.1)-(2.2) must be measured. Indeed, note that $z(t)$ is required for computing $p(t)$ which is used in (3.7)-(3.8), see also (3.6). In this section, we extend the results in Section III by designing a disturbance observer for (2.1)-(2.2) using only output measurement.

We will need the following assumption regarding the existence of a state observer for (2.1)-(2.2).

Assumption 4.1: There exists a state observer for the infinite-dimensional plant (2.1)-(2.2) which, using only the input to the plant and the plant output, generates a trajectory $\hat{z} : [0, \infty) \to Z$ which converges to the state trajectory z of the plant in the following sense:

$$\lim_{t\to\infty} \|z(t) - \hat{z}(t)\|_Z = 0, \tag{4.13}$$

$$\lim_{t\to\infty} \|\Pi\dot{z}(t) - \Pi\dot{\hat{z}}(t)\|_{\mathbb{R}^m} = 0, \tag{4.14}$$

with both the above convergences being exponential.

Remark 4.2: The observer in the above assumption is not a typical state observer since it must ensure that the limits in (4.13)-(4.14) hold even in the presence of an unknown disturbance. Such observers have been proposed for the wave equation in [4], [22] by assuming that the initial error between the plant state and the observer state satisfies certain compatibility conditions. These observers guarantee only the limit in (4.13). While preliminary analysis indicates that (4.13) should imply (4.14) (at least when C is a bounded operator), it is hard to prove. Furthermore, it is also difficult to verify in examples that (4.14) holds. We remark that it may be possible to completely drop (4.14) from Assumption 4.1 by strengthening the results in [9] so as to include exponentially decaying measurement noise in their analysis.

The following theorem presents a disturbance observer which solves Problem 2.2 using output measurement.

Theorem 4.3: Consider the infinite-dimensional plant (2.1)-(2.2) and the ODE (3.3) driven by the plant output. Let Assumptions 2.1, 3.2 and 4.1 hold. Define

$$P = \Pi\hat{z} + w, \tag{4.15}$$

where Π is the linear operator in (3.5), $\hat{z}$ is the observer trajectory for some observer initial state $\hat{z}(0) \in Z$, plant initial state $z(0) \in Z$, input $u \in L^2_{\text{loc}}([0,\infty);\mathbb{R}^m)$ and disturbance $d \in L^2_{\text{loc}}([0,\infty);\mathbb{R}^m)$ and w is the state trajectory of (3.3) for some initial state $w(0) \in \mathbb{R}^m$. Let k_1 and k_2 be as in Theorem 3.3. Then the estimate $\hat{d}$ provided by the disturbance observer

$$\begin{aligned}\dot{\hat{P}}(t) =& Ew(t) + E\Pi\hat{z}(t) + k_1|P(t) - \hat{P}(t)|^{\frac{1}{2}} \\ &+ \text{sign}(P(t) - \hat{P}(t)) + \Pi Bu(t) \\ &+ k_2\int_0^t \text{sign}(P(\tau) - \hat{P}(\tau))\text{d}\tau, \end{aligned}\tag{4.16}$$

$$\hat{d}(t) = k_2(\Pi B)^{-1}\int_0^t \text{sign}(P(\tau) - \hat{P}(\tau))\text{d}\tau \tag{4.17}$$

satisfies

$$\lim_{t\to\infty} \|d(t) - \hat{d}(t)\|_{\mathbb{R}^m} = 0. \tag{4.18}$$

Proof: Define $e_z = z - \hat{z}$. Taking the time derivative of P in (4.15) and using (2.1)-(2.2) and (3.3), it follows after a simple calculation based on the properties of Π mentioned in Lemma 3.1 that

$$\dot{P}(t) = EP(t) + \Pi B(u(t) + d(t)) - \Pi\dot{e}_z(t). \tag{4.19}$$

Let

$$e_P(t) = P(t) - \hat{P}(t),$$

$$q_P(t) = \Pi Bd(t) - k_2\int_0^t \text{sign}(P(\tau) - \hat{P}(\tau))\text{d}\tau.$$

Using (4.16) and (4.19) we get

$$\dot{e}_P(t) = \mu(t) - k_1|e_P(t)|^{\frac{1}{2}}\text{sign}(e_P(t)) + q_P(t), \tag{4.20}$$

$$\dot{q}_P(t) = -k_2\text{sign}(e_P(t)) + \Pi B\dot{d}(t), \tag{4.21}$$

where $\mu(t) = -\Pi\dot{e}_z(t)$. For each $\tau \geq 0$, let

$$\delta(\tau) = \sup_{s\in[\tau,\infty)} \|\mu(s)\|_{\mathbb{R}^m}.$$

It follows from Assumption 4.1 that there exist $M, \omega > 0$ such that

$$\delta(\tau) \leq Me^{-\omega\tau} \qquad \forall\, \tau \geq 0. \tag{4.22}$$

Define

$$\zeta(t) = \begin{bmatrix} |e_P(t)|^{\frac{1}{2}}\text{sign}(e_P(t)) \\ q_P(t) \end{bmatrix}.$$

Applying [9, Theorem 3] to (4.20)-(4.21) it follows that there exist positive constants c_1, c_2 and c_3 independent of the initial conditions $e_P(0)$ and $q_P(0)$ such that the trajectory of (4.20)-(4.21) satisfies

$$\|\zeta(t)\|_{\mathbb{R}^{2m}} \leq c_1\delta(0) \qquad \forall\, t \geq c_2\|\zeta(0)\|_{\mathbb{R}^{2m}} + c_3\delta(0),$$

see the proof of Theorem 3 in [9] and also see the remark after this proof. Using this result, by considering the initial time for the differential equation (4.20)-(4.21) to be some $\tau \geq 0$, we can infer that

$$\|\zeta(t)\|_{\mathbb{R}^{2m}} \leq c_1\delta(\tau) \qquad \forall\, t \geq \tau + c_2\|\zeta(\tau)\|_{\mathbb{R}^{2m}} + c_3\delta(\tau). \tag{4.23}$$

Let $T = c_2\|\zeta(0)\|_{\mathbb{R}^{2m}} + (c_1c_2 + c_3)\delta(0)$. We claim that the following estimate holds for all integers $k \geq 1$:

$$\|\zeta(t)\|_{\mathbb{R}^{2m}} \leq c_1\delta(kT - T) \qquad \forall\, t \in [kT, kT + T]. \tag{4.24}$$

Indeed, taking $\tau = 0$ in (4.23) it follows that (4.24) holds for $k = 1$. Suppose that (4.24) holds for some integer $k = j \geq 1$. Then letting $k = j$ and $t = jT$ in (4.24) and using the fact that δ is a non-increasing function we get

$$\|\zeta(jT)\|_{\mathbb{R}^{2m}} \leq c_1\delta(jT - T) \leq c_1\delta(0).$$

Now taking $\tau = jT$ in (4.23) and using the above inequality and the fact that δ is a non-increasing function we get

$$\|\zeta(t)\|_{\mathbb{R}^{2m}} \leq c_1\delta(jT) \qquad \forall\, t \geq jT + c_1c_2\delta(0) + c_3\delta(0)$$

which, using the definition of T implies that (4.24) holds for $k = j + 1$. It now follows via the principle of mathematical induction that (4.24) holds for all $k \geq 1$.

Combining (4.22) and (4.24) we get

$$\|\zeta(t)\|_{\mathbb{R}^{2m}} \leq c_1Me^{2\omega T}e^{-\omega t} \qquad \forall\, t \geq T.$$

Hence q_P converges to 0 exponentially which along with the definitions of q_P and $\hat{d}$ in (4.17), implies the limit in (4.18). ∎

We remark that [9, Theorem 3] has been derived for scalar equations of the form (4.20)-(4.21), i.e. $e_P(t)$ and $q_P(t)$ are scalars in [9, Theorem 3]. We could use this result in the above proof since (4.20)-(4.21) with $e_P(t) \in \mathbb{R}^m$ and $q_P(t) \in \mathbb{R}^m$ can be regarded as m scalar equations of the form (4.20)-(4.21) which are not coupled to one another.

Remark 4.4: Suppose that A, B and K are as in Remark 3.5 and suppose that there exists a state observer for (2.1)-(2.2) as described in Assumption 4.1. Then taking $u = K\hat{z} - \hat{d}$, where $\hat{z}$ is the observer trajectory and $\hat{d}$ is given in (4.17) will ensure that the state trajectory of (2.1) converges to zero exponentially for any initial state $z(0)$.

V. Illustrative example

In this section, we illustrate the disturbance observer design proposed in Theorem 4.3 using a 1D anti-stable wave equation with matched disturbance. We implement our disturbance observer and a stabilizing output-feedback controller from [4] on the wave equation numerically and show that its state converges to zero asymptotically.

Consider the following 1D anti-stable wave equation: for all $t > 0$,

$$v_{tt}(x,t) = v_{xx}(x,t) \qquad \forall\, x \in (0,1), \tag{5.25}$$

$$v_x(0,t) = -0.5 v_t(0,t), \qquad v_x(1,t) = u(t) + d(t), \tag{5.26}$$

$$y(t) = v(1,t), \tag{5.27}$$

$$y_m(t) = [v_t(0,t) \;\; v_t(1,t) \;\; v(1,t)]^\top. \tag{5.28}$$

Here $z(t) = \begin{bmatrix} v(\cdot,t) \\ v_t(\cdot,t) \end{bmatrix}$ is the state of the wave equation, $u(t) \in \mathbb{R}$ is the control input, $d(t) \in \mathbb{R}$ is the matched disturbance, $y(t)$ is the output used in the disturbance observer and $y_m(t)$ is the measured output used in the state observer.

The wave equation (5.25)-(5.27) can be written as an abstract evolution equation on the state space $Z = H^1(0,1) \times L^2(0,1)$ as follows:

$$\dot{z}(t) = Az(t) + B(u(t) + d(t)),$$
$$y(t) = Cz(t).$$

The state operator A is defined as follows: $D(A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H^2(0,1) \times H^1(0,1) \,\middle|\, f_x(0) = -0.5g(0), f_x(1) = 0 \right\}$ and $A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ f_{xx} \end{bmatrix}$ for all $\begin{bmatrix} f \\ g \end{bmatrix} \in D(A)$. The control operator $B = \begin{bmatrix} 0 \\ \delta_1 \end{bmatrix}$, where δ_1 is the Dirac pulse at $x = 1$ and the observation operator C is defined as $C \begin{bmatrix} f \\ g \end{bmatrix} = f(1)$ for all $\begin{bmatrix} f \\ g \end{bmatrix} \in Z$. Note that A is a Riesz spectral operator which generates an invertible C_0-semigroup $\mathbb{T}$ on Z, $B \in \mathcal{L}(\mathbb{R}, Z_{-1})$ is an admissible control operator for $\mathbb{T}$, $C \in \mathcal{L}(Z, \mathbb{R})$ and the triple (A, B, C) is regular. All this can be established by mimicking the arguments in [16, Section VI] where the constant in the boundary condition is 2.5 instead of -0.5.

We take the disturbance to be a continuous periodic function of unit period determined by the following expression:

$$d(t) = \begin{cases} 1 + 4t & \text{if} \quad t \in [0, 0.25), \\ 3 - 4t & \text{if} \quad t \in [0.25, 0.75], \\ 4t - 3 & \text{if} \quad t \in [0.75, 1]. \end{cases} \tag{5.29}$$

Clearly d satisfies Assumption 2.1 with $R = 4$.

We take $E = -1$ and $F = 1$ in (3.3). We have computed the expression for $(I + A)^{-1}$ explicitly by solving the ODE $\begin{bmatrix} \alpha+\beta \\ \alpha_{xx}+\beta \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$ for $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathcal{D}(A)$ and then verified that $-1 \notin \sigma(A)$. Substituting the computed expression for $(I + A)^{-1}$ in the formula (3.5) we obtain

$$\Pi \begin{bmatrix} f \\ g \end{bmatrix} = \frac{e}{3e^2 - 1} \left[3 \int_0^1 e^y [g(y) - f(y)] \mathrm{d}y + \int_0^1 e^{-y} [g(y) - f(y)] \mathrm{d}y - 2f(0) \right]$$

for all $\begin{bmatrix} f \\ g \end{bmatrix} \in Z$. Using the above formula, we get $\Pi B = 1.09$. So Assumption 3.2 also holds.

We use the following sliding mode observer presented in [4, Theorem 1] for estimating the state of the wave equation (5.25)-(5.26) using the measured output y_m in (5.28): for $t > 0$,

$$\hat{v}_{tt}(x,t) = \hat{v}_{xx}(x,t) \qquad \forall\, x \in (0,1), \tag{5.30}$$

$$\hat{v}_x(0,t) = (c - 0.5)[\hat{v}_t(0,t) - v_t(0,t)] + 0.5\hat{v}_t(0,t), \tag{5.31}$$

$$\begin{aligned} \hat{v}_x(1,t) = u(t) &- c_1[\hat{v}_t(1,t) - v_t(1,t)] - c_2[\hat{v}(1,t) - v(1,t)] \\ &- M_1 \mathrm{sign}[v_t(1,t) - v_t(1,t)] \\ &- M_2 \mathrm{sign}[\hat{v}(1,t) - v(1,t)], \end{aligned} \tag{5.32}$$

where $M_1 = 4$, $M_2 = 8$, $c = 1$ and $c_1 = c_2 = 1$ are the gains. It is shown in [4, Theorem 1] that along any solution of the above observer we have

$$\lim_{t \to \infty} \|v(\cdot,t) - \hat{v}(\cdot,t)\|_{H^1} + \|v_t(\cdot,t) - \hat{v}_t(\cdot,t)\|_{L^2} = 0$$

provided $\begin{bmatrix} v(\cdot,0) \\ v_t(\cdot,0) \end{bmatrix}$ and $\begin{bmatrix} \hat{v}(\cdot,0) \\ \hat{v}_t(\cdot,0) \end{bmatrix}$ satisfy some compatibility conditions. Moreover, the above convergence is exponential. So all the claims in Assumption 4.1, except the limit in (4.14), have been verified (see Remark 4.2). Our simulation results appear to indicate that (4.14) also holds.

We have performed numerical simulations of the closed-loop system consisting of the wave equation (5.25)-(5.28), the state observer (5.30)-(5.32) and the disturbance observer (4.16)-(4.17) by taking u in (5.26), (5.32) and (4.16) to be the following observer-based output-feedback stabilizing control law in [4, Eq. (58)] with $q = 0.5$, $k = 1.4$ and $M = 5$:

$$u(t) = -0.89 v_t(1,t) - 4.45 v(1,t) + 2.48 \hat{v}(0,t) + 4.98 \int_0^1 \hat{v}_t(x,t) \mathrm{d}x - \hat{d}(t).$$

Here $\hat{d}$ is the estimate generated by our disturbance observer (4.16)-(4.17) (and not the $\hat{d}$ generated using ADRC in [4]). In our simulations, we have chosen the initial states for the wave equation to be $v(x,0) = -3x^2(1-x)$ and $v_t(x,0) = 0$ and for the observer to be $\hat{v}(x,0) = 0$ and $\hat{v}_t(x,0) = 0$. These initial states satisfy the compatibility conditions in [4, Theorem 1]. We take $k_1 = 29.3$ and $k_2 = 39.4$ in the disturbance observer (4.16)-(4.17) and choose $\hat{P}(0) = 0$.

We have used the finite-difference scheme for our closed-loop simulations and the results are shown in Figures 1, 2, 3 and 4. Let $z(t) = \begin{bmatrix} v(\cdot,t) \\ v_t(\cdot,t) \end{bmatrix}$ and $\hat{z}(t) = \begin{bmatrix} \hat{v}(\cdot,t) \\ \hat{v}_t(\cdot,t) \end{bmatrix}$ be the states of the wave equation (5.25)-(5.26) and the observer (5.30)-(5.32), respectively. Figure 1 shows that $\|z(t) - \hat{z}(t)\|_Z$

converges to zero. Figure 2 shows that the disturbance estimate generated by the disturbance observer converges to the disturbance d determined by (5.29). Figure 3 shows that the proposed control law u has stabilized the wave equation and its state trajectory converges to zero.

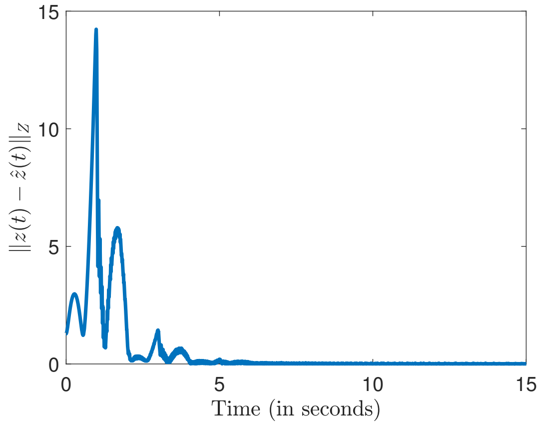

Figure 1. The error between the state trajctory z of the wave equation and its estimate $\hat{z}$ generated by the state observer converges to zero asymptotically.

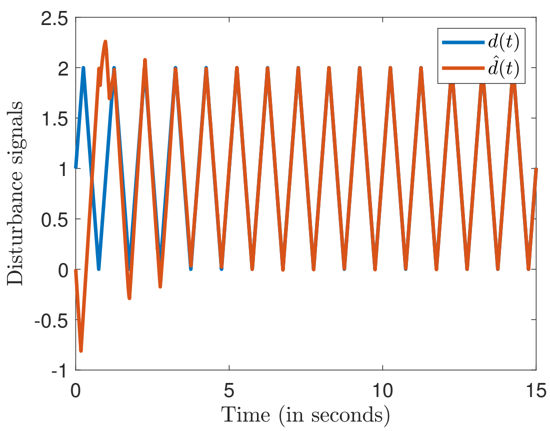

Figure 2. The estimated value of the disturbance converges to the true value asymptotically.

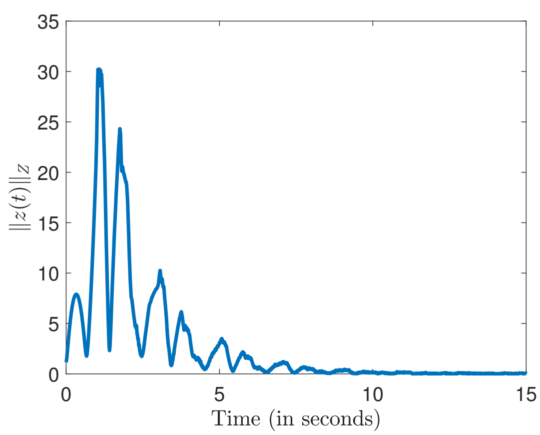

Figure 3. The state trajectory of the stabilized wave equation converges to zero asymptotically.

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