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Stability and Stabilization of Nash Equilibrium for Noncooperative Systems With Vector-valued Payoff Functions

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***Abstract*—A zero-sum tax/subsidy approach and a necessary condition for stabilizing unstable Nash equilibria in pseudo-gradient-based noncooperative dynamical systems with vector-valued payoff functions are proposed. Specifically, we first present a necessary and sufficient condition for the Nash equilibrium of the noncooperative game with vector-valued payoff functions to be bounded. Then we give a sufficient condition for such Nash equilibrium to be stable. After that, we develop a framework where a system manager constructs a zero-sum tax/subsidy incentive structure by collecting taxes from one agent and giving the same amount of subsidy to the other agent to make the incentivized Nash equilibrium stable and bounded, which can make the trajectories converge to the interior of original Nash equilibrium set. Finally, we present a numerical example to illustrate the utility of the zero-sum tax/subsidy approach.**

***Index Terms*—Nash equilibrium, noncooperative system, stabilization, tax/subsidy approach, vector-valued payoff functions**

I. Introduction

Game theory has served as a widely used framework for analyzing noncooperative multi-agent systems, with applications in engineering and economics such as minimizing energy consumption in wireless networks [1], improving social welfare [2], implementing pricing mechanisms [3], among others. In such games, agents make their decisions based on their associated payoff functions.

In noncooperative games, it is assumed that agents prioritize increasing their own payoff functions instead of the interest of others, thus exhibiting selfish behavior. This behavior in noncooperative systems is typically described as the dynamic fictitious play [4] and the pseudo-gradient dynamics [5]. However, such decision-making may often lead to a degradation of social welfare [6]. An example is the tragedy of the commons, which represents a social trap involving a conflict between individual and public interests in resource allocation [7]. In the absence of an individual who is responsible for the entire noncooperative system, each agent demands resources independently based on self-interest, leading to uncontrolled exploitation of limited resources. This ultimately harms the communal good of all agents in shared-resource systems.

To avoid such phenomenon, external policies or explicit incentive mechanisms can be imposed to alter their decision-making tendencies within noncooperative systems [8]. A tax/subsidy approach has been proposed to reward or penalize deviation from the average contribution of other competitors in public goods allocation [9]. Varian proposed a compensation mechanism, where agents are permitted to voluntarily subsidize other agents with prestige when others' decisions have not been made yet [10]. Such a mechanism is considered as a liberal solution since agents are free to avoid the mechanism. This type of mechanisms works as a weak rule for noncooperative systems and is generally expected to be less efficient than coercive solutions.

In this paper, we consider the incentive design method for the game where each agent has vector-valued payoff functions, depending on the decisions of all players. This type of game was initially introduced by Blackwell [11]. Subsequently, Shapley and Rigby extended the concept of Nash equilibrium for two-person zero-sum finite games with vector payoff functions, which is a key concept in games where each agent has multiple objectives [12]. For general n-person multi-objective games, Zhao established the existence of equilibria [13]. Bade characterized the Nash equilibrium set of the multi-objective game as the union of Nash equilibria for certain derived games with complete preference [14]. Patriche explored the existence of equilibrium for multiobjective games in abstract convex spaces [15]. Guo *et al.* characterized the Nash equilibrium in 2-agent games with quadratic payoff functions and provided the stability condition when the interior of the Nash equilibrium is simply connected [16].

In the following we use $E(G)$ to denote the Nash equilibrium set, and there exists some problem not solved in [16], such as the stability property when the interior of $E(G)$ is not simply connected. Also, with the stable dynamics of each agent, the trajectories from the neighborhood of $E(G)$ can only reach its boundary. We wonder if by giving some incentive functions, the trajectories can converge to part of $E(G)$. Then, another important problem arises: can we make an unstable $E(G)$ stable as Yan and Hayakawa did in [17]; and can we make $E(G)$ compact to avoid the tragedy of the commons by incentive functions?

The main contributions of this paper are summarized as follows. Firstly, we give some further results on the characterization (a necessary and sufficient condition for the compactness of Nash equilibrium) and the stability condition of 2-agent games with quadratic vector payoff functions when the dimension of the vector payoff functions is 2. Secondly, we use an incentive function that influences only one element

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in the vector payoff functions to make the Nash equilibrium set compact, stable, and a subset of the original one. We provide the necessary condition for the existence of the incentive function and further discuss the flexibility for such incentive parameters.

Notations: Given a vector $x \in \mathbb{R}^n$, its ith coordinate is denoted by $x_i \in \mathbb{R}$ and its transpose is denoted by x^{T} if there are no special instructions. Given two vectors $a, b \in \mathbb{R}^n$, we write $a \geq b$ if $a_i \geq b_i$, for all $i \in \{1, \ldots, n\}$ and $a \neq b$. Given a matrix $A \in \mathbb{R}^{m \times n}$, A_{ij} denotes the element at ith row and jth column. The ith row of A is denoted by $\mathrm{row}_i(A)$. Finally, the interior of a set S is denoted by $\mathrm{int}(S)$. The set S is called connected if it is not a union of two disjoint nonempty open sets, and a set is called compact if it is closed and bounded. The Hausdorff distance of a point x and a set Y is defined as $d_{\mathrm{H}}(x, Y) \triangleq \min_{y \in Y} \|x - y\|_2$.

II. Problem Formulation

A. System description

Consider the noncooperative system with two agents, denoted by $i \in \{1, 2\}$, where each agent controls its state $x^i \in \mathbb{R}$. We define the overall agents' state profile as $x = (x^i, x^{-i}) \in \mathbb{R}^2$, where $x^{-i} \in \mathbb{R}$ represents the state of the other agent. We can also write x as $x = [x^1, x^2]^{\mathrm{T}}$. We assume that both of the 2 agents have 2 kinds of payoff functions $J^{i1}, J^{i2} : \mathbb{R}^2 \to \mathbb{R}, i \in \{1, 2\}$, so that the vector-valued payoff function of agent i is denoted by $J^i : \mathbb{R}^2 \to \mathbb{R}^2$, where $J^i(x) = [J^{i1}(x), J^{i2}(x)]^{\mathrm{T}}$.

In this paper, we assume that there is a system manager who imposes some incentive mechanisms among the agents to reconstruct the agents' payoff functions and hence alters the agents' decisions. Specifically, let the agents' incentivized payoff functions be given by

$$\begin{aligned} \hat{J}^{i1}(x) &\triangleq J^{i1}(x) + p^i(x), \\ \hat{J}^{i2}(x) &\triangleq J^{i2}(x), \end{aligned} \tag{1}$$

where $p^i : \mathbb{R}^2 \to \mathbb{R}$ denotes the incentive function for agent i. Note that the incentive is applied to only one of the payoff functions and satisfies $p^1(x) + p^2(x) = 0$. We denote the original (resp., incentivized) noncooperative system by $G(J)$ (resp., $G(\hat{J})$), where $J = \{J^1, J^2\}$ $\hat{J} = \{\hat{J}^1, \hat{J}^2\}$ with $\hat{J}^i = [\hat{J}^{i1}, \hat{J}^{i2}]^{\mathrm{T}}$. Here, we assume that $J^{ij}(x)$ and $\hat{J}^{ij}(x), i, j \in \{1, 2\}$, are continuously differentiable and strictly concave with respect to $x^i, i \in \{1, 2\}$. Specifically, the payoff functions are given by quadratic functions represented by

$$J^{ij}(x) = \frac{1}{2} x^{\mathrm{T}} A^{ij} x + b^{ij\mathrm{T}} x + c^{ij}, \tag{2}$$

where $A^{ij} = \begin{bmatrix} A^{ij}_{11} & A^{ij}_{12} \\ A^{ij}_{12} & A^{ij}_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$, with A^{ij}_{ii} being negative, $b^{ij} \in \mathbb{R}^2$, $c^{ij} \in \mathbb{R}$, $i, j \in \{1, 2\}$. In this paper, we consider the incentive functions of the form given by

$$p^i(x) \triangleq \frac{1}{2} x^{\mathrm{T}} P^i x + q^{i\mathrm{T}} x, \tag{3}$$

with $P^i = \begin{bmatrix} P^i_{11} & P^i_{12} \\ P^i_{12} & P^i_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $q^{ij} = \begin{bmatrix} q^i_1 \\ q^i_2 \end{bmatrix} \in \mathbb{R}^2$.

Definition 1 (Shapley [12])**.** The state profile $x^* = (x^{*1}, x^{*2}) \in \mathbb{R}^2$ is called a Nash equilibrium for $G(J)$ with the vector-valued payoff functions $J^i, i \in \{1, 2\}$, if there does not exist $x^i \in \mathbb{R}$ such that

$$J^i(x^i, x^{*-i}) \geq J^i(x^*), \quad i \in \{1, 2\}. \tag{4}$$

As in [16], we use the notion of the weighted games to fully characterize the set of all Nash equilibria $E(G)$ of a given game $G(J)$. Specifically, let $\lambda^i = [\lambda^i_1, \lambda^i_2]^{\mathrm{T}} \in \Delta^2 \triangleq \{\lambda^i \in \mathbb{R}^2 : \lambda^i_1 + \lambda^i_2 = 1, \lambda^i_j \geq 0, j \in \{1, 2\}\}$ denote a vector of weights for agent i. For $\lambda \triangleq (\lambda^1, \lambda^2) \in \Delta \triangleq \Delta^2 \times \Delta^2$, the weighted game is denoted by $G(\tilde{J}_\lambda)$ with $\tilde{J}_\lambda = \{\tilde{J}^1_{\lambda^1}, \tilde{J}^2_{\lambda^2}\}$, where

$$\tilde{J}^i_{\lambda^i}(x) \triangleq \lambda^{i\mathrm{T}} J^i(x) \in \mathbb{R}, \quad i \in \{1, 2\}. \tag{5}$$

We define $E(G_\lambda)$ as the set (which may be a single point set or empty set) of Nash equilibria (equilibrium) for the weighted game $G(\tilde{J}_\lambda)$ with a given λ. According to Theorem 2 in [14] and Theorem 2.5 in [18], since $J^{i1}(x), J^{i2}(x)$ are concave in x^i for $i \in \{1, 2\}$, the set $E(G)$ of Nash equilibria satisfies

$$E(G) = \bigcup\nolimits_{\lambda \in \Delta} E(G_\lambda). \tag{6}$$

Since $J^{ij}(x), i \in \{1, 2\}$, are assumed to be continuously differentiable and concave with respect to x^i, according to the maximum principle and (5), it follows that

$$\left.\frac{\partial \tilde{J}^i_{\lambda^i}(x)}{\partial x^i}\right|_{x = x^*_\lambda} = \lambda^{i\mathrm{T}} \left.\frac{\partial J^i(x)}{\partial x^i}\right|_{x = x^*_\lambda} = 0, \tag{7}$$

holds for $\lambda \in \Delta$, $i = \{1, 2\}$, if and only if $x^*_\lambda \in E(G_\lambda)$.

With the payoff functions defined in (2), according to (7), for a fixed λ, the Nash equilibrium $x^*_\lambda \in E(G_\lambda)$ satisfies

$$\begin{bmatrix} \lambda^1_1 \mathrm{row}_1(A^{11}) + \lambda^1_1 \mathrm{row}_1(A^{12}) \\ \lambda^2_1 \mathrm{row}_2(A^{21}) + \lambda^2_2 \mathrm{row}_2(A^{22}) \end{bmatrix} x^*_\lambda = - \begin{bmatrix} \lambda^1_1 b^{11}_1 + \lambda^1_2 b^{12}_1 \\ \lambda^2_1 b^{21}_2 + \lambda^2_2 b^{22}_2 \end{bmatrix}. \tag{8}$$

Then, by iterating over all elements in Δ, we can get all elements of $E(G)$, which can also be characterized as

$$(\mathrm{row}_1(A^{11}) x^* + b^{11}_1)(\mathrm{row}_1(A^{12}) x^* + b^{12}_1) \leq 0, \tag{9a}$$

$$(\mathrm{row}_2(A^{21}) x^* + b^{21}_2)(\mathrm{row}_2(A^{22}) x^* + b^{22}_2) \leq 0, \tag{9b}$$

for $x^* \in E(G)$. In the context of the 2-agent noncooperative system, we define the best response state x^i as $x^i = \mathrm{BR}^{ij}(x^{-i}) \triangleq \arg\max_{x^i \in \mathbb{R}} J^i_j(x^i, x^{-i}) = -\frac{b^{ij}_i}{A^{ij}_{ii}} - \frac{A^{ij}_{12}}{A^{ij}_{ii}} x^{-i}$ for given x^{-i}. This can also be expressed as $\mathrm{row}_i(A^{ij})x + b^{ij}_i = 0$, which represents agent i's best-response line in relation with its jth payoff function. As there are two agents with two payoff functions each, there are 4 best-response lines in total. The regions divided by the four best-response lines can be described by

$$\begin{aligned} D^i_{k,l} \triangleq \{x \in \mathbb{R}^2 : &\, k(\mathrm{row}_i(A^{i1})x + b^{i1}_i) \geq 0, \\ &\, l(\mathrm{row}_i(A^{i2})x + b^{i2}_i) \geq 0\}, \end{aligned} \tag{10}$$

where $i \in \{1, 2\}$, and $k, l \in \{1, -1\}$. The red, blue and purple regions in Figs. 1–3 depict regions $D^1_{-1,1} \cup D^1_{1,-1}$, $D^2_{-1,1} \cup D^2_{1,-1}$, and the Nash equilibrium set $E(G)$ respectively, with

$$\begin{aligned} E(G) =& (D^1_{1,-1} \cup D^1_{-1,1}) \cap (D^2_{1,-1} \cup D^2_{-1,1}) \\ =& (D^1_{1,-1} \cap D^2_{1,-1}) \cup (D^1_{1,-1} \cap D^2_{-1,1}) \\ & \cup (D^1_{-1,1} \cap D^2_{1,-1}) \cup (D^1_{-1,1} \cap D^2_{-1,1}). \end{aligned} \tag{11}$$

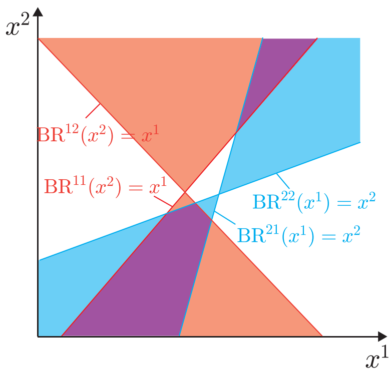

Fig. 1. $\mathcal{A}^{11} < 0$ while $\mathcal{A}^{jk} > 0, (j,k) \in \{(1,2),(2,1),(2,2)\}$. $E(G)$ is noncompact nor connected.

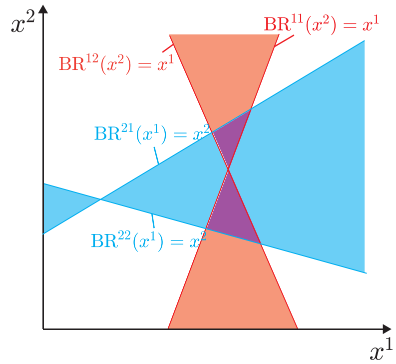

Fig. 2. $\mathcal{A}^{jk} > 0$ for all $j,k \in \{1,2\}$. $E(G)$ is compact and connected, but $\mathrm{int}(E(G))$ is not connected.

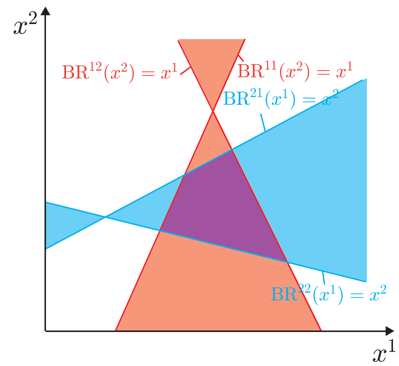

Fig. 3. $\mathcal{A}^{jk} > 0$ for all $j,k \in \{1,2\}$. $\mathrm{int}(E(G))$ is compact and connected.

B. Myopic pseudo-gradient dynamics

In the literature, various decision dynamics are associated with each agent to dynamically alter their states based on other agents' decisions. In our problem setting, we assume that each agent knows her own payoff function and the current state of the both agents. As a result, the frequently considered pseudo-gradient dynamics is applied to the game, in which agents only consider their own payoff and myopically adjust their states based on current information without any foresight into the future states of other agents. Since in the problem there are vector payoff functions and each agent has no preference between the different objectives, she follows the payoff function whose maximal point is closer to her current state. Specifically, the dynamics are given by

$$\dot{x}^i(t) = f^i(x(t)),\ x(0) = x_0 \in \mathbb{R}^2,\ i \in \{1,2\},\ t \geq 0, \quad (12)$$

where $f^i(x)$ is defined to be the piecewise function given by

$$f^i(x) = \begin{cases} \alpha^{i1}\dfrac{\partial J^{i1}(x)}{\partial x^i}, & |x^i - \mathrm{BR}^{i1}(x^{-i})| \leq |x^i - \mathrm{BR}^{i2}(x^{-i})|, \\ & (x^i - \mathrm{BR}^{i1}(x^{-i}))(x^i - \mathrm{BR}^{i2}(x^{-i})) > 0, \\ \alpha^{i2}\dfrac{\partial J^{i2}(x)}{\partial x^i}, & |x^i - \mathrm{BR}^{i1}(x^{-i})| > |x^i - \mathrm{BR}^{i2}(x^{-i})|, \\ & (x^i - \mathrm{BR}^{i1}(x^{-i}))(x^i - \mathrm{BR}^{i2}(x^{-i})) > 0, \\ 0, \text{otherwise}, \end{cases} \quad (13)$$

where α^{i1} and α^{i2} are the sensitivity parameters of agent i for $i \in \{1,2\}$. To ensure that $f^i(x)$ is continuous in $x \in \mathbb{R}^2$, the sensitivity parameters must satisfy $\alpha^{i1}\frac{\partial^2 J^{i1}(x)}{\partial (x^i)^2} = \alpha^{i2}\frac{\partial^2 J^{i2}(x)}{\partial (x^i)^2}$ for $i \in \{1,2\}$. This piecewise function of $f^i(x)$ ensures that each agent has no incentive to change its state if it is in the Nash equilibrium $E(G)$, as defined in Definition 1.

III. Main Results

A. Characterization and stability of $E(G)$

In this subsection, we analyze the condition for the compactness of $E(G)$. We begin by considering the 4 best-response lines in $G(J)$, which are in general position. The term "general position" is defined in [19] and refers to configurations of geometric objects that are most likely to occur. Then, we make the following assumptions.

Assumption 1. The 4 best-response lines characterized by $\mathrm{row}_i(A^{ij})x + b_i^{ij} = 0, i,j \in \{1,2\}$, satisfy:

1) No two lines are parallel,
2) No three lines have a common intersection,
3) No line is vertical or horizontal to the x^1, x^2 axes.

Under Assumption 1, we denote the intersection of $\mathrm{BR}^{i1}(x^{-i}) = x^i$ and $\mathrm{BR}^{i2}(x^{-i}) = x^i$ as x^{p_i}. Let $\mathcal{A}^{jk} \triangleq \begin{bmatrix} A_{11}^{1j} & A_{12}^{1j} \\ A_{12}^{2k} & A_{22}^{2k} \end{bmatrix}, j,k \in \{1,2\}$. Based on Assumption 1, the following proposition is presented.

Proposition 1. *Under Assumption 1, $E(G)$ is compact if and only if* $\det\mathcal{A}^{jk} > 0$ *for all* $j,k \in \{1,2\}$, *or* $\det\mathcal{A}^{jk} < 0$ *for all* $j,k \in \{1,2\}$.

Proof. The proof is omitted due to space limitations. □

From (10), $D_{1,-1}^i$ and $D_{-1,1}^i$ are 2 cones intersected at the vertex x^{p_i}. And according to (11), the interior of $E(G)$ can be either connected or disconnected depending on the patterns of the four intersecting regions of $D_{k,l}^i, i \in \{1,2\}, k,l \in \{1,-1\}$, which are divided by the 4 best response lines. In Figs. 1–3, we show three distinct patterns for the Nash equilibrium set $E(G)$ represented by purple regions. Fig. 1 exhibits a noncompact Nash equilibrium set, and Fig. 2 shows a compact Nash equilibrium set with disconnected interior, while Fig. 3 depicts a compact Nash equilibrium set with connected interior. Proposition 1 highlights the difference between Fig. 1 and Fig. 2 (Fig. 3), and demonstrates the consistency between Fig. 2 and Fig. 3. The distinction between Figs. 2 and 3 is the position of $x^{p_i}, i \in \{1,2\}$. Here, we present the following lemma to be a necesary condition for $E(G)$ to be compact.

Lemma 2. *Under Assumption 1, $E(G)$ is compact, only if $x^{p_1} \in E(G)$ and $x^{p_2} \in E(G)$ do not hold simultaneously.*

Proof. The proof is omitted due to space limitations. □

And we give the following assumption.

Assumption 2. $E(G)$ is compact.

Under Assumptions 1 and 2, the situation for $E(G)$ to be compact can be divided by the positions of x^{p_1} and x^{p_2}

into two cases corresponding to the cases in Figs. 3 and 2 respectively, according to Lemma 2:

1) $x^{p_1} \notin E(G)$ and $x^{p_2} \notin E(G)$,
2) $x^{p_1} \in E(G)$ and $x^{p_2} \notin E(G)$, or $x^{p_1} \notin E(G)$ and $x^{p_2} \in E(G)$.

In the following we use the Hausdorff distance $d_{\mathrm{H}}(x_0, E(G))$ to define stability of $E(G)$.

Definition 2. Consider the dynamical system given by (12). We say that the set $E(G)$ is stable if for every $\epsilon > 0$, there exists $\delta > 0$, such that $d_{\mathrm{H}}(x_0, E(G)) < \delta$ implies $d_{\mathrm{H}}(x(t), E(G)) < \epsilon$, $t \geq 0$. Furthermore, $E(G)$ is asymtotically stable, if $E(G)$ is stable and there exists $\delta > 0$ such that $d_{\mathrm{H}}(x_0, E(G))$ implies $\lim_{t \to \infty} d_{\mathrm{H}}(x(t), E(G)) = 0$.

Note that under Assumptions 1 and 2, case 1) implies $E(G)$ is convex according to [16], in which stability properties in case 1) has also been thoroughly examined. As such, we focus on case 2) where $x^{p_1} \in E(G)$ and $x^{p_2} \notin E(G)$ without loss of generality. Since $x^{p_1} \in E(G)$ means $D^1_{1,-1} \cap D^1_{-1,1} \subset E(G)$, it follows that $E(G) \subset D^1_{1,-1} \cup D^1_{-1,1}$ instead of $E(G) \subset D^1_{1,-1}$ or $E(G) \subset D^1_{-1,1}$ based on Assumption 1. Also, since $x^{p_2} \notin E(G)$ and Assumption 1, $E(G)$ is a subset of $(D^2_{1,-1} \cup D^2_{-1,1}) \backslash x^{p_2}$. And $(D^2_{1,-1} \cup D^2_{-1,1}) \backslash x^{p_2}$ is disconnected. According to [16] and Proposition 1, $E(G)$ is connected when it is compact. Then, it follows from (11) that $E(G)$ is a subset of the connected part of $(D^2_{1,-1} \cup D^2_{-1,1}) \backslash x^{p_2}$. As a result we let $E(G) = (D^1_{1,-1} \cup D^1_{-1,1}) \cap D^2_{1,-1}$ without loss of generality. Thus, to perform stability analysis, we can partition $\mathbb{R}^2$ based on the ordinate of x^{p_1} and abscissa of x^{p_2}, creating the 4 sets $S^1_k \triangleq \{x \in \mathbb{R}^2 : k(x^2 - x_2^{p_1}) \leq 0\}$ and $S^2_k \triangleq \{x \in \mathbb{R}^2 : k(x^1 - x_1^{p_2}) \leq 0\}$, where $k \in \{-1, 1\}$. Here, we provide an example of how we characterize the areas of interest around $E(G)$.

Example 1. Consider Fig. 4, where the horizontal line $x^2 = x_2^{p_1}$ has an intersection x^D with the best-response line $\mathrm{BR}^{21}(x^1) = x^2$. Let $D \triangleq \{x \in \mathbb{R}^2 : x^1 \geq x_1^D, -\frac{A^{11}_{11} x_1^D + b^{11}_1}{A^{11}_{12}} \leq x^2 \leq -\frac{A^{12}_{11} x_1^D + b^{12}_1}{A^{12}_{12}}\}$. We note that $D \backslash E(G)$ is divided into 10 domains given by

$$
\begin{aligned}
&D^{(1)} \triangleq D \cap D^1_{1,1} \cap D^2_{-1,-1}, \quad D^{(2)} \triangleq D \cap D^1_{1,1} \cap D^2_{1,-1} \cap S^1_{-1}, \\
&D^{(3)} \triangleq D \cap D^1_{1,1} \cap D^2_{1,-1} \cap S^1_{1}, \quad D^{(4)} \triangleq D \cap D^1_{1,1} \cap D^2_{1,1}, \\
&D^{(5)} \triangleq D \cap D^1_{-1,1} \cap D^2_{1,1}, \quad D^{(6)} \triangleq D \cap D^1_{-1,-1} \cap D^2_{1,1}, \\
&D^{(7)} \triangleq D \cap D^1_{-1,-1} \cap D^2_{1,-1} \cap S^1_{1}, \\
&D^{(8)} \triangleq D \cap D^1_{-1,-1} \cap D^2_{1,-1} \cap S^1_{-1}, \\
&D^{(9)} \triangleq D \cap D^1_{-1,-1} \cap D^2_{-1,-1}, \quad D^{(10)} \triangleq D \cap D^1_{1,-1} \cap D^2_{-1,-1}.
\end{aligned} \tag{14}
$$

Let $x^{Di} \triangleq x^{Djk} \in D^{(i)}$ be the intersection point of $\mathrm{BR}^{1j}(x^2) = x^1$ and $\mathrm{BR}^{2k}(x^1) = x^2$, where $i \in \{1, 4, 6, 9\}$. The corresponding relation can be seen from Fig. 4. For example, $x^{D1} = x^{D21} \in D^{(1)}$, and the dynamics (12) in $D^{(1)}$ can be rewritten as

$$
\dot{x}(t) = \begin{bmatrix} \alpha^{12} & 0 \\ 0 & \alpha^{21} \end{bmatrix} \mathcal{A}^{21}(x(t) - x^{D21}). \tag{15}
$$

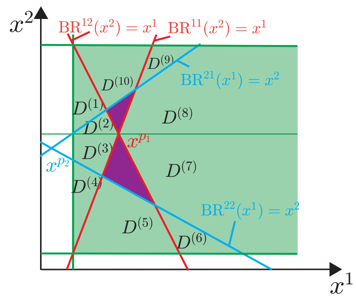

Fig. 4. An example where $x^{p_1} \in E(G)$ and $\mathrm{int}(E(G))$ is not connected. We can see how the domain D denoted by green color around $E(G)$ is partitioned.

The dynamics (12) can be also rewritten in the form of (15) to $D^{(i)}, i \in \{4, 6, 9\}$. In $D^{(i)}, i \in \{2, 3, 7, 8\}$, the state moves horizontally, while in $D^{(i)}, i \in \{5, 10\}$, the state moves vertically.

Note that when the sign of slopes of the best response lines changes, the partitions in Fig. 4 may change. By checking the different cases of the slopes of the best response lines, we get the following theorem based on Lemma 4 in [16].

Theorem 3. *Consider the noncooperative dynamical system* (12) *satisfying* (2) *under Assumptions 1 and 2. If* $\det \mathcal{A}^{jk} > 0$ *for* $j, k \in \{1, 2\}$, *then* $E(G)$ *is asymptotically stable. Conversely, if* $\det \mathcal{A}^{jk} < 0$ *for* $j, k \in \{1, 2\}$, *then* $E(G)$ *is unstable.*

Proof. The proof is omitted due to space limitations. □

Remark 1. Theorem 3 is a generalized result of the stability property of $E(G)$ with connected interior analyzed in [16], and can be used for our incentive design in the next subsection. Note that the sensitivity parameters $\alpha^{jk}, j, k \in \{1, 2\}$, in (12) do not influence the stability property. As a result, if the incentive functions do not change the slopes of the best-response lines, the stability property of the Nash equilibrium set does not change. Also, when $E(G)$ is not compact, according to our simulation, the trajectories converge to $E(G)$. But the stability property depends on the connectedness in such a case, which can be seen from the numerical example.

B. Incentive design

From Proposition 1 and Theorem 3, we know that the compactness and stability of $E(G)$ are determined by the signs of the determinants of $\mathcal{A}^{jk}$, $j, k \in \{1, 2\}$. By applying the incentive function $p^i(x)$ to the original noncooperative system, we can obtain the incentivized payoff functions of the game $G(\hat{J})$ as given in (1). To simplify the analysis, we combine (1)–(3) to express $\hat{J}^{ij}$ in the form of:

$$
\hat{J}^{ij}(x) = \frac{1}{2} x^{\mathrm{T}} \hat{A}^{ij} x + \hat{b}^{ij\mathrm{T}} x + c^{ij}, \tag{16}
$$

where $\hat{A}^{i1} = A^{i1} + P^i$, $\hat{b}^{i1} = b^{i1} + q^i$, $\hat{A}^{i2} = A^{i2}$, and $\hat{b}^{i2} = b^{i2}$ for $i \in \{1, 2\}$. Note that $p^i(x)$ can alter the position of the best-response lines of $\mathrm{BR}^{i1}(x^{-i}) = x^i, i \in \{1, 2\}$, and the resulting $E(\hat{G})$ may differ from the original $E(G)$. Firstly, we aim to find the condition when $E(\hat{G})$ is a subset of $E(G)$, i.e.,

$$E(\hat{G}) \subset E(G). \tag{17}$$

To address this problem, we define 4 pairs of cones given by

$$C_1 = \{x \in \mathbb{R}^2 \backslash \{0\} : x = c_1 \left[-A_{12}^{11} \; A_{11}^{11} \right]^{\mathrm{T}} + d_1 \left[-A_{12}^{12} \; A_{11}^{12} \right]^{\mathrm{T}}, \quad c_1, d_1 \in \mathbb{R}, c_1 d_1 \geq 0\}, \tag{18a}$$

$$C_2 = \{x \in \mathbb{R}^2 \backslash \{0\} : x = c_2 \left[A_{22}^{21} \; -A_{21}^{21} \right]^{\mathrm{T}} + d_2 \left[A_{22}^{22} \; -A_{21}^{22} \right]^{\mathrm{T}}, \quad c_2, d_2 \in \mathbb{R}, c_2 d_2 \geq 0\}, \tag{18b}$$

$$\hat{C}_1 = \{x \in \mathbb{R}^2 \backslash \{0\} : x = c_1 \left[-\hat{A}_{12}^{11} \; \hat{A}_{11}^{11} \right]^{\mathrm{T}} + d_1 \left[-\hat{A}_{12}^{12} \; \hat{A}_{11}^{12} \right]^{\mathrm{T}}, \quad c_1, d_1 \in \mathbb{R}, c_1 d_1 \geq 0\}, \tag{18c}$$

$$\hat{C}_2 = \{x \in \mathbb{R}^2 \backslash \{0\} : x = c_2 \left[\hat{A}_{22}^{21} \; -\hat{A}_{21}^{21} \right]^{\mathrm{T}} + d_2 \left[\hat{A}_{22}^{22} \; -\hat{A}_{21}^{22} \right]^{\mathrm{T}}, \quad c_2, d_2 \in \mathbb{R}, c_2 d_2 \geq 0\}. \tag{18d}$$

The pairs C_1 and C_2 represent the cones with tips at the origin by translating $D_{1,-1}^1 \cup D_{-1,1}^1$ and $D_{1,-1}^2 \cup D_{-1,1}^2$ respectively. The cones $\hat{C}_1$ and $\hat{C}_2$ are defined similarly for the incentivized payoff functions. Then, we have the following lemma.

Lemma 4. *Under Assumption 1, if* $q^1 = \begin{bmatrix} -\mathrm{row}_1(P^1)x^{p_1} \\ -\mathrm{row}_2(P^1)x^{p_2} \end{bmatrix}$, $\hat{C}_1 \subset C_1$, *and* $\hat{C}_2 \subset C_2$, *then* $E(\hat{G}) \subset E(G)$.

Proof. The proof is omitted due to space limitations. □

Then, we try to guarantee that $E(\hat{G})$ is both compact and asymptotically stable. First, we present the following corollary.

Corollary 5. *Under Assumption 1, if* $\det\mathcal{A}^{22} < 0$, *then there is no incentive function in the form of* (3) *such that* $E(\hat{G})$ *is both compact and stable.*

Proof. The proof is omitted due to space limitations. □

Based on Corollary 5, we have the following assumption.

Assumption 3. $\det\mathcal{A}^{22} > 0$ for the original game $G(J)$.

Note that the reason we do not consider the situation when $\det\mathcal{A}^{jk} = 0$ is that it has been excluded in Assumption 1. Here, we give the following lemma for the incentive design.

Lemma 6. *Under Assumption 1 for the game* $G(\hat{J})$, $E(\hat{G})$ *is compact if and only if* $\hat{C}_1 \cap \hat{C}_2 = \emptyset$.

Proof. This result can be obtained directly from Proposition 1 and Propositions 1 and 2 from [16]. □

The existence of $p^i(x), i \in \{1, 2\}$, is derived from the following theorem, based on Assumption 3 and Lemma 6.

Theorem 7. *Under Assumptions 1 and 3 for the game* $G(J)$, *there exist* $p^i(x), i \in \{1, 2\}$, *such that* $E(\hat{G})$ *is a compact subset of* $E(G)$, *and asymptotically stable.*

Proof. The proof is omitted due to space limitations. □

Remark 2. Here we show how we design $p^i(x)$ to fulfill our objective. In the case when A_{12}^{11} and A_{12}^{12} have the same sign, while A_{12}^{21} and A_{12}^{22} have different signs, we let $P_{12}^1 = 0$, P_{11}^1 be a number between 0 and $\frac{A_{12}^{11} A_{11}^{12}}{A_{12}^{12}} - A_{11}^{11}$, and P_{22}^2 small enough. Then, based on Lemma 4 by satisfying the condition of q^1, $E(\hat{G})$ is compact, connected, and asymptotically stable due to Lemma 6. The other cases can be handled similarly.

The design method in Remark 2 is not the only way, since Lemma 4 does not characterize a necessary condition for $E(\hat{G}) \subset E(G)$. We can also design $E(\hat{G})$ satisfying Lemma 6 to be a quadrilateral. If the 4 corners of quadrilateral $E(\hat{G})$ are in one convex subset of $E(G)$, then $E(\hat{G}) \subset E(G)$.

In the above analysis, we have 3 restrictions, including the concavity of $\hat{J}^{i1}$ with respect to x^i, the compactness and stability of $E(\hat{G})$, and $E(G) \subset E(\hat{G})$. At last, we discuss mathematically whether we can design appropriate incentive functions for given slopes under the concavity restriction. In this part we only consider A^{11} and A^{21}, since b^{11} and b^{21} do not influence stability and compactness. If we want the incentived best-response lines to have slopes $\frac{1}{a}$ and b, where $a, b \neq 0$, then for convenience we let

$$A^{11} + A^{21} = \begin{bmatrix} A & B \\ B & C \end{bmatrix} = \hat{A}^{11} + \hat{A}^{21} \triangleq \alpha \begin{bmatrix} -1 & a \\ a & x \end{bmatrix} + \beta \begin{bmatrix} y & b \\ b & -1 \end{bmatrix}, \tag{19}$$

with $\alpha, \beta > 0$, and solve α, β, x, y to satisfy the basic concavity condition. Now, we have the following proposition.

Proposition 8. *For given* a, b *in* (19), *there do not exist* $\alpha, \beta > 0$ *and* x, y *such that* (19) *holds if and only if* $(B > 0, a < 0, b < 0)$, $(B < 0, a > 0, b > 0)$, *or* $(B = 0, ab > 0)$.

Proof. The proof is omitted due to space limitations. □

Proposition 8 confirms the correctness of Theorem 7 from the side, since it shows that the rotations of the best-response lines are very flexible. If $A^{i1}, i \in \{1, 2\}$, are negative definite, we can solve the linear programming problem in the α, β space, since x, y can be expressed as a linear form of α, β. We can also find a sufficient condition for when the solution can be found as shown below.

Proposition 9. *For given* a, b *in* (19), *if* $0 < \frac{B}{a} < \min\{-A, \frac{-C}{a^2}\}$, $0 < \frac{B}{b} < \min\{-C, \frac{-A}{b^2}\}$, *or* $(B = 0, ab < 0)$, *there exist* $\alpha, \beta > 0$ *and* x, y *such that* $\hat{A}^{i1}, i \in \{1, 2\}$, *are negative definite.*

Proof. The proof is omitted due to space limitations. □

IV. Numerical Example

Consider two companies that manufacture different types of environmentally friendly cars. Both companies aim to maximize their profits by increasing production, which will also help in reducing carbon dioxide emissions. However, the market for these cars is limited, and an increase in production of cars leads to a decrease in the average profit per unit sale. Furthermore, excessive production that remains unsold can result in greater carbon dioxide emissions during the manufacturing process. As the system manager, we aim to promote healthy competition between the two companies through a tax/subsidy approach. Specifically, the payoff functions are assumed to be

$$J^{11}(x) = \frac{1}{2} x^{\mathrm{T}} \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 55 & 30 \end{bmatrix} x,$$

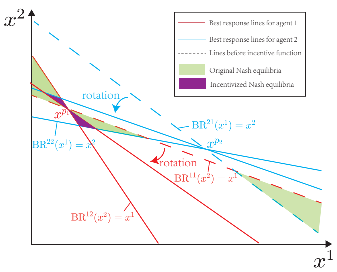

Fig. 5. Nash equilibrium before and after the incentive design in numerical example.

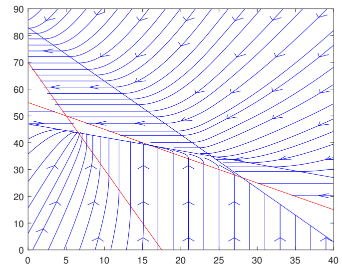

Fig. 6. Trajectories before the incentive design in numerical example. There is one stable connected part and one unstable connected part of $E(G)$.

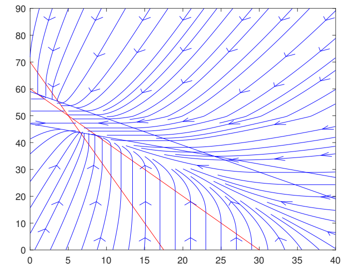

Fig. 7. Trajectories after the incentive design in numerical example. $E(\hat{G})$ is a compact subset of original $E(G)$, and asymptotically stable.

$$J^{12}(x) = \frac{1}{2}x^{\mathrm{T}}\begin{bmatrix} -4 & -1 \\ -1 & -3 \end{bmatrix}x + \begin{bmatrix} 70 & 20 \end{bmatrix}x,$$
$$J^{21}(x) = \frac{1}{2}x^{\mathrm{T}}\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}x + \begin{bmatrix} 35 & 83 \end{bmatrix}x,$$
$$J^{22}(x) = \frac{1}{2}x^{\mathrm{T}}\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}x + \begin{bmatrix} 65 & 47 \end{bmatrix}x.$$

The Nash equilibria for this example is illustrated in Fig. 5, where the green part represents the original Nash equilibrium set. However, it should be noted that the unbounded parts of $E(G)$ indicate that agents are caught in a dilemma where one of the agent has to maintain the current production.

By rotating the best-response lines of the agents with respect to their first payoff functions, a new set of Nash equilibrium set $E(\hat{G})$ can be obtained. Specifically, we design the incentive function as

$$p^1(x) = \frac{1}{2}x^{\mathrm{T}}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} 5 & -35 \end{bmatrix}x,$$

and $p^2(x) = -p^1(x)$. Then, the phase portrait of the original game $G(J)$ and the incentivized game $G(\hat{J})$ are given by Figs. 6 and 7, respectively.

Before implementing our tax/subsidy approach, note that $E(G)$ has one stable connected set and one unstable connected set. Although all trajectories ultimately converge to $E(G)$, those states starting near x^{D11} would initially move away from the right part of $E(G)$ before converging to the left connected part, as discussed in Remark 1. After applying our tax/subsidy approach, the trajectories will converge to a compact $E(\hat{G})$, which is also a subset of $E(G)$.

V. Conclusion

We proposed a zero-sum tax/subsidy approach, which can be used to modify or stablize the original Nash equilibrium set, and presented a necessary condition for stabilizing unstable or unbounded Nash equilibria in pseudo-gradient-based noncooperative dynamical systems with vector-valued payoff functions. Specifically, we first present a necessary and sufficient condition to ensure that the Nash equilibrium set is compact, and a sufficient condition for the Nash equilibrium to be asymptotically stable. After that, we give a necessary condition and relevant design method, to make the incentivized Nash equilibrium set a subset of the original Nash equilibrium, compact and stable. Furthermore, we discuss flexibility about the incentive parameters under different restrictions. Finally, we give an example to show our design method.

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