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Capturability of the Pulsed Guidance Law Based on Differential Game Theory*

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Abstract— **The capturability analysis of the pulsed guidance law using differential game theory is presented in this paper. The engagement of an interceptor with pulsed guidance constraint and a maneuvering target with bounded acceleration is considered. The differential game guidance laws for both the interceptor and the target are proposed. Then the engagement is converted into four different cases according to the guidance law parameters, and the capture boundary conditions for these four cases are given as functions of the guidance law parameters and the system parameters, including acceleration constraints, the acceptable miss distance and initial values of the engagement. Afterwards, the capture zone is given according to the boundary conditions and is decided by the guidance law parameters and the system parameters. Finally, various specific examples show that the interception can be guaranteed with lower acceleration constraint ratio and higher guidance law parameter ratio.**

I. Introduction

Various guidance laws have been studied extensively, such as the classical proportional navigation guidance law, the sliding mode guidance law, the optimal guidance law and the differential guidance law[1], [2]. The capturability analysis has also drawn great attention, aiming at evaluating guidance laws through theoretical analysis. Ghose *et al.*[3] proposed a 3D proportional navigation guidance law with a negative navigation constant to intercept the non-maneuvering target with higher speed and analyzed its nontrivial capture zone. Guaranteed capture zones of augmented pure proportional navigation were given and compared with classical proportional navigation guidance law in [4]. Mukherjee and Ghose[5] used a cyclic scheme to ensure global reachability against a maneuvering target. Li *et al.*[6] analyzed the capturability of 3D realistic true proportional navigation guidance law. Performance of pure proportional navigation guidance law was analyzed in [7] for the maneuvering target with higher speed and was later given in [8] considering the arbitrarily maneuvering target. The capturability of finite-time sliding mode guidance law was addressed in [9].

Differential game theory considers strategies of both the interceptor and the target; thus, it is widely used in the deduction and analysis of guidance laws[10]. Turetsky and Shinar[11] analyzed the capture zone of differential game guidance law with bounded controls. Rubinsky and Gutman utilized differential game theory to perform parameter analysis in [12], proposed guidance laws with different time-to-go estimation in [13], and further discussed the impact of time-to-go estimation in [14]. A linear quadratic guidance law with a terminal body angle constraint was presented in [15] and a combined linear–quadratic/bounded control differential game guidance law was given in [16]. Hayoun and Shima[17] analyzed the relationship between the independent capture zone of a pursuer and the joint capture zone of a pursuing team. Qi *et al.*[18], [19] addressed evasion and pursuit game among a target, a defender and an attacker. Liu *et al.*[20], [21] proposed differential game guidance laws for multiple attackers. Liang *et al.*[22] presented winning regions for cooperative target defense game.

Nevertheless, most of guidance laws were proposed and analyzed with continuous guidance command, while only a few papers designed guidance laws for pulsed thrusters. Yeh[23] designed a 3D sliding mode guidance law for the pulse-type system. Zhang *et al.*[24] designed a predictive pulsed guidance law considering the measurement noise. Yu *et al.*[25] developed the optimal terminal guidance law considering the final velocity vector constraint and transformed it into a pulsed command. However, these papers only designed pulsed guidance laws but did not analyze their capturability.

Motivated by the above mentioned, this paper utilizes differential game theory to analyze the capturability of pulsed guidance law. Specifically, the engagement of an interceptor with pulsed guidance constraint and a maneuvering target with bounded acceleration is considered and differential guidance laws for both the interceptor and the target are proposed. Then

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the engagement is converted to four different cases according to guidance law parameters, and capture boundary conditions for these four cases are given. Afterwards, capture zones are presented and the impact of system parameters on capture zones is shown through various examples. The main contributions of this paper are as follows.

a) Contributions to the capturability analysis: The engagement of an interceptor with the pulsed guidance constraint and a maneuvering target with bounded acceleration is considered, extending the capturability analysis with the capture conditions for pulsed guidance laws.

b) Contributions to the pulsed guidance law: Capturability analysis is proposed for the pulsed guidance law, revealing the impact of the guidance law parameters and the system parameters on the interception condition. Capture conditions obtained in this paper provides a theoretical reference for the design and parameter selection of pulsed guidance laws.

The remainder of this paper is organized as follows. Section 2 describes the engagement model and designs the differential guidance laws. The capture boundary conditions are given in Section 3. Section 4 presents the capture zones and the impact of system parameters. Section 5 concludes the paper.

II. Guidance Law Design

In this section, the engagement model of an interceptor and a maneuvering target in the two-dimensional space is presented and reduced to a scalar one. Afterwards, the differential game guidance laws are proposed based on the engagement model.

A. Engagement Geometry

The planar interception model of an interceptor with pulsed guidance constraint and a maneuvering target with bounded acceleration is established under the following assumptions[11]:

a) The interceptor dynamics and the target dynamics are expressed by first-order transfer functions.

b) The interceptor acceleration is pulsed and perpendicular to the interceptor velocity, and the interceptor velocity is constant.

c) The target acceleration is saturated and perpendicular to the target velocity, and the target velocity is constant.

d) The line-of-sight angle and front angles are sufficiently small angle, i.e., the engagement model can be linearized along the initial line-of-sight.

Concerning above assumptions, the interception geometry is shown in Figure 1. (x_T, y_T) and (x_M, y_M) are the positions of the target and the interceptor. v_T, a_T, ϕ_T and v_M, a_M, ϕ_M are the velocities, the accelerations and the front angles of the target and the interceptor respectively.

Denote the initial positions as (x_{T0}, y_{T0}) and (x_{M0}, y_{M0}). t_0 is the initial time. Then the final time

$$t_f = t_0 + x_{R0}/v_R \tag{1}$$

where $x_{R0} = x_{T0} - x_{M0}, v_R = v_T + v_M$.

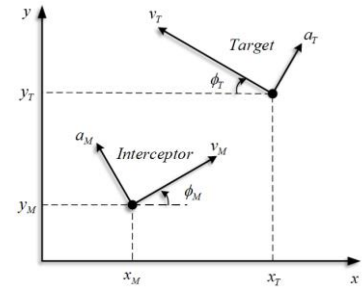


Figure 1. Engagement geometry

Considering the assumptions above and the engagement geometry in Figure 1, the interception dynamics is shown as follows[11]

$$\begin{aligned} &\dot{x}_1 = x_2, \quad x_1(t_0) = 0, \quad t_0 \le t \le t_f, \\ &\dot{x}_2 = x_3 - x_4, \;\; x_2(t_0) = v_T\phi_{T,0} - v_M\phi_{M,0}, \\ &\dot{x}_3 = (a_{T,\max}u_T - x_3)\tau_T, \quad x_3(t_0) = 0, \\ &\dot{x}_4 = (a_{M,\max}u_M - x_4)\tau_M, \quad x_4(t_0) = 0, \end{aligned} \tag{2}$$

where $x_1 = y_T - y_M$ denotes the distance between the target and interceptor normal to the initial line of sight. $\phi_{T0}, a_{T,\max}, \tau_T$ and $\phi_{M0}, a_{M,\max}, \tau_M$ are the initial front angles, the maximum accelerations, the guidance commands and the time constants of the target and the interceptor. u_T and u_M are the target guidance command and the interceptor guidance command, satisfying

$$u_M \in \{\pm 1, 0\}, |u_T| \le 1 \tag{3}$$

In order to reduce the system order, denote zero-effort miss distance (ZEM) as

$$Z(t) = \boldsymbol{D\Phi}(t_f, t)\boldsymbol{x} \tag{4}$$

where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^{\mathrm{T}}, \boldsymbol{D} = [1,0,0,0]$. $\boldsymbol{\Phi}(t_f, t)$ is the transition matrix of (2).

Therefore, the derivative of ZEM is given as[1],[11]

$$\dot{Z}(t) = h_T(t)u_T(t) - h_M(t)u_M(t) \tag{5}$$

where

$$h_T(t) = \tau_T\left(e^{-(t_f - t)/\tau_T} + (t_f - t)/\tau_T - 1\right)a_{T,\max} \tag{6}$$

$$h_M(t) = \tau_M\left(e^{-(t_f - t)/\tau_M} + (t_f - t)/\tau_M - 1\right)a_{M,\max} \tag{7}$$

The derivatives of h_T and h_M are non-positive. Therefore, h_T and h_M decrease monotonically, satisfying

$$h_T > 0, h_M > 0, \forall t < t_f \tag{8}$$

Denote ZEM at the final time as $Z(t_f) = Z_f$. The capture boundary constraint is

$$|Z_f| \le Z_m \tag{9}$$

where Z_m is the acceptable miss distance.

B. Guidance Laws

In this engagement, the purpose of the target is to minimize its fuel consumption, maximize the interceptor's fuel consumption and maximize ZEM, while the aim of the interceptor's aim is opposite. Then the cost function is

$$J = \frac{1}{2}\left[Z\left(t_f\right)\right]^2 + \frac{1}{2}\alpha\int_{t_0}^{t_f} u_M^2\left(t\right)dt - \frac{1}{2}\beta\int_{t_0}^{t_f} u_T^2\left(t\right)dt \quad (10)$$

where $\alpha > 0, \beta > 0$.

The problem then transfers to designing u_T and u_M under constraints (3) to realize

$$\min_{u_m}\max_{u_T} J \quad (11)$$

Considering the system state (5), the Hamilton function corresponding with (10) is

$$\begin{aligned} H &= \frac{1}{2}\alpha u_M^2 - \frac{1}{2}\beta u_T^2 + \lambda\left(h_T u_T - h_M u_M\right) \\ &= H_M + H_T \end{aligned} \quad (12)$$

where λ is the Lagrange multiplier, and

$$H_M = \frac{1}{2}\alpha u_M^2 - \lambda h_M u_M \quad (13)$$

$$H_T = -\frac{1}{2}\beta u_T^2 + \lambda h_T u_T \quad (14)$$

Then, the problem further converts from (11) to

$$\min_{u_M}\max_{u_T} H = \min_{u_M} H_M + \max_{u_T} H_T \quad (15)$$

The adjoint equation is

$$\dot{\lambda} = -\frac{\partial H}{\partial Z} = 0, \lambda\left(t_f\right) = Z\left(t_f\right) \quad (16)$$

From (16), the Lagrange multiplier can be written as

$$\lambda\left(t\right) = Z_f \quad (17)$$

For the inceptor, combining the constraint (3) and the Hamilton function (13) yields

$$H_M = \begin{cases} 0 & , u_M = 0 \\ 0.5\alpha - \lambda h_M , u_M = 1 \\ 0.5\alpha + \lambda h_M , u_M = -1 \end{cases} \quad (18)$$

To design u_M to minimize (18), the optimal guidance command for the interceptor is

$$u_M^* = \arg\min_{u_M} H_M = \begin{cases} 0 & , \left|\lambda h_M\right| \le \alpha/2 \\ \operatorname{sign}\left(\lambda h_M\right) & , \left|\lambda h_M\right| > \alpha/2 \end{cases} \quad (19)$$

Substituting (8) and (17) into (19) yields

$$u_M^* = \begin{cases} 0 & , h_M \le \alpha/\left(2\left|Z_f\right|\right) \\ \operatorname{sign}\left(Z_f\right), & h_M > \alpha/\left(2\left|Z_f\right|\right) \end{cases} \quad (20)$$

Similarly, for the target, the first order and second order derivatives of Hamilton function (14) are

$$\frac{\partial H_T}{\partial u_T} = -\beta u_T + \lambda h_T, \frac{\partial^2 H_T}{\partial u_T^2} = -\beta < 0 \quad (21)$$

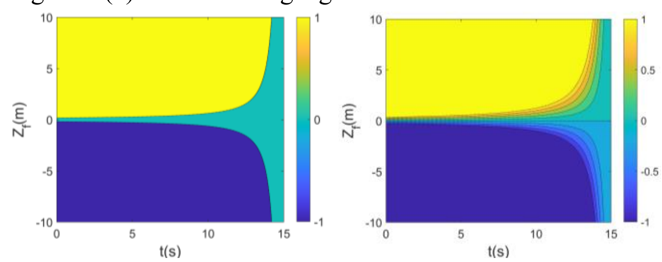
Therefore, the optimal guidance command for the target can be obtained by combining (3), (17) and (21), given as

$$u_T^* = \arg\max_{u_T} H_T = \min\left\{\frac{Z_f h_T}{\beta}, 1\right\} \quad (22)$$

Considering that $h_T > 0$, (22) can be written as

$$u_T^* = \begin{cases} Z_f h_T/\beta & , h_T \le \beta/\left|Z_f\right| \\ \operatorname{sign}\left(Z_f\right), & h_T > \beta/\left|Z_f\right| \end{cases} \quad (23)$$

Considering the engagement dynamics as (2) and guidance command constraints as (3), the differential guidance law can be designed as (20) for the interceptor and (23) for the target. Choose the time constants $\tau_M = 1s, \tau_T = 1s$, the maximum accelerations $a_M = 1, a_T = 1$, the guidance law parameters $\alpha = 5, \beta = 5$ and the time duration $t_f - t_0 = 15$ as an example. The corresponding differential guidance laws with respect to time t and the terminal ZEM Z_f are shown in Figure 2, where Figure 2 (a) shows the interceptor guidance law and Figure 2 (b) shows the target guidance law.



(a) Interceptor guidance law (b) Target guidance law

Figure 2. Differential guidance laws

Figure 2 demonstrates that the interceptor guidance law is discrete while the target guidance law is continuous. The absolute values of both guidance laws decrease with time, but increase with the terminal ZEM Z_f. Both guidance laws tend to zero when t tends to t_f.

III. BOUNDARY CONDITIONS

In this section, the interception problem is converted to four cases according to the guidance law parameters α, β. Then the critical values of β in four cases to realize interception are given, which are decided by the system parameters including acceleration constraints, the acceptable miss distance and initial values of the engagement. Afterwards, the capture zone is given regarding the system parameters $\tau_T, \tau_M, a_{T,\max}, a_{M,\max}, t_0, t_f$, the guidance law parameters α, β, and the acceptable miss distance Z_m.

A. Capture Cases

Considering the capture constraint (9), the terminal ZEM corresponding with the capture boundary is

$$Z_f = \pm Z_m \quad (24)$$

If $Z_m \le \alpha/\left(2h_M\left(t_0\right)\right)$, consider that h_M decreases monotonically, $\forall t \in \left[t_0, t_f\right], Z_m \le \alpha/\left(2h_M\left(t\right)\right)$, leading to $u_M^* = 0$.

Similarly, if $Z_m > \alpha/(2h_M(t_0))$, then $\exists t_M \in [t_0, t_f]$, $h_M(t_M) = \alpha/(2Z_m)$. According to (20), the interceptor guidance law is

$$u_M^* = \begin{cases} 0 & ,t_M \le t \le t_f \\ \mathrm{sign}(Z_f), & t_0 \le t < t_M \end{cases} \tag{25}$$

If $Z_m \le \beta/h_T(t_0)$, consider that h_T decreases monotonically, $\forall t \in [t_0, t_f], Z_m \le \beta/h_T(t)$, leading to $u_T^* = Z_f h_T/\beta$.

Similarly, if $Z_m > \beta/h_T(t_0)$, then $\exists t_T \in [t_0, t_f]$, $h_T(t_T) = \beta/Z_m$. According to (23), the target guidance law is

$$u_T^* = \begin{cases} Z_f h_T/\beta & ,t_T \le t \le t_f \\ \mathrm{sign}(Z_f), & t_0 \le t < t_T \end{cases} \tag{26}$$

In convenience, the interception problem can be analyzed in four cases described in the following definition of Capture Cases. Denote

$$\alpha_0 = 2Z_m h_M(t_0), \beta_0 = Z_m h_T(t_0) \tag{27}$$

***Definition 1. Capture Cases*:** the range of the guidance law parameters $\alpha > 0, \beta > 0$ can be divided by α_0, β_0, given as

Case 1: $\alpha \ge \alpha_0,\ \beta \ge \beta_0$;

Case 2: $\alpha \ge \alpha_0,\ \beta < \beta_0, when\ \alpha/\beta > 2h_M(t_0)/h_T(t_0)$;

Case 3: $\alpha < \alpha_0,\ \beta \ge \beta_0, when\ \alpha/\beta < 2h_M(t_0)/h_T(t_0)$;

Case 4: $\alpha < \alpha_0,\ \beta < \beta_0$.

where the expressions of $h_T(t)$ and $h_M(t)$ are given as (6) and (7) respectively.

According to (6) and (7), $h_T(t)$ and $h_M(t)$ are decided by the system parameters τ_T, τ_M, $a_{T,\max}$, $a_{M,\max}$, t_0, t_f. Consequently, these four capture cases are divided by the relationship between the guidance law parameters α, β, the system parameters, and the acceptable miss distance Z_m.

B. *Critical Values of the Guidance Law Parameters*

Critical values of β for the four capture cases given in Definition 1 are denoted as $\beta_i\ (i = 1,2,3,4)$ respectively. Corresponding values of t_M and t_T are denoted as $t_{Mi}\ (i = 3,4)$ and $t_{Tj}\ (j = 2,4)$.

In *Case 1*, $u_M^* = 0$, $u_T^* = Z_f h_T/\beta$, then on the capture boundary, the terminal ZEM satisfies

$$\left|Z_f^*\right| = \left|\int_{t_0}^{t_f} h_T(t) u_T^*(t)\,dt\right| = \frac{Z_m}{\beta_1}\int_{t_0}^{t_f} h_T^2(t)\,dt \tag{28}$$

Therefore, for a given $\alpha \ge \alpha_0$, $\left|Z_f^*\right| = Z_m$ is satisfied when

$$\beta_1 = \int_{t_0}^{t_f} h_T^2(t)\,dt \tag{29}$$

In *Case 2*, $u_M^* = 0$. Substituting the capture boundary $Z_f^* = \pm Z_m$ into (26) yields

$$u_T^* = \begin{cases} \mathrm{sign}(Z_f) Z_m h_T/\beta_2 & ,t_{T2} \le t \le t_f \\ \mathrm{sign}(Z_f) & ,t_0 \le t < t_{T2} \end{cases} \tag{30}$$

Then, the terminal ZEM can be written as

$$\left|Z_f^*\right| = \frac{Z_m}{\beta_2}\int_{t_{T2}}^{t_f} h_T^2(t)\,dt + \int_{t_0}^{t_{T2}} h_T(t)\,dt \tag{31}$$

Therefore, for a given $\alpha \ge \alpha_0$, $\left|Z_f^*\right| = Z_m$ is satisfied when

$$\beta_2 = Z_m\int_{t_{T2}}^{t_f} h_T^2(t)\,dt \Big/ \left[Z_m - \int_{t_0}^{t_{T2}} h_T(t)\,dt\right] \tag{32}$$

where $t_{T2} \in [t_0, t_f)$ satisfies

$$\begin{gathered} f_T(t_{T2}) = Z_m \\ f_T(\tau) = \int_\tau^{t_f} h_T^2(t)\,dt \Big/ h_T(\tau) + \int_{t_0}^{\tau} h_T(t)\,dt, t_0 \le \tau < t_f \end{gathered} \tag{33}$$

If τ tends to t_f, the limitation of $f_T(\tau)$ is

$$\lim_{\tau \to t_f} f_T(\tau) = \int_{t_0}^{t_f} h_T(t)\,dt \tag{34}$$

The derivative of $f_T(\tau)$ is

$$f_T^{'}(\tau) = -h_T^{'}(\tau)\int_\tau^{t_f} h_T^2(t)\,dt \Big/ h_T^2(\tau) \ge 0 \tag{35}$$

Therefore, t_{T2} exists when $f_T(t_0) \le Z_m < \lim_{\tau\to t_f} f_T(\tau)$, leading to

$$\int_{t_0}^{t_f} h_T^2(t)\,dt \Big/ h_T(t_0) \le Z_m < \int_{t_0}^{t_f} h_T(t)\,dt \tag{36}$$

In *Case 3*, considering the capture boundary $Z_f^* = \pm Z_m$, the guidance laws can be written as

$$\begin{gathered} u_T^* = \mathrm{sign}(Z_f) Z_m h_T/\beta_3 \\ u_M^* = \begin{cases} 0 & ,t_{M3} \le t \le t_f \\ \mathrm{sign}(Z_f), & t_0 \le t < t_{M3} \end{cases} \end{gathered} \tag{37}$$

Substituting (37) into (5) and integrating the derivative of ZEM from t_0 to t_f yields

$$Z_f^* = \mathrm{sign}(Z_f)\left[\frac{Z_m}{\beta_3}\int_{t_0}^{t_f} h_T^2(t)\,dt - \int_{t_0}^{t_{M3}} h_M(t)\,dt\right] \tag{38}$$

Therefore, the terminal ZEM boundary $\left|Z_f^*\right| = Z_m$ can be satisfied when

$$\beta_3 = Z_m\int_{t_0}^{t_f} h_T^2(t)\,dt \Big/ \left[Z_m + \int_{t_0}^{t_{M3}} h_M(t)\,dt\right] \tag{39}$$

where for a given $0 < \alpha < \alpha_0$, $t_{M3} \in [t_0, t_f]$ satisfies

$$h_M(t_{M3}) = \alpha/(2Z_m) \tag{40}$$

In *Case 4*, considering the optimal ZEM boundary $Z_f^* = \pm Z_m$, the guidance laws (25) and (26) can be written as

$$\begin{gathered} u_M^* = \begin{cases} 0 & ,t_{M4} \le t \le t_f \\ \mathrm{sign}(Z_f), & t_0 \le t < t_{M4} \end{cases} \\ u_T^* = \begin{cases} \mathrm{sign}(Z_f) Z_m h_T/\beta_4 & ,t_{T4} \le t \le t_f \\ \mathrm{sign}(Z_f) & ,t_0 \le t < t_{T4} \end{cases} \end{gathered} \tag{41}$$

Substituting (41) into (5) and integrating the derivative of ZEM from t_0 to t_f yields

$$Z_f^* = \operatorname{sign}(Z_f)\left[\frac{Z_m}{\beta_4}\int_{t_{T4}}^{t_f} h_T^2(t)\,dt + \int_{t_0}^{t_{T4}} h_T(t)\,dt - \int_{t_0}^{t_{M4}} h_M(t)\,dt\right] \quad (42)$$

Therefore, $|Z_f^*| = Z_m$ is satisfied when

$$\beta_4 = \frac{Z_m\int_{t_{T4}}^{t_f} h_T^2(t)\,dt}{Z_m - \int_{t_0}^{t_{T4}} h_T(t)\,dt + \int_{t_0}^{t_{M4}} h_M(t)\,dt} \quad (43)$$

where for a given $0 < \alpha < \alpha_0$, t_{T4} and t_{M4} satisfy

$$h_M(t_{M4}) = \alpha/(2Z_m),\ f_T(t_{T4}) = Z_m + \int_{t_0}^{t_{M4}} h_M(t)\,dt \quad (44)$$

Because of (35) and $\int_{t_0}^{t_{M4}} h_M(t)\,dt \ge 0$, t_{T4} exists when $f_T(t_0) \le Z_m + \int_{t_0}^{t_{M4}} h_M(t)\,dt < \lim_{\tau\to t_f} f_T(\tau)$, resulting in

$$\int_{t_0}^{t_f} h_T^2(t)\,dt \Big/ h_T(t_0) \le Z_m + \int_{t_0}^{t_{M4}} h_M(t)\,dt < \int_{t_0}^{t_f} h_T(t)\,dt \quad (45)$$

The boundary conditions in four cases can be summarized as following *Theorem 1*.

Theorem 1. Boundary Conditions: Considering four Capture Cases in *Definition 1*, the terminal ZEM capture boundary $|Z_f^*| = Z_m$ can be realized when

Case 1: $\beta = \beta_1$;

Case 2: (36) and $\beta = \beta_2$;

Case 3: $\beta = \beta_3$;

Case 4: (45) and $\beta = \beta_4$;

where the expressions of $\beta_i\ (i = 1,2,3,4)$ are given in (29), (32), (39) and (43) respectively.

Theorem 1 gives the critical values of the guidance law parameter β to realize boundary constraints for given engagement parameters. The capture zone is given in the next section based on the boundary conditions summarized above.

IV. Capture Zones Analysis

This section aims to obtain the capture zones regarding *Theorem 1* and demonstrate some examples of the capture zones to analyze the impact of the system parameters and the guidance law parameters on capturability.

A. Capture Zones

It can be obtained from (28), (31), (38) and (42) that $\partial|Z_f^*|/\partial\beta_i < 0\ (i = 1,2,3,4)$. Then, in four capture cases, the optimal terminal ZEM decreases monotonically regarding the guidance law parameter β.

Therefore, to realize the capture boundary $|Z_f^*| \le Z_m$, the guidance law parameter β should satisfy

$$\beta \ge \beta_i\ (i = 1,2,3,4) \quad (46)$$

The capture conditions in four cases can be derived, as shown in *Theorem 2*.

Theorem 2. Capture Conditions: Concerning the interception engagement and the differential guidance laws presented in *Theorem 1*, the capture constraint $|Z_f^*| \le Z_m$ can be realized when one of the following conditions holds:

Case 1: (a) $\beta_1 \le \beta_0$; (b) $\beta \ge \beta_1 > \beta_0$;

Case 2: (c) (36), $\beta_2 < \beta_0$ and $\beta \ge \beta_2$;

Case 3: (d) $\beta_3 \le \beta_0$; (e) $\beta \ge \beta_3 > \beta_0$;

Case 4: (f) (45), $\beta_4 < \beta_0$ and $\beta \ge \beta_4$.

The non-capture conditions, i.e., the conditions that cannot ensure interception, are complementary to the capture conditions in *Theorem 2*.

Therefore, the non-capture conditions for four cases can be given as *Theorem 3*.

Theorem 3. Non-Capture Conditions: Concerning the interception engagement and the differential guidance laws in Sec. II, the capture constraint $|Z_f^*| \le Z_m$ cannot be ensured when one of the following conditions holds:

Case 1: (a) $\beta < \beta_1$;

Case 2: (b) (36) cannot be satisfied; (c) $\beta < \beta_2$;

Case 3: (d) $\beta < \beta_3$;

Case 4: (e) (45) cannot be satisfied; (f) $\beta < \beta_4$.

The overall capture zones can be summarized as follows: Concerning the interception engagement and the differential guidance laws, the capture constraint $|Z_f^*| \le Z_m$ can be realized if one of the following conditions holds:

(a) $\alpha \ge \alpha_0$ and $\beta \ge \beta_0 \ge \beta_1$;

(b) $\alpha \ge \alpha_0$ and $\beta \ge \beta_1 > \beta_0$;

(c) $\alpha \ge \alpha_0$, (36) and $\beta_2 \le \beta < \beta_0$;

(d) $\alpha < \alpha_0$ and $\beta \ge \beta_1 > \beta_0$;

(e) $\alpha < \alpha_0$ and $\beta \ge \beta_3 > \beta_0$;

(f) $\alpha < \alpha_0$, (45) and $\beta_4 \le \beta < \beta_0$.

This gives the capture zones of the differential guidance laws when the interceptor guidance command is pulsed and the target guidance command is bounded. Obviously, the capturability is decided by the system parameters τ_T, τ_M, $a_{T,\max}$, $a_{M,\max}$, t_0, t_f, the guidance law parameters α, β, and the acceptable miss distance Z_m.

B. Examples of Capture Zones

In the following examples, CC and NC followed by letters represent corresponding condition in *Theorem 2* and *Theorem 3*, respectively.

In order to show the impact of the guidance law parameters α, β, choose $\alpha \in [0,1000]$, $\beta \in [0,1500]$. The other parameters are chosen as $\tau_T = 1s$, $\tau_M = 1s$, $a_{T,\max} = 10\,m/s^2$, $a_{M,\max} = 20\,m/s^2$, $v_R = 7km/s$, $t_0 = 0s$, $x_{R0} = 100km$, $Z_m = 2m$. The capture zone under these parameters is shown in Figure 3.

Choose the guidance law parameter ratio $\beta/\alpha \in [0,20]$ and the acceleration constraint ratio $a_{T,\max}/a_{M,\max} \in [0,10]$. The other parameters are chosen as $a_{M,\max} = 10\,m/s^2$, $t_0 = 0s$, $x_{R0} = 100km$, $v_R = 7km/s$, $Z_m = 2m$, $\tau_T = 1s$, $\tau_M = 1s$, $\alpha = 200$. Its corresponding capture zone is shown in Figure 4.

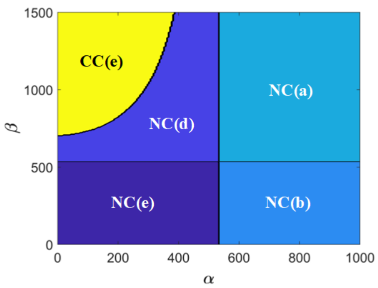


Figure 3. Capture zones with different guidacne law parameters

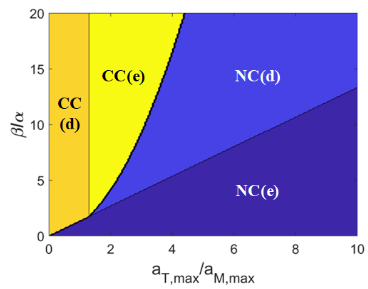


Figure 4. Capture zones with different acceleration constraints

Figure 3 gives the capture zones of the interception with higher maneuvering, and it demonstrates that in this engagement, it is easier for the interceptor to capture the target with lower α and higher β. Figure 4 shows that it is easier for the interceptor to capture a target with lower maneuvering. However, the interception can also be guaranteed with high guidance law parameter ratio β/α, as elaborated in Figure 3.

V. Conclusion

This paper utilizes the differential game theory to analyze the capturability of the pulsed guidance law against the maneuvering target with bounded acceleration. According to the capture conditions, the capturability is decided by the guidance law parameters α, β, the system parameters $\tau_T, \tau_M, a_{T,\max}, a_{M,\max}, t_0, t_f$ and the acceptable miss distance Z_m. Through two specific examples, it can be seen that the interception can be guaranteed with lower acceleration constraint ratio $a_{T,\max}/a_{M,\max}$ and higher guidance law parameter ratio β/α. Note that the capture condition given in this paper is the sufficient but not necessary condition for interception. Therefore, in the future work, both the sufficient and necessary condition will be further analyzed and more examples will be given to demonstrate the impact of the parameters on the capture zones. Moreover, capture zones of other guidance laws will be further studied and compared with the differential game guidance law.

References

[1] T. Shima, "Optimal Cooperative Pursuit and Evasion Strategies Against a Homing Missile," *J. Guid. Control Dyn.*, vol. 34, no. 2, pp. 414–425, Mar. 2011.

[2] V. Shaferman and T. Shima, "Cooperative Multiple-Model Adaptive Guidance for an Aircraft Defending Missile," *J. Guid. Control Dyn.*, vol. 33, no. 6, pp. 1801–1813, Nov. 2010.

[3] S. Ghosh, D. Ghose, and S. Raha, "Capturability Analysis of a 3-D Retro-PN Guidance Law for Higher Speed Nonmaneuvering Targets," *IEEE Trans. Control Syst. Technol.*, vol. 22, no. 5, pp. 1864–1874, Sep. 2014.

[4] S. Ghosh, D. Ghose, and S. Raha, "Capturability of Augmented Pure Proportional Navigation Guidance Against Time-Varying Target Maneuvers," *J. Guid. Control Dyn.*, vol. 37, no. 5, pp. 1446–1461, Sep. 2014.

[5] D. Mukherjee and D. Ghose, "Target Capturability Using Agents in Cyclic Pursuit," *J. Guid. Control Dyn.*, vol. 39, no. 5, pp. 1034–1045, May 2016.

[6] K. Li, Z. Bai, H.-S. Shin, A. Tsourdos, and M.-J. Tahk, "Capturability of 3D RTPN guidance law against true-arbitrarily maneuvering target with maneuverability limitation," *Chin. J. Aeronaut.*, vol. 35, no. 7, pp. 75–90, Jul. 2022.

[7] K.-B. Li, H.-S. Shin, A. Tsourdos, and M.-J. Tahk, "Capturability of 3D PPN Against Lower-Speed Maneuvering Target for Homing Phase," vol. 56, no. 1, 2020.

[8] K.-B. Li, H.-S. Shin, A. Tsourdos, and M.-J. Tahk, "Performance of 3-D PPN Against Arbitrarily Maneuvering Target for Homing Phase," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 56, no. 5, pp. 3878–3891, Oct. 2020.

[9] K.-B. Li, H.-S. Shin, and A. Tsourdos, "Capturability of a Sliding-Mode Guidance Law With Finite-Time Convergence," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 56, no. 3, pp. 2312–2325, Jun. 2020.

[10] S. Battistini and T. Shima, "Differential games missile guidance with bearings-only measurements," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 50, no. 4, pp. 2906–2915, Oct. 2014.

[11] V. Turetsky and J. Shinar, "Missile guidance laws based on pursuit–evasion game formulations," *Automatica*, vol. 39, no. 4, pp. 607–618, Apr. 2003.

[12] S. Rubinsky and S. Gutman, "Three-Player Pursuit and Evasion Conflict," *J. Guid. Control Dyn.*, vol. 37, no. 1, pp. 98–110, Jan. 2014.

[13] S. Rubinsky and S. Gutman, "Vector Guidance Approach to Three-Player Conflict in Exoatmospheric Interception," *J. Guid. Control Dyn.*, vol. 38, no. 12, pp. 2270–2286, Dec. 2015.

[14] S. Gutman and S. Rubinsky, "Exoatmospheric Thrust Vector Interception Via Time-to-Go Analysis," *J. Guid. Control Dyn.*, vol. 39, no. 1, pp. 86–97, Jan. 2016.

[15] I. Rusnak, H. Weiss, R. Eliav, and T. Shima, "Missile guidance with constrained intercept body angle," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 50, no. 2, pp. 1445–1453, Apr. 2014.

[16] V. Turetsky, M. Weiss, and T. Shima, "A Combined Linear–Quadratic/Bounded Control Differential Game Guidance Law," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 57, no. 5, pp. 3452–3462, Oct. 2021.

[17] S. Y. Hayoun and T. Shima, "On guaranteeing point capture in linear n -on-1 endgame interception engagements with bounded controls," *Automatica*, vol. 85, pp. 122–128, Nov. 2017.

[18] Q. Sun, C. Zhang, N. Liu, W. Zhou, and N. Qi, "Guidance laws for attacking defended target," *Chin. J. Aeronaut.*, vol. 32, no. 10, pp. 2337–2353, Oct. 2019.

[19] N. Qi, Q. Sun, and J. Zhao, "Evasion and pursuit guidance law against defended target," *Chin. J. Aeronaut.*, vol. 30, no. 6, pp. 1958–1973, Dec. 2017.

[20] F. Liu, X. Dong, Q. Li, and Z. Ren, "Cooperative differential games guidance laws for multiple attackers against an active defense target," *Chin. J. Aeronaut.*, vol. 35, no. 5, pp. 374–389, May 2022.

[21] F. Liu, X. Dong, Q. Li, and Z. Ren, "Robust multi-agent differential games with application to cooperative guidance," *Aerosp. Sci. Technol.*, vol. 111, p. 106568, Apr. 2021.

[22] L. Liang, F. Deng, Z. Peng, X. Li, and W. Zha, "A differential game for cooperative target defense," *Automatica*, vol. 102, pp. 58–71, Apr. 2019.

[23] F.-K. Yeh, "Design of nonlinear terminal guidance/autopilot controller for missiles with pulse type input devices," *Asian J. Control*, vol. 12, no. 3, pp. 399–412, Mar. 2010.

[24] Guojiang Zhang, Baoqing Yang, and Yu Yao, "The design of predictive guidance law and stability analysis of guidance system for Exo-atmospheric interceptor," in *2010 8th World Congress on Intelligent Control and Automation*, Jinan, China: IEEE, Jul. 2010, pp. 2383–2387.

[25] W. Yu, W. Chen, L. Yang, X. Liu, and H. Zhou, "Optimal terminal guidance for exoatmospheric interception," *Chin. J. Aeronaut.*, vol. 29, no. 4, pp. 1052–1064, Aug. 2016.