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Regret Analysis with Almost Sure Convergence for OBF-ARX Filter

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Abstract— **This paper considers the output prediction problem for an unknown Linear Time-Invariant (LTI) system. In particular, we focus our attention on the OBF-ARX filter, whose transfer function is a linear combination of Orthogonal Basis Functions (OBFs), with the coefficients determined by solving a least-squares regression. We prove that the OBF-ARX filter is an accurate approximation of the Kalman Filter (KF) by quantifying its online performance. Specifically, we analyze the average regret between the OBF-ARX filter and the KF, proving that the average regret over N time steps converges to the asymptotic bias at the speed of $O(N^{-0.5+\epsilon})$ almost surely for all $\epsilon > 0$. Then, we establish an upper bound on the asymptotic bias, demonstrating that it decreases exponentially with the number of OBF bases, and the decreasing rate $\tau(\boldsymbol{\lambda}, \boldsymbol{\mu})$ explicitly depends on the poles of both the KF and the OBF. Numerical results on diffusion processes validate the derived bounds.**

I. Introduction

The problem of modeling unknown systems has long been a focal point of extensive study within the control community. Recent interest in data-driven methods has spurred a significant amount of research into adaptive modeling algorithms, such as online output prediction algorithms for unknown systems and the quantification of their online performance [1–3]. The output prediction algorithms aim to minimize the output prediction error, commonly measured by the Mean Square Error (MSE), based on historical inputs and outputs [4]. Such algorithms play a crucial role in controller design, such as in model-predictive control problems [5] and direct data-driven control methods [6], as well as offering substantial benefits in practical scenarios, such as GPS navigation [3]. However, the task of output prediction for an unknown partially observed LTI systems is challenging due to the problem's nonlinearity and nonconvexity [7].

It is widely established that given the true system parameters and the noise statistics, the Kalman Filter (KF) is the optimal output predictor for LTI systems among all linear predictors [8]. As a result, for unknown LTI systems, several recent works focus on identifying the systems' KF using historical samples and quantifying the online performance of the algorithms, for instance, the algorithms' regret over time [3, 9]. Moreover, several other works are concerned with the online performance of the linear-quadratic-Gaussian regulator, where the identification of KF is also included [10, 11].

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On the other hand, another line of work leverages the well-studied Orthogonal Basis Function (OBF) methods to parametrize the online prediction filter in the frequency domain. These OBF-based filters are shown to require fewer parameters than the conventional FIR, ARX and ARMAX filters and to mitigate parameter inconsistency issues encountered in ARX and ARMAX filters [12]. Specifically, this type of methods rely on OBF, a classic framework in system identification. The OBF method proposes to approximate the transfer function of an LTI system using the linear combination of a group of OBFs, such as the Laguerre bases, the Kautz bases, and the Generalized OBF (GOBF), with the poles of bases selected based on *a priori* system knowledge. With a set of predetermined OBFs, the identification problem reduces to a least-squares regression. Notably, Van den Hof et al. [13] and Heuberger et al. [14] quantify the asymptotic bias of the GOBF, revealing that the bias decreases exponentially w.r.t. the number of OBF bases. They also prove that the solutions to the least-squares regression converge with probability 1 [15].

Subsequently, various researches propose online output predictors based on the OBF. One type of methods introduce the OBF-ARX filter by parametrizing the ARX filter using the OBF bases [15–17], hence reducing the problem to least-squares regression. The resulting filter is proved to converge in L^2 to the asymptotic solution of the regression for stable SISO systems [15, Chapter 4]. Madakyaru et al. [5] further introduce the state-space form of the OBF-ARX filter for MIMO systems by splitting the filter into multiple multi-input single-output filters. Additionally, Tufa et al. [12] propose to parametrize the rational function before the input in ARX and ARMAX structures with OBF, introducing the ARX-OBF and OBF-ARMAX methods, respectively. However, to the best of our knowledge, the connection between these OBF-based filters and the KF has not been extensively studied, which naturally raises a research question:

Is the OBF-based filter an accurate approximation to the KF with online performance guarantees?

In this paper, we study the online average regret between the OBF-ARX filter and the KF. The steady-state KF is considered as an LTI system, with past inputs and outputs as its inputs and the prediction value as its output. As a result, we approximate the transfer function of KF using a linear combination of OBF bases, resulting in an analogous form as the OBF-ARX filter for SISO systems [15, Chapter 4]. Moreover, we quantify the average regret $\mathcal{R}_N$ of the filter over N time steps, demonstrating that $\mathcal{R}_N \sim \bar{\alpha}\tau(\boldsymbol{\lambda}, \boldsymbol{\mu})^{-(\#\text{bases})} + O(N^{-0.5+\epsilon})$ almost surely for a constant $\bar{\alpha} > 0$ and any small number $\epsilon > 0$. The bound shows that the average regret converges at the rate of $O(N^{-0.5+\epsilon})$ to the asymptotic bias almost surely, where the bias can be exponentially reduced by increasing the number of the OBF bases. Additionally, the exponential decreasing rate $\tau(\boldsymbol{\lambda}, \boldsymbol{\mu})$ explicitly depends on the pole locations $\boldsymbol{\lambda}$ of the KF and those locations $\boldsymbol{\mu}$ of the GOBF. This result also demonstrates that *a priori* knowledge of the KF can facilitate the pole selection of the OBF-ARX filter. The main contribution of the paper is threefold:

- We study the relationship between the OBF-ARX filter and the KF, showing that OBF-ARX is an accurate approximation of the KF.
- We prove that the online average regret of the OBF-ARX filter over N time steps converges to the asymptotic bias at the speed of $O(N^{-0.5+\epsilon})$ almost surely.
- We derive an upper bound on the asymptotic bias of the average regret between the OBF-ARX filter and KF, demonstrating that the bias deceases exponentially w.r.t. the number of bases.

Numerical examples on diffusion processes are provided to validate the theoretical results.

Paper Structure: Section II briefly introduces the OBF bases and the OBF-ARX filter in the literature. In Section III, we conduct an analysis on the average regret of the OBF-ARX filter. Section IV leverages numerical results to validate the derived bounds. Finally, Section V concludes the paper.

Notations: $\mathbb{R}$ is the set of all real numbers. $\mathbb{C}$ is the set of all complex numbers. $\mathbb{R}^{a\times b}$ is the set of $a \times b$ real matrices, and $\mathbb{C}^{a\times b}$ is the set of $a \times b$ complex matrices. I_n denotes the n-dimensional unit matrix. $\mathbf{0}_n$ denotes an n-dimensional all-zero column vector and $\mathbf{0}_{m\times n}$ represents an $m \times n$ zero matrix. $\mathbf{1}_n$ and $\mathbf{1}_{m\times n}$ are similarly defined. $A^\top$ denotes the matrix transpose and A^H denotes the conjugate transpose of A. $\|A\|$ denotes the 2-norm for a vector A or a matrix A, and $\mathcal{H}_2$ norm for a system. For Hermitian matrices $A \in \mathbb{C}^{n\times n}$, $A \succeq 0$ implies that A is positive semi-definite. We say that $|f(k)| \sim O(g(k))$ for $g(k) > 0$ if there exists a constant $M > 0$, such that $\lim_{k\to\infty} \frac{|f(k)|}{g(k)} \leq M$ for all $k = 1, 2, \cdots$. The extended expectation $\bar{\mathbb{E}}$ is defined as $\bar{\mathbb{E}}(\phi(t)) = \lim_{t\to\infty} \mathbb{E}(\phi(t))$ for a proper ϕ.

II. Preliminary

The Orthogonal Basis Function (OBF) method tackles the identification problem for stable LTI systems. Consider an n-dimensional stable LTI system with p inputs and m outputs:

$$y(t) = G(z)u(t) + v(t), \tag{1}$$

where $G \in \mathcal{H}_2$, u is a quasistationary signal and v is a martingale difference sequence. z denotes the time-shift operator, i.e., $zu(t) = u(t + 1)$. The objective of the OBF method is to identify the system's transfer function $G(z)$ by approximating it through a linear combination of a set of predefined OBFs, $V_k(z)$:

$$\check{G}(z) = \sum_{k=1}^{\check{q}} L_k V_k(z). \tag{2}$$

The OBFs $V_k(z)$ can be selected as either scalar functions or, given additional information regarding each instance of the transfer function, as matrix-valued functions [18]. In this paper, for ease of discussion, we focus on scalar OBFs. Consequently, the coefficients of linear combination are complex matrices $L_k \in \mathbb{C}^{p\times m}$. Common examples of OBF include Laguerre bases, Kautz bases, and Generalized OBF (GOBF) [13]. See [15] for more details. Specifically, both the Laguerre and Kautz bases are special cases of the GOBF [13]:

$$\begin{aligned} V_{j+(k-1)n_b}(z) &= e_j^\top (zI - A_b)^{-1} B_b [G_b(z)]^{k-1}, \\ k &= 1, 2, \cdots, j = 1, \cdots, n_b, \end{aligned} \tag{3}$$

where the inner function $G_b(z)$ denotes an n_b-th order all-pass filter with poles $\mu_1, \cdots, \mu_{n_b}$, e_j is the j-th canonical vector and (A_b, B_b, C_b, D_b) denotes a minimal balanced realization of $G_b(z)$ [13].

In the OBF method, leveraging *a priori* information of the system, the system operator first selects the order n_b, the pole locations $\mu_1, \cdots, \mu_{n_b}$ of the inner function $G_b(z)$, and the number of the bases $\check{q}$, ensuring that the resulting GOBFs $V_1(z), \cdots, V_{\check{q}}(z), V_{\check{q}+1}(z), \cdots$ are complete over the $\mathcal{H}_2$ space [15]. These parameters subsequently allow for the computation of the GOBF $V_1(z), \cdots, V_{\check{q}}(z)$ in accordance with (3). The primary objective then becomes the following least-squares problem:[1]

$$\min_{L_1,\cdots,L_{\check{q}}} \bar{\mathbb{E}} \left\| y(t) - \sum_{k=1}^{\check{q}} L_k V_k(z) u(t) \right\|^2, \tag{4}$$

using input and output samples $\{u(t), y(t)\}$ of the system $G(z)$. Specifically, given N sample pairs $u(1 : N) = \{u(1), \cdots, u(N)\}, y(1:N) = \{y(1), \cdots, y(N)\}$, the resulting identification algorithm solves the following least-squares regression:

$$\min_{L_1,\cdots,L_{\check{q}}} \frac{1}{N} \sum_{t=1}^{N} \left\| y(t) - \sum_{k=1}^{\check{q}} L_k V_k(z) u(t) \right\|^2, \tag{5}$$

with its optimal solution denoted as $L_1(N), \cdots, L_{\check{q}}(N)$, and the identified system model becomes:

$$\check{G}(z) = \sum_{k=1}^{\check{q}} L_k(N) V_k(z). \tag{6}$$

The least-squares problem (5) can be further derived into a recursive formulation [15]. Additionally, the asymptotic identification bias for SISO systems using the GOBF is quantified [13], which is summarized as follows. Consider the identification of the SISO system $G(z)$ with n poles $\lambda_1, \cdots, \lambda_n$, using $\check{q}$ GOBF bases $V_1(z), \cdots, V_{\check{q}}(z)$ with the inner function $G_b(z)$, whose poles are chosen as $\mu_1, \cdots, \mu_{n_b}$, and it is assumed without loss of generality

[1]Notice that both the closed-loop system and the OBF bases are stable, the extended expectation is well-defined.

that $\check{q} = qn_b$. Moreover, denote the asymptotic estimate as

$$\begin{aligned} \begin{bmatrix}(L_1^*)^\top & \cdots & (L_{\check{q}}^*)^\top\end{bmatrix}^\top &= \bar{\mathbb{E}}(\phi(t)\phi(t)^H)^{-1}\bar{\mathbb{E}}(\phi(t)y(t)^H), \\ \phi(t) &= \begin{bmatrix}(V_1(z)u(t))^\top & \cdots & (V_{\check{q}}(z)u(t))^\top\end{bmatrix}^\top. \end{aligned} \tag{7}$$

The following theorem provides an upper bound on the asymptotic approximation bias between G and $\check{G}$.

Theorem 1 (Asymptotic bias of GOBF [13])**.** *For a small constant $\delta > 0$, let*

$$\tau(\boldsymbol{\lambda}, \boldsymbol{\mu}) \triangleq \left(\max_{j=1,\cdots,n} \prod_{k=1}^{n_b} \left|\frac{\lambda_j - \mu_k}{1 - \bar{\lambda}_j \mu_k}\right|\right)^{2/n_b} + \delta. \tag{8}$$

Then, there exists a constant $c > 0$, such that the system approximation bias satisfies

$$\left\|\sum_{k=1}^{\check{q}} L_k^* V_k(z) - G(z)\right\|^2 \leq \tilde{c}\frac{\tau(\boldsymbol{\lambda}, \boldsymbol{\mu})^{(q+1)n_b}}{1 - \tau(\boldsymbol{\lambda}, \boldsymbol{\mu})^{n_b}}, \tag{9}$$

where $\|\cdot\|$ denotes the system $\mathcal{H}_2$ norm, $\tilde{c} = c\left(1 + \frac{\operatorname{ess\,sup}_\omega \Phi_u(\omega)}{\operatorname{ess\,inf}_\omega \Phi_u(\omega)}\right)^2$ and $\Phi_u(\omega)$ denotes the power spectral density of the input signal $u(t)$.

Additionally, parametrizing the ARX filter via OBF methods leads to the OBF-ARX filter for SISO systems introduced in [15, Chapter 4]:

$$\check{y}(t) = \sum_{k=1}^{\check{q}_u} L_k^u V_k^u(z)u(t) + \sum_{k=1}^{\check{q}_y} L_k^y V_k^y(z)y(t), \tag{10}$$

where $V_k^u(z)$ and $V_k^y(z)$ can be selected as two different sets of OBFs, and the coefficients L_k^u, L_k^y are obtained by solving a least-squares problem given N samples $\{u(t), y(t)\}_{t=1}^N$:

$$\min_{L_k^u, L_k^y} \frac{1}{N}\sum_{t=1}^N \|y(t) - \check{y}(t)\|^2. \tag{11}$$

The least-squares problem can be further derived into a recursive manner [15, Chapter 4].

III. Main Results

To facilitate the regret analysis, we first derive an OBF-ARX filter by approximating the transfer function of KF using the linear combination of OBFs and describe the corresponding online prediction algorithm. Subsequently, we analyze the online average regret of the derived OBF-ARX filter.

A. Problem Formulation

Consider an n-dimensional LTI system with m inputs and p outputs taking the following form:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t), \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{12}$$

where $w(t)$ and $v(t)$ represent i.i.d. zero mean random noise with covariances $Q \succeq 0$ and $R \succ 0$, respectively. $w(t)$ and $v(t)$ are mutually independent.

Assumption 1. *The system is controlled by a stabilizing linear control law, formulated by an n_u-dimensional LTI system:*

$$\begin{aligned} \psi(t+1) &= A_u\psi(t) + B_u y(t) + w_u(t), \\ u(t) &= C_u\psi_u(t) + v_u(t), \end{aligned} \tag{13}$$

where $w_u(t)$ and $v_u(t)$ are i.i.d. random noise with zero mean and covariances $Q_u \succeq 0$ and $R_u \succeq 0$, respectively. $w_u(t)$ and $v_u(t)$ are mutually independent.

Remark 1. *The controller described in* (13) *encompasses typical control inputs, for instance, the white noise inputs commonly used in system identification and linear controllers such as PID.*

For simplicity of the analysis, we assume that the system has reached the steady state after operating for a sufficiently long time. Now we formulate the output prediction problem [3], which aims to use historical inputs and outputs to minimize the output prediction error in the minimum MSE sense. In this paper, we focus on linear predictors, where the predicted output is a linear combination of past inputs and outputs. The problem is formulated as:

$$\hat{y}^*(t \mid t-1) = \arg\min_{y \in \mathcal{L}_t} \mathbb{E}[\|y(t) - y\|^2], \tag{14}$$

where $\mathcal{L}_t$ denotes the set of all possible linear combinations of past inputs $u(-\infty : t-1)$ and outputs $y(-\infty : t-1)$.

Definition 1 (Average regret)**.** *The average regret over N time steps for an online linear output predictor $\check{y}(t \mid 1 : t-1)$ using finite past information $u(1 : t-1), y(1 : t-1)$ is defined as:*

$$\mathcal{R}_N = \frac{1}{N}\sum_{t=1}^N \|\check{y}(t \mid 1 : t-1) - \hat{y}^*(t \mid t-1)\|^2. \tag{15}$$

It is well established that the steady-state KF provides an optimal solution to (14) given the true system parameters A, B, C and covariance matrices Q, R [8]:

$$\begin{aligned} \hat{x}(t+1) &= A(I - KC)\hat{x}(t) + Bu(t) + AKy(t), \\ \hat{y}^*(t) &= C\hat{x}(t), \end{aligned} \tag{16}$$

where $K = PC^H(CPC^H + R)^{-1}$ and P is the solution to the algebraic Riccati equation:

$$P = APA^H - APC^H(CPC^H + R)^{-1}CPA^H + Q.$$

Hence, the average regret of the filter $\check{y}$ can be reformulated as follows leveraging the optimality of the steady-state KF among all linear predictors:

$$\mathcal{R}_N = \frac{1}{N}\sum_{t=1}^N \|\check{y}(t) - \hat{y}^*(t)\|^2, \tag{17}$$

where $\hat{y}^*(t)$ is the predicted output of the steady-state KF using $u(-\infty : t-1), y(-\infty : t-1)$ and we write $\check{y}(t \mid 1 : t-1)$ as $\check{y}(t)$ for simplicity of notations.

B. OBF-ARX Filter Inspired by KF

For the output prediction problem without relying on explicit system parameters, we derive an OBF-ARX filter inspired by KF. By viewing the KF as a stable LTI system that takes inputs $u(t), y(t)$ to produce the estimated output $\hat{y}^*(t)$, we can approximate the transfer function of KF $\hat{G}(z)$ using the linear combination of a set of scalar GOBFs $V_1(z), \cdots, V_{\check{q}}(z)$ with its inner function as the n_b-th order all-pass filter G_b and the number of bases $\check{q} = qn_b$ without loss of generality. This formulation leads to a variant of the OBF-ARX filter:

$$\begin{aligned}\check{y}(t) &= \left(\sum_{k=1}^{\check{q}} L_k V_k(z)\right)\begin{bmatrix}u(t)\\ y(t)\end{bmatrix} \\ &= \sum_{k=1}^{\check{q}} L_k^u V_k(z)u(t) + \sum_{k=1}^{\check{q}} L_k^y V_k(z)y(t),\end{aligned} \tag{18}$$

where $L_k = [L_k^u \; L_k^y], L_k^u \in \mathbb{C}^{p\times m}, L_k^y \in \mathbb{C}^{p\times p}, V_k(z)u(t) = [V_k(z)u(t)_1, \cdots, V_k(z)u(t)_p]$ with $u(t)_k$ denoting the k-th element of $u(t)$. Let $L = [L_1 \; \cdots \; L_{\check{q}}]$,

$$\check{x}_k(t) = \begin{bmatrix}V_k(z)u(t)\\ V_k(z)y(t)\end{bmatrix}, \check{x}(t) = \begin{bmatrix}\check{x}_1(t)^\top & \cdots & \check{x}_{\check{q}}(t)^\top\end{bmatrix}^\top. \tag{19}$$

To construct the online prediction algorithm, after selecting the OBFs $V_k(z)$ in a manner similar to the procedure in system identification, we aim to solve the following least-squares problem to optimize the coefficients L using online samples:[2]

$$L^* = \arg\min_L \bar{\mathbb{E}}\|\hat{y}^*(t) - L\check{x}(t)\|^2. \tag{20}$$

Since KF provides the optimal output prediction among all linear estimators, it follows that

$$\begin{aligned}L^* = \arg\min_L \{&\bar{\mathbb{E}}\|y(t) - \hat{y}^*(t)\|^2 + \bar{\mathbb{E}}\|\hat{y}^*(t) - L\check{x}(t)\|^2 \\ &+ 2\bar{\mathbb{E}}[(y(t) - \hat{y}^*(t))^\top(\hat{y}^*(t) - L\check{x}(t))]\} \\ = \arg\min_L \; &\bar{\mathbb{E}}\|y(t) - L\check{x}(t)\|^2.\end{aligned} \tag{21}$$

Hence, we can determine L by solving the latter least-squares problem, which has a direct relationship with input and output samples. The online prediction algorithm using the OBF-ARX filter is summarized as:

1) Update the state of the filter for $k = 1, \cdots, \check{q}$ with initial states $\check{x}(0) = \mathbf{0}_{\check{q}(p+m)}, \mathcal{W}(0) = \mathbf{0}_{(p+\check{q}(p+m))\times(p+\check{q}(p+m))}$:[3]

$$\check{x}_k(t) = \begin{bmatrix}(V_k(z)u(t))^\top & (V_k(z)y(t))^\top\end{bmatrix}^\top, \tag{22}$$

$$\check{\mathcal{W}}(t+1) = \frac{t}{t+1}\check{\mathcal{W}}(t) + \frac{1}{t+1}\begin{bmatrix}y(t)\\ \check{x}(t)\end{bmatrix}\begin{bmatrix}y(t)\\ \check{x}(t)\end{bmatrix}^H,$$

and predict the output $\check{y}(t) = L(t)\check{x}(t)$. Since $V_k(z)$ is casual, $\check{x}(t)$ only possesses information of $y(1:t-1)$ and $u(1:t-1)$.

2) With $L(0) = \mathbf{0}_{p\times\check{q}(p+m)}$, update $L(t)$ by solving a least-squares problem if $\check{\mathcal{W}}(t+1)$ is invertible. Otherwise, $L(t) = L(t-1)$:[4]

$$L(t+1) = \text{argmin}_L \begin{bmatrix}I_p & -L\end{bmatrix}\check{\mathcal{W}}(t+1)\begin{bmatrix}I_p & -L\end{bmatrix}^H. \tag{23}$$

Remark 2. *The derived OBF-ARX filter coincides with the predictor introduced in [19], despite being obtained through an entirely different approach. In subsequent discussions, we shall show that the intrinsic connection between* (18) *and the KF facilitates the regret analysis.*

C. Average Regret Analysis

In this subsection, we quantify the average regret of the derived OBF-ARX filter. Let $\check{y}^*(t)$ denote the predicted output of the asymptotic OBF-ARX filter, expressed as:

$$\check{y}^*(t) = \sum_{k=1}^{\check{q}} L_k^* \check{x}_k(t), \tag{24}$$

where $L^* = [L_1^* \; \cdots \; L_{\check{q}}^*]$ is the solution to the following least-squares problem:

$$L^* = \arg\min_L \begin{bmatrix}I_p & -L\end{bmatrix}\bar{\mathbb{E}}\left(\begin{bmatrix}y(t)\\ \check{x}(t)\end{bmatrix}\begin{bmatrix}y(t)\\ \check{x}(t)\end{bmatrix}^H\right)\begin{bmatrix}I_p\\ -L^H\end{bmatrix}. \tag{25}$$

Then, the average regret (15) is upper bounded by

$$\mathcal{R}_N \le \frac{2}{N}\sum_{t=1}^N \|\check{y}(t) - \check{y}^*(t)\|^2 + \frac{2}{N}\sum_{t=1}^N \|\check{y}^*(t) - \hat{y}^*(t)\|^2. \tag{26}$$

We first introduce another assumption that is common in the OBF literature [15]:

Assumption 2 (Persistent excitation)**.** *Assuming that the input $u(t)$ is persistently exciting of a sufficiently high order, combined with the previous assumption that $R \succ 0$, guarantees that $\bar{\mathbb{E}}(\check{x}(t)\check{x}(t)^H)$ and $\check{\mathcal{W}}(N)$ in the algorithm are invertible for sufficiently large N. Additionally, there exists a constant $\underline{\alpha} > 0$, such that $\bar{\mathbb{E}}(\check{x}(t)\check{x}(t)^H) \ge \underline{\alpha} I_m$.*

The following theorem demonstrates that the first term in (26), quantifying the gap between the identified filter and the asymptotic filter, converges to 0 almost surely, and the second term converges to the asymptotic bias of the filter. The proof of the theorem is provided in Appendix I.

Theorem 2 (Almost sure convergence)**.** *Under Assumption 1 and 2, the squared error between the regressed filter and the asymptotic OBF-ARX filter satisfies*

$$\lim_{N\to\infty} \frac{\frac{1}{N}\sum_{t=1}^N \|\check{y}(t) - \check{y}^*(t)\|^2}{N^{-1+\epsilon}} = 0 \; a.s., \tag{27}$$

[2] Notice that the closed-loop true system, the system's KF, and the derived OBF-ARX filter, being stable LTI systems, each converges to their unique steady states over time. As a result, the following limits all exist: $\lim_{t\to\infty}\mathbb{E}[\phi(t)\phi(t)^H], \phi \in \{x, y, \hat{y}^*, \check{x}, \check{y}\}$.

[3] The update of $\check{x}$ can be further written into a recursive form using the state-space realization of the OBFs [15]. The explicit form is omitted due to the space limit.

[4] The least-squares algorithm can be further written into a recursive least-squares form, which is omitted due to the space limit.

for all $\epsilon > 0$. Moreover, the squared error averaged over time between the prediction of KF and that of the asymptotic OBF-ARX filter converges to the asymptotic bias with

$$\lim_{N\to\infty} \frac{\frac{1}{N}\sum_{t=1}^{N}\|e^*(t)\|^2 - \bar{\mathbb{E}}\|e^*(t)\|^2}{N^{-0.5+\epsilon}} = 0 \ a.s., \qquad (28)$$

for all $\epsilon > 0$, where $e^*(t) = \check{y}^*(t) - \hat{y}^*(t)$.

It remains to quantify the asymptotic bias $\tilde{e} \triangleq \lim_{t\to\infty} \mathbb{E}\|\check{y}^*(t) - \hat{y}^*(t)\|^2$. Let the transfer function of (12) be $G(z) = C(zI - A)^{-1}B$. Then, we rewrite (12) as:

$$y(t) = G(z)u(t) + \epsilon(t),$$

where $\epsilon(t)$ is the system noise composed of $v(t), w(t-1), w(t-2), \cdots$. We now introduce a commonly adopted assumption in the OBF literature [15]:

Assumption 3. *The system noise sequence $\epsilon(t)$ is a martingale difference sequence.*

Theorem 3 (Asymptotic bias)**.** *Under Assumption 1 and Assumption 3, there exists a constant $\alpha > 0$ independent of q and n_b, such that the asymptotic bias between the true system's Kalman filter and the OBF-ARX filter satisfies*

$$\lim_{t\to\infty} \mathbb{E}[\|\check{y}^*(t) - \hat{y}^*(t)\|^2] \le \alpha \frac{\tau(\boldsymbol{\lambda},\boldsymbol{\mu})^{(q+1)n_b}}{1-\tau(\boldsymbol{\lambda},\boldsymbol{\mu})^{n_b}}, \qquad (29)$$

where the expression of $\tau(\boldsymbol{\lambda},\boldsymbol{\mu})$ is defined in Theorem 1, $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_n\}$ denotes the eigenvalues of the KF and $\boldsymbol{\mu} = \{\mu_1, \cdots, \mu_{n_b}\}$ are the poles of $G_b(z)$ in GOBF.

The proof of this theorem is neglected due to the space limit. Please refer to our complete version [20] for the detailed proof.

Remark 3. *Theorem 3 reveals that the asymptotic bias of the OBF-ARX filter has an exponential decay rate w.r.t. the number of the OBF bases $\check{q} = qn_b$. Specifically, the decay rate of the asymptotic bias, denoted as $\tau(\boldsymbol{\lambda},\boldsymbol{\mu})$, relies on the pole locations of OBFs and their relationship with the eigenvalues of the KF. This result indicates that* a prior *knowledge of the KF can facilitate the pole selection of the OBF-ARX filter.*

Finally, we synthesize the results in Theorem 2 and Theorem 3 into the following theorem:

Theorem 4 (Average regret)**.** *Under Assumption 1, 2 and 3, there exists a constant $\bar{\alpha} > 0$, such that the average online regret $\mathcal{R}_N$ of the OBF-ARX filter over N time steps satisfies*

$$\mathcal{R}_N \sim \bar{\alpha}\tau(\boldsymbol{\lambda},\boldsymbol{\mu})^{\check{q}} + O(N^{-0.5+\epsilon}) \ a.s. \qquad (30)$$

for all $\epsilon > 0$, where $\check{q}$ denotes the number of GOBF bases and the expression of $\tau(\boldsymbol{\lambda},\boldsymbol{\mu}) < 1$ is defined in Theorem 1.

IV. Simulations

This section provides numerical examples to verify the derived bounds. We consider a heat diffusion process [21] in a $(3 \times 3)\mathrm{m}^2$ square region, with certain obstacles inside, as shown in Fig. 1a. The two small circles center at $(0.75, 2.25)$ and $(2.25, 2.25)$ respectively, with radius $0.1\mathrm{m}$, and the half circle centers at $(1.5, 0.95)$ with radius $0.55\mathrm{m}$.

We denote the temperature at (x, y) and time t as $s(x, y, t)$, and the dynamics of the diffusion process in the square region can be characterized by the following Partial Differential Equation (PDE):

$$\frac{\partial s}{\partial t} = \alpha(x,y)\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}\right), \qquad (31)$$

with the boundary condition

$$s(x, y, t) = 0, \forall (x, y) \in \mathcal{B},$$

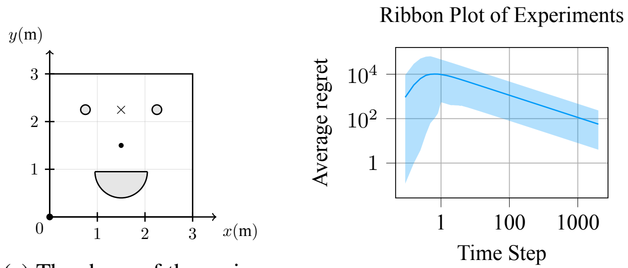
where $\alpha(x, y)$ denotes the diffusion constant at (x, y), and $\mathcal{B}$ denotes the boundary of the region. A heat source in the region is located at $(1.5, 2.25)$, and it is set to the temperature $u_k \sim \mathcal{N}(0, 1)$ at each time step. A sensor is placed at $(1.5, 1.5)$. We then discretize the system in a 10×10 grid into a 100-dimensional LTI system with the following parameters:

- Diffusion constant randomly selected from $\alpha(x, y) \in [0.005, 0.02]$.
- Covariance of the noises: $Q = 0, R = 0.01$.
- Parameters of the discretization:
  - Time step: 0.1s;
  - Space step: 0.3m;
- Total simulation time: 200s.

In our simulations, by injecting i.i.d. Gaussian noise with zero mean and unit covariance as input at each time step and recording the output, we leverage the OBF-ARX algorithm with 10 Laguerre bases for simulations across 100 distinct experiments. Each of these experiments is conducted on a different diffusion process, with its diffusion constant being randomly selected from the interval $[0.005, 0.02]$, to ensure a broad representation of system behaviors. The average regret for each system, defined in (15), is plotted in Fig.1b, which demonstrates the consistency of the filter's regret with the theoretical bounds in Theorem 4. Moreover, the average asymptotic bias over the 100 experiments is 9.27×10^{-5}, which is negligible compared to the overall regret.

V. Conclusion

This paper demonstrates that the OBF-ARX filter accurately approximates the optimal KF in the output prediction of an unknown LTI system by quantifying its online average regret. Specifically, we demonstrate that the average regret of the OBF-ARX filter over N time steps converges to the asymptotic bias at the speed of $O(1/N^{-0.5+\epsilon})$ almost surely for all $\epsilon > 0$. Furthermore, we prove that the bias of the average regret decays exponentially w.r.t. the number of OBF bases. Finally, numerical simulations demonstrate that the average regret of the OBF-ARX filter aligns with the theoretical bounds. Notice that the decreasing rate of the asymptotic bias explicitly depends on the relationship between the poles of the KF and those of the OBF, the pole selection of the OBF-ARX filter leveraging *a priori* knowledge of the KF is left for future work.



(a) The shape of the region considered in the diffusion process. The obstacles are shown in grey, the heat sources are denoted as $\times$, and the sensor is denoted as the black dot.

(b) The average regret of the OBF-ARX filter with 10 Laguerre bases for the heat diffusion process in a log-log ribbon plot. The average asymptotic bias over the 100 experiments is 9.27×10^{-5}.

Fig. 1: The shape of the region considered in the diffusion process and the average regret.

Appendix I
Proof of Theorem 2

For simplicity of notations, for a random vector or matrix $x(t)$, we denote that $x(t) \sim \mathcal{C}_t(\beta)$ if for all $\epsilon > 0$, we have that $x(t) \sim O(t^{\beta+\epsilon})$ $a.s.$ as t tends to infinity, i.e., $\lim_{t\to\infty} \frac{\|x_t\|}{t^{\beta+\epsilon}} = 0$ almost surely. Moreover, define the average function as $\mathcal{S}_N(\phi(t)) = \frac{1}{N}\sum_{t=1}^{N} \phi(t)$.

Define the covariance matrix

$$\mathcal{W}(t) = \left[y(t)^\top \ \hat{x}(t)^\top \ \check{x}(t)^\top\right]^\top \left[y(t)^H \ \hat{x}(t)^H \ \check{x}(t)^H\right],$$

where $\hat{x}(t)$ denotes the state of the KF and $\check{x}(t)$ is defined in (19). The following theorem demonstrates the convergence property of $\mathcal{S}_N(\mathcal{W}(t))$ and $L(N)$. Proofs of this theorem and Theorem 6 are omitted due to the space limit. Please refer to the full version of this paper in [20] for details.

Theorem 5. *Under Assumption 1 and 2, $\mathcal{S}_N(\mathcal{W}(t))$ and $L(N)$ satisfies*

$$\begin{aligned} &\mathcal{S}_N(\mathcal{W}(t)) - \bar{\mathbb{E}}(\mathcal{W}(t)) \sim \mathcal{C}_N(-0.5), \\ &L(N) - L^* \sim \mathcal{C}_N(-0.5). \end{aligned} \tag{32}$$

Combining Theorem 5 with Lemma 3 2) in [22], we can quantify the convergence speed of the second term in the average regret (26) leveraging the fact that $\tilde{A}_{n+\check{n}}(X)$ is differentiable at $\bar{\mathbb{E}}(\mathcal{W}(t))$:

$$\mathcal{S}_N(\|\hat{y}^*(t) - \check{y}^*(t)\|^2) - \bar{\mathbb{E}}\|\hat{y}^*(t) - \check{y}^*(t)\|^2 \sim \mathcal{C}_N(-0.5). \tag{33}$$

Next, we prove a result that directly leads to (27). Consider the following stable linear time-variant system alongside its time-invariant counterpart:

$$\begin{aligned} &x(t+1) = Ax(t) + w(t), \\ &y(t) = C(t)x(t) + v(t), \ y^*(t) = C^* x(t) + v(t), \end{aligned} \tag{34}$$

where $C(t)$ is a time-variant matrix that depends on past noise signals $w(-\infty : t-1), v(-\infty : t-1)$ and $C(t) - C^* \sim \mathcal{C}_t(-\beta), -0.5 \le \beta < 0$. $w(t)$ and $v(t)$ are i.i.d. random signals with zero mean and covariance Q and R respectively. $w(t)$ and $v(t)$ are mutually independent.

Theorem 6. *Suppose $C(t)$ satisfies $C(t) - C^* \sim \mathcal{C}_t(\beta), -0.5 \le \beta < 0$, then*

$$\frac{1}{N}\sum_{t=1}^{N} \|y^*(t) - y(t)\|^2 \sim \mathcal{C}_N(2\beta). \tag{35}$$

Therefore, the theorem is proved combining (26) with (33) and Theorem 6.

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