2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

On Simultaneous Implementability Problem in the Behavioral Framework

Rei Ishii[1] and Osamu Kaneko[1]

Abstract— **The dynamic characteristics of a plant vary with time-related deterioration and external factors. Therefore, it is very useful to know how much change in the dynamic characteristics of the plant can be tolerated. This paper seeks a parametrization of a plant that satisfies the given control specification using the same controller within the behavioral framework. Since this is regarded as the problem whether a single controller can achieve (implement) the given control specification for multiple plants, this problem is referred to as simultaneous implementability problem. As for this problem, we give parametrizations of a plant with respect to both of a controller and a specification. We also consider the stabilization case. Finally, we give some illustrative examples to show the validity of our results.**

I. INTRODUCTION

The dynamic characteristics of a plant vary due to age-related deterioration and external factors. Therefore, in a situation where a plant and control specification are given and a controller that satisfies that control specification is required, it is useful to know how much change in the dynamic characteristics of the plant can be tolerated. In control engineering, obtaining the parametrization of a controller is very useful because it allows the structure of the controller to be expressed precisely to suit its purpose with no excess or deficiency. One particularly well-known parametrization is the Youla-Kucera parametrization [1], [2], which represents a stabilizing controller by using a mathematical model of a plant and solutions of the related Bezout equation. This result was developed in the polynomial ring, and was extended to the stable rational function case in the reference [3]. We can also find that there are many related studies on the parametrization of controllers in the conventional control system theory in which a controller is assumed to be implemented as feedback.

In [5][6], the behavioral approach was proposed to construct systems and control theory from more broader and generalized view point. In this approach, a dynamical system is regarded as the set of trajectories that obey the dynamical property of the system. Thus, a mathematical model used in this approach is a not conventional input (with state) and output relation but a mathematical restriction on variables without the input and output relation. The advantage of the behavioral approach is that it can handle systems that are not proper, so the class of system interconnections to be considered can include interconnections that are not limited to feedback control (e.g. [7], Ex. 2.2). In this approach, the parametrization of a stabilizing controller was proposed in [7] and [8]. Furthermore, a parametrization of a controller that achieves regular implementability was obtained by Praagman et. al. [8]. In the behavioral approach a control is regarded as an interconnection of a plant and a controller to achieve a given specification. As mentioned in the above, it is important to clarify the class of plants which can achieve a given specification, that is, it is also meaningful to obtain the parametrization of a plant that achieve the given specification.

Based on these considerations, this paper seeks a parametrization of a plant that satisfies the control specification using the same controller within the framework of a behavioral approach. We refer to such a problem as *simultaneous implementability*. Particularly, this paper focus on the full interconnection case in which all of the varaibles interacting with an external environment are used in the interconnection with a controller. As for another type of interconnection, the partial interconnection case, we consider the case in which a control specification is given by all of the variables. We also mention the difficulty for the case in which a control specification is given for only manifest variables (to be controlled variables) with respect to the observability.

This paper is organized as follows. In Section 2, we give basic preliminaries of behavioral approach and control as an interconnection of systems. In this section, we also give a brief review on required preliminaries on regular implementability [8]. In Section 3, we present our problem formulation. In Section 4, we present our main result in this paper. Mainly, we present three main theorems. The first theorem provides the parametrization of a plant that yields the same controlled behavior after the interconnection with the same controller. The second theorem restate the first theorem by using a specification instead of controllers. The third theorem presents the parametrozation of a plant that yields a stabilized behavior with the same characteristic stable polynomials. In this section, we also mention the partial interconnection case. In Section 5, we give illustrative examples to show the validities of our ain results. In Section 6, we give concluding remarks.

We introduce the notation using in this paper. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real number and the set of complex number. Let $\mathbb{R}^{p\times q}[\xi]$ denote the set of polynomial matrix of size $p\times q$ consisting of polynomials with real coefficients, where ξ is a variable. The polynomials that all of roots of its are located in open left half plane is called Hurwitz polynomial. Let $\mathbb{R}_H[\xi]$ denote the set of Hurwitz polynomials. For nonsingular

*This work was not supported by any organization

[1]The authors are with The University of Electro-Communications、1-5-1, Chofugaoka Chof, Tokyo, 1828585, Japan i2332006@edu.cc.uec.ac.jp, o.kaneko@uec.ec.jp

matrix $R \in \mathbb{R}^{p\times p}[\xi]$, R is said to be Hurwitz polynomial matrix if $\det(R) \in \mathbb{R}_H[\xi]$. Let $\mathbb{R}_H[\xi]^{p\times p}[\xi]$ denote the set of Hurwitz polynomial matrix of size $p \times p$. For nonsingular matrix $U \in \mathbb{R}^{p\times p}[\xi]$, U is said to be unimodular matrix if $\det(U)$ is in $\mathbb{R}$ except $\{0\}$. Let $\mathbb{R}_U[\xi]^{p\times p}$ denote the set of unimodular matrix of size $p \times p$. Let $C^\infty(\mathbb{R}, \mathbb{R}^q)$ denote the set of infinitely differential functions from $\mathbb{R}$ to $\mathbb{R}^q$.

II. Preliminaries

A. Behavioral Approach

In this paper, we assume that a dyamical system is linear time-invariant system. This section describes the basics of the behavioral approach for analysis and synthesis of linear time invariant dynamical systems based on the references [4][5].

A behavior $\mathscr{P}$ of a system is the set of that systems's trajectories. $\mathscr{P}$ is subspace of $\mathscr{W}^q$, where $\mathscr{W}$ is the signal space of the system, and q is signal number of the system. In this paper, we consider $\mathscr{W}^q := C^\infty(\mathbb{R}, \mathbb{R}^q)$. A linear time-invariant system are represented by multiple differential equations:$R_0 w + R_1 \frac{dw}{dt} + \cdots + R_N \frac{d^N w}{dt^N} = 0$, where $R_i \in \mathbb{R}^{p\times q}$ and $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$. This equation is called kernel representation, also w is called the manifest variable. Now, differential operator $\frac{d}{dt}$ is regarded as variable ξ. So, using polynomial matrix $R(\xi) := R_0 + R_1\xi + \cdots + R_N\xi^N \in \mathbb{R}^{p\times q}[\xi]$, we get $Rw = 0$. Thus, a behavior of linear time-invariant system $\mathscr{P}$ denotes $\mathscr{P} := \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid Rw = 0\} \subset \mathscr{W}^q$. To characterise a behavior $\mathscr{P}$ of a system by kernel representation, we denote $\mathscr{P} = \mathrm{Ker}(R)$. There are many kernel representation of a behavior. Among them, for $\mathscr{P} = \mathrm{Ker}(R)$, the kernel representation is said to be a minimal representation if R has full row rank ([4], Th. 3.6.4). The number of rows in the minimal representation is called the output number of $\mathscr{P}$ and is denoted by $\mathrm{p}(\mathscr{P})$. Next, we describe the controllability of the behavior ([4], Sect. 5.2). $\mathscr{P} = \mathrm{Ker}(R)$ is said to be controllable if there exist $w \in \mathscr{P}$ and $t_1, t_2 \in \mathbb{R}$ such that $w(t) = w_1(t), t < t_1$ and $w(t) = w_2(t), t > t_2$ for any $w_1, w_2 \in \mathscr{P}$. $\mathscr{P} = \mathrm{Ker}(R)$ is controllable if and only if $R(\lambda)$ has full row rank for $^\forall\lambda \in \mathbb{C}$ ([4], Th. 5.2.5). Furthermore, $\mathscr{P} = \mathrm{Ker}(R)$ is stabilizable if and only if $R(\lambda)$ has full row rank for $^\forall\lambda \in \mathbb{C}^+ = \{\lambda \in \mathrm{C} \mid \mathrm{Re}(\lambda) \geq 0\}$ ([4], Th. 5.2.30).

Furthermore, given two behaviors, the following lemma is presented as a necessary and sufficient condition for the two behaviors to be equivalent ([4], Th. 3.6.2, also [5], Prop. 3.3).

Lemma 1: Let $\mathscr{P} = \mathrm{Ker}(R), \mathscr{P}' = \mathrm{Ker}(R')$ and $R, R' \in \mathbb{R}^{p\times q}[\xi]$. $\mathscr{P} = \mathscr{P}'$ if and only if there exists $U \in \mathbb{R}_U^{p\times p}[\xi]$ such that $R = UR'$.

The autonomy and stability of the behaviroal approach can be formalized as follows ([4], Sect. 3.2 and 7.2). $\mathscr{P} = \mathrm{Ker}(R)$ is said to be autonomous if for any $w_1, w_2 \in \mathscr{P}$, $w_1(t) = w_2(t), t \leq 0$ imples $w_1(t) = w_2(t)$ for $t > 0$. $\mathscr{P} = \mathrm{Ker}(R)$ is autonomous if and only if R has full column rank ([5], Prop. 5.7). Thus, when R is full row rank, $\mathscr{P}$ is autonomous if and only if R is square and nonsingular. Assume that $\mathscr{P} = \mathrm{Ker}(R)$ is autonomous, then $\mathscr{P}$ is said to be stable if any $w \in \mathscr{P}$ goes to 0 as $t \to \infty$. A minimal representation $\mathscr{P} = \mathrm{Ker}(R), R \in \mathbb{R}^{q\times q}[\xi]$ is stable if and only if $R \in \mathbb{R}_H^{q\times q}[\xi]$, i.e., $\det(R) \in \mathbb{R}_H[\xi]$ ([4], Th.7.2.2).

B. Control in the behavioral approach

This subsection describes control in the behavioral approach based on references [6], [10]. Particularly, we focus on the full interconnection. In this interconnection, all of the manifest variables are used for the interconnection with a controller. Let $\mathscr{P} = \mathrm{Ker}(R), R \in \mathbb{R}^{p\times q}[\xi]$ denote the behavior of a plant and $\mathscr{C} = \mathrm{Ker}(C), C \in \mathbb{R}^{c\times q}[\xi]$ denote the behavior of the controller. Then, the full interconnection of $\mathscr{P}$ and $\mathscr{C}$ denotes $\mathscr{P} \cap \mathscr{C}$. The kernel representation of the full interconnection $\mathscr{P} \cap \mathscr{C}$ is expressed by the following equation:

$$\begin{pmatrix} R \\ C \end{pmatrix} w = 0. \tag{1}$$

If $\mathrm{p}(\mathscr{P}) + \mathrm{p}(\mathscr{C}) = \mathrm{p}(\mathscr{P} \cap \mathscr{C})$, we say that the full interconnection $\mathscr{P} \cap \mathscr{C}$ is regular ([6], Sect. 7). In other word, if $\mathscr{P} = \mathrm{Ker}(R)$,$\mathscr{C} = \mathrm{Ker}(C)$ are minimal representation, the full interconnection $\mathscr{P} \cap \mathscr{C}$ is regular if and only if the matrix of Eq. (1) has full row rank. Let $\mathscr{K} = \mathrm{Ker}(K), K \in \mathbb{R}^{k\times q}[\xi]$ denote the behavior of a control specification. For given $\mathscr{P}$ and $\mathscr{K}$, we say that $\mathscr{K}$ is implementable if there exists the behavior of a controller $\mathscr{C}$ such that $\mathscr{P} \cap \mathscr{C} = \mathscr{K}$. To use another expression, $\mathscr{C}$ implements $\mathscr{K}$ with respect to $\mathscr{P}$ by full interconnection. Furthermore, we say that $\mathscr{K}$ is regularly implementable if there exists the behavior of a controller $\mathscr{C}$ such that $\mathscr{P} \cap \mathscr{C} = \mathscr{K}$ and $\mathrm{p}(\mathscr{P}) + \mathrm{p}(\mathscr{C}) = \mathrm{p}(\mathscr{P} \cap \mathscr{C})$ [10]. To use another expression, $\mathscr{C}$ regularly implements $\mathscr{K}$ with respect to $\mathscr{P}$ by full interconnection. In the case that R and C have full row rank, and $p + c = q$, $\mathscr{C}$ stabilizes $\mathscr{P}$ if and only if $\det(\begin{pmatrix} R^T & C^T \end{pmatrix}^T) \in \mathbb{R}_H[\xi]$.

In addition, the parametrization of a controller that achieves regular implementability is presented in the following theorem ([8], Th. 11).

Theorem 1: Let a minimal representation of the plant $\mathscr{P} = \mathrm{Ker}(R), R \in \mathbb{R}^{p\times q}[\xi]$. Let a control specification $\mathscr{K} = \mathrm{Ker}(K), K \in \mathbb{R}^{k\times q}[\xi]$ be regularly implementable. The polynomial matrix inducing the minimal representation $\mathscr{C} = \mathrm{Ker}(C), C \in \mathbb{R}^{c\times q}[\xi]$ of a controller that regularly implements $\mathscr{K}$ is parametrized by the following equation using arbitrary $F \in \mathbb{R}^{c\times p}[\xi]$ and arbitrary $U \in \mathbb{R}_U^{c\times c}[\xi]$:

$$C = FR + UWK \tag{2}$$

where W is created by the following procedure:

1) Let :$M \in \mathbb{R}^{q\times(q-p)}[\xi]$ be a matrix where $RM = 0$ and $M(\lambda)$ has full column rank for $^\forall\lambda \in \mathbb{C}$.
2) Let $Q \in \mathbb{R}^{(k-c)\times k}[\xi]$ be a full row rank matrix that satisfies $QKM = 0$.
3) Let $W \in \mathbb{R}^{c\times k}[\xi]$ be a matrix such that $\begin{pmatrix} Q \\ W \end{pmatrix} \in \mathbb{R}_U^{k\times k}[\xi]$.

Furthermore, the parametrization of a stabilizing controller is presented by the following theorem ([8], Th. 12).

Theorem 2: Let a minimal representation of a stabilizable plant $\mathscr{P} = \mathrm{Ker}(R), R \in \mathbb{R}^{p\times q}[\xi]$. Let a minimal representation of a controllable part of $\mathscr{P}$ be $\mathscr{P}_{\mathrm{c}} = \mathrm{Ker}(R_{\mathrm{c}}), R_{\mathrm{c}} \in \mathbb{R}^{p\times q}[\xi]$. The polynomial matrix inducing the controller $\mathscr{C} = \mathrm{Ker}(C), C \in \mathbb{R}^{c\times q}[\xi]$ such that $\mathscr{P} \cap \mathscr{C}$ is regular, autonomous and stable is parametrized by the following equation using arbitrary $F \in \mathbb{R}^{(q-p)\times p}[\xi]$ and arbitrary $D \in \mathbb{R}_H^{(q-p)\times(q-p)}[\xi]$:

$$C = FR + DQ \tag{3}$$

where $Q \in \mathbb{R}^{(q-p)\times q}[\xi]$ is a matrix such that $\begin{pmatrix} R_{\mathrm{c}} \\ Q \end{pmatrix} \in \mathbb{R}_U^{q\times q}[\xi]$.

III. PROBLEM FORMULATION

We formulate the problem we consider here as follows. First, we consider a linear time-invariant plant $\mathscr{P} = \mathrm{Ker}(R), R \in \mathbb{R}^{p\times q}[\xi]$ and a control specification $\mathscr{K} = \mathrm{Ker}(K), K \in \mathbb{R}^{k\times q}[\xi]$. We assume that we obtain the controller $\mathscr{C} = \mathrm{Ker}(C), C \in \mathbb{R}^{c\times q}[\xi]$ that regularly implements $\mathscr{K}$, i.e., the matrix C meets

$$\mathrm{Ker}\begin{pmatrix} R \\ C \end{pmatrix} = \mathrm{Ker}(K) \tag{4}$$

and this is full row rank. Now we consider another plant behavior $\mathscr{P}' = \mathrm{Ker}(R'), R' \in \mathbb{R}^{p\times q}[\xi]$. Then, find the parametrization of $\mathscr{P}'$ that meets

$$\mathrm{Ker}\begin{pmatrix} R' \\ C \end{pmatrix} = \mathrm{Ker}(K) \tag{5}$$

and this is full row rank. Moreover, find the parametrization of the above setting for the case that $\mathscr{P}$ is stabilizable and $\mathscr{K} = \mathrm{Ker}(K), K \in \mathbb{R}^{k\times q}[\xi]$ is autonomous and stable.

IV. MAIN RESULTS

A. Parametrization of a plant that regularly implements $\mathscr{K}$ by using a representation $Cw = 0$

We derive the following parametrization on based previous section.

Theorem 3: Let $\mathscr{P}$, R, $\mathscr{K}$ and K be as in Theorem 1. Assume that a minimal representation of a controller $\mathscr{C} = \mathrm{Ker}(C), C \in \mathbb{R}^{c\times q}[\xi]$ regularly implements the control specification $\mathscr{K}$ w.r.t. $\mathscr{P}$ by full interconnection. Then, the polynomial matrix inducing the minimal representation $\mathscr{P}' = \mathrm{Ker}(R'), R' \in \mathbb{R}^{p\times q}[\xi]$ such that $\mathscr{P}' \cap \mathscr{C} = \mathscr{K}$ is parametrized as follows using an arbitrary $U \in \mathbb{R}_U^{p\times p}[\xi]$ and arbitrary $V \in \mathbb{R}^{p\times c}[\xi]$:

$$R' = UR + VC. \tag{6}$$

Proof: First, sufficiency is shown. Let $R' = UR + VC$ with arbitrary unimodular matrix U and arbitrary polynomial matrix V. In this case, the following calculation can be performed:

$$\begin{pmatrix} R' \\ C \end{pmatrix} = \begin{pmatrix} UR + VC \\ C \end{pmatrix} = \begin{pmatrix} U & V \\ 0 & I \end{pmatrix}\begin{pmatrix} R \\ C \end{pmatrix}$$

Because U is a unimodular matrix, $\mathscr{P}' \cap \mathscr{C} = \mathscr{K}$ by Lemma 1.

Next, necessity is shown. Let $\mathscr{P}' \cap \mathscr{C} = \mathscr{K}'$ for $\mathscr{P}' = \mathrm{Ker}(R')$ and $\mathscr{K}' = \mathrm{Ker}(K')$. As we are considering necessity, $\mathscr{K} = \mathscr{K}'$ for $\mathscr{P} \cap \mathscr{C} = \mathscr{K}$. Therefore, from Lemma 1, there exists a unimodular matrix U' satisfying $K' = U'K$, and the following calculation can be performed:

$$\begin{pmatrix} R' \\ C \end{pmatrix} = K' = U'K = U'\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} R \\ C \end{pmatrix}$$

In the above calculation, U' is divided according to the number of rows of R and C. In this case, we obtain the following two equations:

$$\begin{aligned} R' &= U_{11}R + U_{12}C \\ C &= U_{21}R + U_{22}C \end{aligned} \tag{7}$$

Here, as we assume the regular implementability of $\mathscr{P}$ and $\mathscr{C}$, $\begin{pmatrix} R \\ C \end{pmatrix}$ has full row rank. Noting that the second equation of Eq. (7) can be rewritten as

$$\begin{pmatrix} -U_{21} & I - U_{22} \end{pmatrix}\begin{pmatrix} R \\ C \end{pmatrix} = 0, \tag{8}$$

we see that $U_{21} = 0, U_{22} = I$. Moreover, as U' is a unimodular matrix, $U_{11} = U, U_{12} = V$, where U can be any unimodular matrix and V can be any polynomial matrix. Thus, we obtain $R' = UR + VC$. (Q.E.D.)

B. Parametrization of a plant that regularly implements $\mathscr{K}$ by using a representation $Kw = 0$

Theorem 3 is a controller-dependent parametrization like Eq. (6). Now, from Theorems 1 and 3, we can derive the following parametrization that depends on the control specification.

Theorem 4: Let $\mathscr{P}$, R, $\mathscr{K}$, K and W be as in Theorem 1. Let $\mathscr{C}$ and C be as in Theorem 3. Then, the polynomial matrix inducing the minimal representation $\mathscr{P}' = \mathrm{Ker}(R'), R' \in \mathbb{R}^{p\times q}[\xi]$ such that $\mathscr{P}' \cap \mathscr{C} = \mathscr{K}$ is parametrized as follows using arbitrary $U \in \mathbb{R}_U^{p\times p}[\xi]$, $U' \in \mathbb{R}_U^{c\times c}[\xi]$, $V \in \mathbb{R}^{p\times c}[\xi]$ and $F \in \mathbb{R}^{c\times p}[\xi]$:

$$R' = (U + VF)R + VU'WK. \tag{9}$$

Proof: Necessity follows from the necessity of Theorems 1 and 3.

Here, sufficiency is shown. By assumption, the controller $\mathscr{C} = \mathrm{Ker}(C)$ that regularly implements $\mathscr{K}$ for $\mathscr{P}$ is obtained from Theorem 1 as $C = FR + UWK$ using an arbitrary unimodular matrix U and an arbitrary polynomial matrix F, where W is created as in Theorem 1. Here, $\mathscr{P}' = \mathrm{Ker}(R')$ is calculated as $R' = (U+VF)R + VU'WK$ with arbitrary unimodular matrices U and U', and arbitrary polynomial matrices F and V In this case, the following calculation can be performed:

$$\begin{aligned} \begin{pmatrix} R' \\ C \end{pmatrix} &= \begin{pmatrix} (U+VF)R + VU'WK \\ FR + U'WK \end{pmatrix} \\ &= \begin{pmatrix} U+VF & VU'W \\ F & U'W \end{pmatrix}\begin{pmatrix} R \\ K \end{pmatrix} \\ &= \begin{pmatrix} U & V \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ F & U'W \end{pmatrix}\begin{pmatrix} R \\ K \end{pmatrix} \end{aligned}$$

Meanwhile, the following calculation can be performed:

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} R \\ FR + U'WK \end{pmatrix} = \begin{pmatrix} I & 0 \\ F & U'W \end{pmatrix} \begin{pmatrix} R \\ K \end{pmatrix}$$

From the above calculations, $\mathscr{P}' \cap \mathscr{C} = \mathscr{K}$ is obtained from Lemma 1 (Q.E.D.)

C. Parametrization of a plant that regularly implements stable $\mathscr{K}$

In this section, we discuss the case from the previous section in which the characteristic polynomial of the polynomial matrix inducing the control specification is stable. From Theorems 2 and 3, the following stabilization parametrization can be derived.

Theorem 5: Let $\mathscr{P}$, R, $\mathscr{P}_c$, R_c and Q be as in Theorem 2. Let $\mathscr{K}$, $\mathscr{C}$ and C be as in Theorem 3. Let $\mathscr{K}$ is autonomous and stable. Then, the polynomial matrix inducing the minimal representation $\mathscr{P}' = \mathrm{Ker}(R'), R' \in \mathbb{R}^{p\times q}[\xi]$ such that $\mathscr{P}' \cap \mathscr{C} = \mathscr{K}$ is parametrized by the following equation using arbitrary $U \in \mathbb{R}_U^{p\times p}[\xi]$, $V \in \mathbb{R}^{p\times(q-p)}[\xi]$, $F \in \mathbb{R}^{(q-p)\times p}[\xi]$ and $D \in \mathbb{R}_H^{(q-p)\times(q-p)}[\xi]$:

$$R' = (U + VF)R + VDQ. \tag{10}$$

Proof: Necessity follows from the necessity of Theorems 2 and 3.

Here, sufficiency is shown. Since the controller $\mathscr{C} = \mathrm{Ker}(C)$ that regulary implements autonomous and stable $\mathscr{K}$ w.r.t. $\mathscr{P}$ by full interconnection, using Theorem 2, we get $C = FR + BD$ using an arbitrary polynomial matrix F and arbitrary Hurwitz polynomial matrix D. Let the polynomial matrix inducing $\mathscr{P}' = \mathrm{Ker}(R')$ be $R' = (U+VF)R+VDQ$ using an arbitrary unimodular matrix U, arbitrary polynomial matrices F, V, and an arbitrary Hurwitz polynomial matrix D. Furthermore, F and D of R' and C can be assumed to be the same due to their arbitrariness. Let $R = D_c R_c$ with $D_c \in \mathbb{R}_H^{p\times p}[\xi]$ be the same as the proof of ([8], Th. 12). Then, the following calculation can be performed:

$$\begin{aligned}
\det \begin{pmatrix} R' \\ C \end{pmatrix} &= \det \begin{pmatrix} (U+VF)R + VDQ \\ FR + DQ \end{pmatrix} \\
&= \det \left\{ \begin{pmatrix} (U+VF) & VD \\ F & D \end{pmatrix} \begin{pmatrix} D_c R_c \\ Q \end{pmatrix} \right\} \\
&= \det \left\{ \begin{pmatrix} U & V \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ F & D \end{pmatrix} \begin{pmatrix} D_c & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} R_c \\ Q \end{pmatrix} \right\} \\
&= \det(U) \cdot \det(D) \cdot \det(D_c) \cdot \det \begin{pmatrix} R_c \\ Q \end{pmatrix} \\
&= a \cdot \det(D) \cdot \det(D_c).
\end{aligned}$$

In the above equation, $a \in \mathbb{R}\backslash\{0\}$. According to the statements of this theorem, U and $\begin{pmatrix} R_c^T & Q^T \end{pmatrix}^T$ are unimodular matrices, so $a := \det(U) \cdot \det(\begin{pmatrix} R_c^T & Q^T \end{pmatrix}^T)$. In addition, the following calculation can be performed:

$$\begin{aligned}
\det \begin{pmatrix} R \\ C \end{pmatrix} &= \det \begin{pmatrix} D_c R_c \\ FR + DQ \end{pmatrix} \\
&= \det \left\{ \begin{pmatrix} I & 0 \\ F & D \end{pmatrix} \begin{pmatrix} D_c & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} R_c \\ Q \end{pmatrix} \right\} \\
&= a' \cdot \det(D) \cdot \det(D_c).
\end{aligned}$$

In the above equation, $\det(\begin{pmatrix} R_c^T & Q^T \end{pmatrix}^T) =: a' \in \mathbb{R}\backslash\{0\}$. Thus, we obtain $\mathscr{P}' \cap \mathscr{C} = \mathscr{P} \cap \mathscr{C}$.(Q.E.D.)

D. The partial interconnection case

This paper focuses on only full interconnection. We consider whether the results in this paper can be extend to the partial interconnection case. At first glance, it seems that this extention may be trivial. However, necessity is not obvious.

In the case of the partial interconnection case, the behavior of plant is denoted by $\mathscr{P}_{\mathrm{full}} \subset \mathscr{W}^q \times \mathscr{W}^l$ and induced the following kernel representation:

$$Rw + Mc = 0 \tag{11}$$

where $R \in \mathbb{R}^{p\times q}[\xi], M \in \mathbb{R}^{p\times l}[\xi], w \in C^\infty(\mathbb{R}, \mathbb{R}^q), c \in C^\infty(\mathbb{R}, \mathbb{R}^l)$. c is called control variable which is used to the interconnection.

Let be $\mathscr{P}_{\mathrm{full}} = \mathrm{Ker}\begin{pmatrix} R & M \end{pmatrix}$. In this case, c is observable from w if and only if $M(\lambda)$ has full column rank for $^\forall\lambda \in \mathbb{C}$ ([4], Th. 5.3.3). Here, we recall the elimination ([4], Th. 6.2.6). Let $\mathscr{P}_w$ denote the behavior that eliminates c from $\mathscr{P}_{\mathrm{full}}$. We choose unimodular matrix U such that $UM = \mathrm{col}\begin{pmatrix} M_1 & 0 \end{pmatrix}$ with M_1 full row rnak. Now, we can get $UR = \mathrm{col}\begin{pmatrix} R_1 & R_2 \end{pmatrix}$. Then, we have $\mathscr{P}_w = \mathrm{Ker}(R_2)$.

$\mathscr{C} = \mathrm{Ker}\begin{pmatrix} 0 & C \end{pmatrix}, C \in \mathbb{R}^{c\times l}[\xi]$ denote the behavior of controller. Considering the interconnection of $\mathscr{P}_{\mathrm{full}}$ and $\mathscr{C}$, i.e. $\mathscr{P}_{\mathrm{full}} \cap \mathscr{C}$, the kernel representation of this interconnetion describe as the following equation:

$$\begin{pmatrix} R & M \\ 0 & C \end{pmatrix} \begin{pmatrix} w \\ c \end{pmatrix} = 0 \tag{12}$$

This interconnection is called the partial interconnection. If $\mathrm{p}(\mathscr{P}_{\mathrm{full}}) + \mathrm{p}(\mathscr{C}) = \mathrm{p}(\mathscr{P}_{\mathrm{full}} \cap \mathscr{C})$, we say that the partial interconnection of $\mathscr{P}_{\mathrm{full}}$ and $\mathscr{C}$ is regular. We denote $\mathscr{K}_{\mathrm{full}}$ and $(\mathscr{K}_{\mathrm{full}})_w$ full control specification and control specification. For given $\mathscr{P}_{\mathrm{full}}$ and $(\mathscr{K}_{\mathrm{full}})_w$, we say that $(\mathscr{K}_{\mathrm{full}})_w$ is implementable if there exists $\mathscr{C}$ such that $(\mathscr{P}_{\mathrm{full}} \cap \mathscr{C})_w = (\mathscr{K}_{\mathrm{full}})_w$. Necessary and sufficient condition for implementability of $(\mathscr{K}_{\mathrm{full}})_w$ was obtained in ([9], Th. 1), furthermore, in the case of general system, was obtained in [15]. Also, we say that $(\mathscr{K}_{\mathrm{full}})_w$ is regulary implementable if there exists $\mathscr{C}$ such that $(\mathscr{P}_{\mathrm{full}} \cap \mathscr{C})_w = (\mathscr{K}_{\mathrm{full}})_w$ and $\mathrm{p}(\mathscr{P}_{\mathrm{full}}) + \mathrm{p}(\mathscr{C}) = \mathrm{p}(\mathscr{P}_{\mathrm{full}} \cap \mathscr{C})$ [10]. A necessary and sufficient condition for regularly implementability of $(\mathscr{K}_{\mathrm{full}})_w$ was obtained in ([10], Th. 4). Here, we give the following theorem.

Theorem 6: Let a minimal representation of the plant $\mathscr{P}_{\mathrm{full}} = \mathrm{Ker}\begin{pmatrix} R & M \end{pmatrix}, R \in \mathbb{R}^{p\times q}[\xi], M \in \mathbb{R}^{p\times l}[\xi]$ and a specification for the full control $\mathscr{K}_{\mathrm{full}}$. Assume that we obtain a minimal representation $\mathscr{C} = \mathrm{Ker}\begin{pmatrix} 0 & C \end{pmatrix}, C \in \mathbb{R}^{c\times l}[\xi]$ such that $\mathscr{P}_{\mathrm{full}} \cap \mathscr{C} = \mathscr{K}_{\mathrm{full}}$ and $\mathrm{p}(\mathscr{P}_{\mathrm{full}}) + \mathrm{p}(\mathscr{C}) = \mathrm{p}(\mathscr{P}_{\mathrm{full}} \cap \mathscr{C})$. Assume that R has full row rank. Then, the polynomial matrix inducing the minimal representation $\mathscr{P}'_{\mathrm{full}} = \mathrm{Ker}\begin{pmatrix} R' & M' \end{pmatrix}, R' \in \mathbb{R}^{p\times q}[\xi], M' \in \mathbb{R}^{p\times l}[\xi]$ such that $\mathscr{P}'_{\mathrm{full}} \cap \mathscr{C} = \mathscr{K}_{\mathrm{full}}$ is parametrized as follows using arbitrary $U \in \mathbb{R}_U^{p\times p}[\xi]$ and $V \in \mathbb{R}^{p\times c}[\xi]$:

$$R' = UR, \; M' = UM + VC \tag{13}$$

Proof: Frist, sufficiency is shown. Let be $R' = UR, M' = UM + VC$ with arbitrary unimodular matrix U and arbitrary polynomial matrix V. In this case, the following calculation can be performed:

$$\begin{pmatrix} R' & M' \\ 0 & C \end{pmatrix} = \begin{pmatrix} UR & UM+VC \\ 0 & C \end{pmatrix} = \begin{pmatrix} U & V \\ 0 & I \end{pmatrix} \begin{pmatrix} R & M \\ 0 & C \end{pmatrix}$$

Because U is unimodular matrix, $\mathscr{P}'_{\text{full}} \cap \mathscr{C} = \mathscr{K}_{\text{full}}$ by Lemma 1.

Next, necessity is shown. Assume that $\mathscr{P}'_{\text{full}} \cap \mathscr{C} = \mathscr{K}_{\text{full}}$. So, we can get $\mathscr{P}_{\text{full}} \cap \mathscr{C} = \mathscr{P}'_{\text{full}} \cap \mathscr{C}$. By Lemma 1, there exists a unimodular matrix U' that satisfies the following equation:

$$\begin{pmatrix} R' & M' \\ 0 & C \end{pmatrix} = U' \begin{pmatrix} R & M \\ 0 & C \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} R & M \\ 0 & C \end{pmatrix}$$

In the above calculation, U' is divided according to the size of R, M and C. Then, we obtain the following four equations:

$$\begin{gathered} R' = U_{11}R, 0 = U_{21}R \\ M' = U_{11}M + U_{12}C, C = U_{21}M + U_{22}C \end{gathered} \tag{14}$$

Second equation of Eq. (14) is solved by $U_{21} = 0$ because R have full row rank. Then, forth equation iof Eq. (14) is $C = U_{22}C$. So, we can obtain $U_{22} = I$. Moreover, as U' is a unimodular matrix, $U_{11} = U, U_{22} = V$, where U can be any unimodular matrix and V can be any polynomial matrix. Thus, we obtain $R' = UR, M' = UM + VC$.(Q.E.D.)

Given two behaviors $\mathscr{K}_{\text{full}}, \mathscr{K}'_{\text{full}} \subset \mathscr{P}_{\text{full}}$, it is obviously that $(\mathscr{K}_{\text{full}})_w = (\mathscr{K}'_{\text{full}})_w$ if $\mathscr{K}_{\text{full}} = \mathscr{K}'_{\text{full}}$. So, by using theorem 6 we can say that $(\mathscr{P}'_{\text{full}} \cap \mathscr{C})_w = (\mathscr{K}_{\text{full}})_w = (\mathscr{P}_{\text{full}} \cap \mathscr{C})_w$ if the polynomial matrix inducing the minimal representation $\mathscr{P}'_{\text{full}}$ is represented by $R' = UR, M' = UM + VC$ where U is arbitrary unimodular matrix and V is arbitrary polynomial matrix. In order to say $(\mathscr{K}_{\text{full}})_w = (\mathscr{K}'_{\text{full}})_w$ only if $\mathscr{K}_{\text{full}} = \mathscr{K}'_{\text{full}}$, we need the condition that c is obsevable from w in $\mathscr{P}_{\text{full}}$ and $\mathscr{P}'_{\text{full}}$ ([8], Lem.18). However, it should be noted that c is not necessarily observable from w in $\mathscr{P}'_{\text{full}}$, even if c is observable from w in $\mathscr{P}_{\text{full}}$.

V. EXAMPLES

First, we consider a numerical example for Theorem 4. Let the polynomial matrix inducing the plant behavior $\mathscr{P} = \text{Ker}(R)$ be

$$R = \begin{pmatrix} 1 & \xi - 1 \end{pmatrix} \tag{15}$$

Let the polynomial matrix inducing the control specification $\mathscr{K} = \text{Ker}(K)$ be

$$K = \begin{pmatrix} 2 & \xi - 3 \\ 3 & \xi - 5 \end{pmatrix}. \tag{16}$$

Now, we parametrize the polynomial matrix inducing the plant behavior $\mathscr{P}' = \text{Ker}(R')$ such that $\mathscr{P}' \cap \mathscr{C} = \mathscr{K}$. First, we find W using Theorem 1. So, we can get

$$M = \begin{pmatrix} \xi - 1 \\ -1 \end{pmatrix}. \tag{17}$$

Moreover, we find the polynomial matrix Q satisfying next equation and have full row rank.

$$QKM = Q \begin{pmatrix} 2 & \xi - 3 \\ 3 & \xi - 5 \end{pmatrix} \begin{pmatrix} \xi - 1 \\ -1 \end{pmatrix} = Q \begin{pmatrix} \xi + 1 \\ 2(\xi + 1) \end{pmatrix} = 0. \tag{18}$$

We can get $Q = \begin{pmatrix} -2 & 1 \end{pmatrix}$. Next, we find W such that $\begin{pmatrix} Q^T & W^T \end{pmatrix}^T$ is unimodular. We can get

$$W = \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{19}$$

Now, using Theorem 4, we can get the polynomial matrix

$$\begin{aligned} R' &= (U + VF)R + U'VWK \\ &= \begin{pmatrix} (U + VF) - VU' & (U + VF)(\xi - 1) + 2VU' \end{pmatrix}, \end{aligned} \tag{20}$$

where U and U' be any real number other than 0, F, V be any polynomial. For example, choosing $U = 1, U' = 1, F = \xi, V = \xi$, we get

$$R' = \begin{pmatrix} \xi^2 - \xi + 1 & \xi^3 - \xi^2 + 3\xi - 1 \end{pmatrix}. \tag{21}$$

Also, we can get the polynomial matrix inducing the regular implementability controller $\mathscr{C} = \text{Ker}(C)$ as

$$C = FR + U'WK = \begin{pmatrix} \xi - 1 & \xi^2 - \xi + 2 \end{pmatrix} \tag{22}$$

In fact, the determinant of the kernel representation of the interconnection is calculated as follows:

$$\begin{aligned} &\det \begin{pmatrix} R \\ C \end{pmatrix} = \det \begin{pmatrix} 1 & \xi - 1 \\ \xi - 1 & \xi^2 - \xi + 2 \end{pmatrix} \\ &= (\xi^2 - \xi + 2) - (\xi^2 - 2\xi + 1) = \xi + 1, \end{aligned}$$

$$\begin{aligned} &\det \begin{pmatrix} R' \\ C \end{pmatrix} = \det \begin{pmatrix} \xi^2 - \xi + 1 & \xi^3 - \xi^2 + 3\xi - 1 \\ \xi - 1 & \xi^2 - \xi + 2 \end{pmatrix} \\ &= (\xi^4 - 2\xi^3 + 4\xi^2 - 3\xi + 2) - (\xi^4 - 2\xi^3 + 4\xi^2 - 4\xi + 1) \\ &= \xi + 1. \end{aligned}$$

From this, we see that the characteristic polymomials of both of the interconnected systems are the same.

Next, we consider a numerical example for theorem 6. Let the polynomial matrix inducing the full plant behavior $\mathscr{P}_{\text{full}} = \text{Ker} \begin{pmatrix} R & M \end{pmatrix}$ be

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M = \begin{pmatrix} 1 & \xi - 1 \\ \xi & 0 \end{pmatrix}. \tag{23}$$

Let the polynomial matrix inducing the controller behavior $\mathscr{C} = \text{Ker} \begin{pmatrix} 0 & C \end{pmatrix}$ be

$$C = \begin{pmatrix} 2\xi + 1 & \xi^2 - 1 \\ \xi(\xi + 1) & 0 \end{pmatrix}. \tag{24}$$

Then, the polynomial matrix inducing the full control specification $\mathscr{P}_{\text{full}} \cap \mathscr{C} = \mathscr{K}_{\text{full}} = \text{Ker}(K_f)$ be

$$K_f = \begin{pmatrix} 1 & 0 & 1 & \xi - 1 \\ 0 & 1 & \xi & 0 \\ 0 & 0 & 2\xi + 1 & \xi^2 - 1 \\ 0 & 0 & \xi(\xi + 1) & 0 \end{pmatrix}. \tag{25}$$

Then, premultiplying the matrix of Eq. (25) by appropriate unimodular matrix, we get the kernel representation of $(\mathscr{K}_{\text{full}})_w$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi+1 & 1 & -1 & 0 \\ 0 & \xi+1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \xi-1 \\ 0 & 1 & \xi & 0 \\ 0 & 0 & 2\xi+1 & \xi^2-1 \\ 0 & 0 & \xi(\xi+1) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & \xi+1 \\ 0 & 1 & \xi & 0 \\ \xi+1 & 1 & 0 & 0 \\ 0 & \xi+1 & 0 & 0 \end{pmatrix}. \tag{26}$$

From Eq. (26), we can get $(\mathscr{K}_{\text{full}})_w = \text{Ker}(K)$:

$$K = \begin{pmatrix} \xi+1 & 1 \\ 0 & \xi+1 \end{pmatrix}. \tag{27}$$

Now, using Theorem 6, we can get the polynomial matrices of $\mathscr{P}'_{\text{full}} = \text{Ker}\begin{pmatrix} R' & M' \end{pmatrix}$:

$$\begin{aligned} R' &= U \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ M' &= U \begin{pmatrix} 1 & \xi-1 \\ \xi & 0 \end{pmatrix} + V \begin{pmatrix} 2\xi+1 & \xi^2-1 \\ \xi(\xi+1) & 0 \end{pmatrix}, \end{aligned} \tag{28}$$

where arbitrary $U \in \mathbb{R}_U^{2\times 2}[\xi]$ and $V \in \mathbb{R}^{2\times 2}[\xi]$. For example, we choose $U = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$. So, we can get:

$$R' = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}, M' = \begin{pmatrix} 2(\xi+1) & (\xi-1)(\xi+2) \\ \xi(\xi^2+\xi+2) & \xi(\xi-1) \end{pmatrix}. \tag{29}$$

Then, the polynomial matrix inducing the full control specification $\mathscr{P}'_{\text{full}} \cap \mathscr{C} = \mathscr{K}'_{\text{full}} = \text{Ker}(K'_f)$ be:

$$K'_f = \begin{pmatrix} 1 & 0 & 2(\xi+1) & (\xi-1)(\xi+2) \\ \xi & 1 & \xi(\xi^2+\xi+2) & \xi(\xi-1) \\ 0 & 0 & 2\xi+1 & (\xi-1)(\xi+1) \\ 0 & 0 & \xi(\xi+1) & 0 \end{pmatrix} \tag{30}$$

Then, premultiplying the matrix of Eq. (30) by appropriate unimodular matrix, we get the kernel representation of $(\mathscr{K}'_{\text{full}})_w$:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ -\xi & 1 & \xi & -\xi \\ 1 & 1 & -2 & -\xi \\ -\xi(\xi+1) & \xi+1 & \xi(\xi+1) & -(\xi^2+\xi+1) \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 2(\xi+1) & (\xi-1)(\xi+2) \\ \xi & 1 & \xi(\xi^2+\xi+2) & \xi(\xi-1) \\ 0 & 0 & 2\xi+1 & (\xi-1)(\xi+1) \\ 0 & 0 & \xi(\xi+1) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & \xi-1 \\ 0 & 1 & \xi & 0 \\ \xi+1 & 1 & 0 & 0 \\ 0 & \xi+1 & 0 & 0 \end{pmatrix}. \tag{31}$$

Thus, from Eq. (27) and Eq. (31), we can get $(\mathscr{P}_{\text{full}} \cap \mathscr{C})_w = (\mathscr{P}'_{\text{full}} \cap \mathscr{C})_w$.

VI. CONCLUSIONS

In this paper, in a situation where a plant and control specification are given and a controller that satisfies that control specification is obtained, we found a parametrization of the plant that satisfies the same control specification without changing the controller. Future challenges include seeking parametrizations with higher generality without considering regularity. Also, in partial interconnection case, we introduced only sufficient condition. So, the issue of considering sufficient and necessary condition with respect to partial interconnection case remains. Another future direction is to clarify the relationship with simultaneous stabilization problem, which is known as one of the unsolved problems in the control theory. Finding conditions that single controller stabilizes multiple plants is simultaneous stabilization problem[12][13][14]. In [12][13][14], study simultaneous stabilization problem using stable proper functions. Also, in [11], study simultaneous stabilization problem using behavior approach, i.e., polynomial theory.

REFERENCES

[1] D. C. Youla, H. A. Jabr and J. J. Bongiorno: Modern Wiener-Hopf design of optimal controllers, Part2: The multivariable case; *IEEE Transactions on Automatic Control*, Vol. 21, pp. 319-338 (1976)

[2] V. Kucera, Discrete Linear Control : Polynomial Equation Approach, John Willey and Sons Ltd., (1979)

[3] M. Vidyasagar, Cotrol System Synthesis : Factozation Approach, MIT Press (1985)

[4] J. W. Polderman and J. C. Willems: Introduction to mathematical systems theory: a behavioral approach, Vol. 26, Springer (1997)

[5] J. C. Willems, Paradigms and Puzzles in the Theory of Dynamical Systems, *IEEE Transactions on Automatic Control*, Vol. 36, No.3, pp.259-294 (1991)

[6] J. C. Willems: On Interconnections, Control and Feedback; *IEEE Transactions on Automatic Control*, Vol. 42, pp. 326-339 (1997)

[7] M. Kujiper: Why do stabilizing controllers stabilize?; *Automatica*, Vol. 31, No. 4, pp. 621-625 (1995)

[8] C. Praagman, H. L. Trentelman and R. Zavala Yoe: On the parametrization of all regularly implementing and stabilizing controllers; *SIAM Journal on Control and Optimization*, Vol. 45, No. 6, pp. 2035-2054 (2007)

[9] J. C. Willems and H. L. Trentelman, Syntheis of Dissiptive Systems using Quadrtic Differential Forms: Part 1, *IEEE Transaction on Automatic Control*, Vol. 47, No. 1, pp. 53-69 (2002)

[10] M. N. Belur and H. L. Trentelman, Stabilization, pole placement and regular implementability; *IEEE Trans. Aut. Control*,Vol. 47, No. 5,pp. 735-744 (2002)

[11] O. Kaneko, Behavioral Approach to Simultaneous Stabilization, Mathematical Control Theory 2, Behavioral Systems and Robust Control, Springer, pp. 65-80, ISBN 978-3-319-21002-5 (2015)

[12] H. Maeda and M. Vidyasagar, Some results on simultaneous stabilization, *Systems and Control Letters*, Vol.5, pp.205-208 (1984)

[13] V. D. Blondel, *Simultaneous stabilization of linear systems*, Springer (1994)

[14] M. Vidyasagar and N. Viswanadham, Algebraic design techniques for reliable stabilization, *IEEE. Transactions on Automatic Control*, vol. 27, no. 5, pp. 1085–1095 (1982).

[15] A. J. van der Schaft, Achivable behavior of general system, *Systems and Control Letters*, Vol.49, pp.141-149 (2003)