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Linear Stability of Plane Poiseuille Flow in the Sense of Lyapunov

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Abstract— **In this paper, we present a linear stability analysis formulation for a plane Poiseuille flow developed in a continuous time domain. Contrary to a conventional approach based on an eigenvalue analysis, which can only proof stability with respect to certain solutions that are assumed to be time harmonics modulated by an exponentially growing or decaying amplitude, the presented methodology does not make any assumptions on a solution form. By analyzing all time-varying solutions and not only the ones restricted to a specific functional form, the developed stability test provides a stronger condition with regard to the system stability. Stability analysis is performed by first casting the corresponding linearized partial differential equation into a partial integral equation (PIE) form, and subsequently employing a linear partial inequality (LPI) stability test, which searches for a corresponding Lyapunov function parameterized through polynomial expansions to prove or disprove stability. Stability results of the continuous-time formulation for the plane Poiseuille flow are compared with a traditional eigenvalue-based analysis, demonstrating that the developed methodology represents a stricter condition on stability.**

I. INTRODUCTION

Properties of a laminar-to-turbulent transition in fluid flows is a subject of strong interest due to its critical importance, both from a canonical perspective, and in practical applications. To analyze stability of fluid flows around an underlying laminar profile, a linear stability theory (LST) is often employed, which considers small perturbations to a laminar profile and investigates asymptotic stability of such perturbations in the linearized equations [1], [2]. Traditionally, ever since a pioneering work of Orr and Sommerfeld in the beginning of the 19th century [3], [4], linear stability analysis has been approached from an eigenvalue-based perspective. In an eigenvalue-based analysis, a solution of a certain form, specifically, in a form of an exponentially growing or a decaying harmonic perturbation is assumed. Correspondingly, this form of perturbations is substituted into the underlying governing equations (linearized Navier-Stokes equations), and the resulting eigenvalue problem, with the eigenvalues representing a complex frequency of the assumed harmonic perturbation, is solved. If an eigenvalue with a positive imaginary part is found, a system is pronounced unstable, otherwise it is stable.

While this approach provides important insights into stability properties of fluid flows, it is fundamentally restricted to an analysis of disturbances that assume a form of exponentially growing or decaying complex exponentials (normal modes). Thus, it represents a weaker condition for stability in comparison to, for example, a test, in which all possible solution forms, and not just the normal modes, could be analyzed. For example, a system, which is (linearly) stable with respect to the normal mode perturbations, can still be (linearly) unstable with respect to some other functional form of the perturbations. In applications, a laminar-to-turbulent transition typically occurs earlier (with respect to, for example, an increasing Reynolds number) than at a threshold predicted by an eigenvalue analysis within the LST theory [2], [5], [6]. While it can be attributed to a variety of reasons, including boundary imperfections [7], [8], transient amplifications [9], [10], non-linear effects [2], [11], a possibility of a *linear* instability with respect to a wider range of disturbances than previously analyzed should not be eliminated.

The current paper develops a new stability analysis framework for fluid flows, where stability with respect to an arbitrary time-varying disturbance in a continuous time domain is investigated. While continuous form of control through backstepping has been previously applied to stabilize linearly unstable perturbations in a Poiseuille flow at certain conditions [12], [13], a stability analysis of such flows in the sense of Lyapunov with respect to a wide range of parameters is lacking. To this end, we introduce a novel approach to analyze stability of the linearized Navier-Stokes equations that leverages a recently developed partial-integral equation (PIE) framework [14] that allows to apply a Lyapunov-based LPI stability test to a continuous form of equations (both in space, here represented by a wall-normal direction, and in time), so that neither a mode decomposition nor a spatial discretization are required to analyze stability. We show how to reformulate the corresponding two-dimensional LNS equation, Fourier-transformed in the streamwise direction, in a PIE format, extend the stability proof from [14] to this new (modified) form of the PIE equation, and demonstrate that the corresponding LPI stability condition yields a stronger stability test as compared to the conventional eigenvalue stability analysis.

II. PROBLEM FORMULATION

We consider a two-dimensional (2D) fluid flow between two parallel plates governed by the incompressible Navier-Stokes equations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned} \tag{1}$$

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where $\mathbf{u} = (u, v)$ is the velocity in x (streamwise) and y (wall-normal) directions, respectively, p is the pressure, and $Re = U_c \delta/\nu$ is the Reynolds number based on the characteristic velocity U_c and the channel half-width δ. With this definition, the velocities in (1) are normalized with U_c, and the spatial variables are normalized with δ. Boundary conditions are the no-slip at the plate walls $\mathbf{u}|_{y=\pm 1} = 0$. We decompose instantaneous variables into a sum of a corresponding laminar solution and a perturbation as $\mathbf{u} = \mathbf{U} + \mathbf{u}'$, $p = P + p'$, with the parallel mean flow assumption $\mathbf{U} = \{U(y), 0\}$, and linearize Eqs. (1) around the laminar solution to yield

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} = -\nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}', \qquad \nabla \cdot \mathbf{u}' = 0. \tag{2}$$

Taking a curl of the momentum equation and using the continuity equation allows one to eliminate the pressure from the system (2) and arrive at a single linear PDE to fully describe the 2D LNS operator

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - \frac{d^2 U}{d y^2} \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^4 \right] \psi = 0, \tag{3}$$

where the stream function $\psi(x, y, t)$ has been introduced, such that $u' = \partial \psi / \partial y, v' = -\partial \psi / \partial x$. Owing to the problem periodicity in streamwise direction, we perform a Fourier transform of (3) in x yielding, for a streamwise wave-number k, a one-dimensional PDE,

$$\left[\left(\frac{\partial}{\partial t} + i\, k\, U \right) \hat{\Delta}^2 - i\, k \frac{d^2 U}{d y^2} - \frac{1}{Re} \hat{\Delta}^4) \right] \hat{\psi} = 0, \tag{4}$$

where $\hat{\psi}(y, t)$ is the corresponding Fourier coefficient of the stream function for the wave-number k, i is an imaginary unit, and the one-dimensional differential operator $\hat{\Delta}^2 = \partial^2 / \partial y^2 - k^2$ with $\hat{\Delta}^4 = (\hat{\Delta}^2)^2$. Boundary conditions on $\hat{\psi}$ can be derived from the boundary conditions on $\mathbf{u}$ as

$$B(\hat{\psi}) : \hat{\psi}|_{\pm 1} = \hat{\psi}_y|_{\pm 1} = 0. \tag{5}$$

III. DEFINITION OF STABILITY

We introduce the following definition of an exponential stability in the sense of Lyapunov for the solutions of Eq. (4):

Definition 1: The PDE (4) with boundary conditions (5) is exponentially stable in L_2 in the sense of Lyapunov if there exist constants δ, K, $\gamma > 0$ such that for any $\hat{\psi}(y, 0) \in B(\hat{\psi})$: $\|\hat{\psi}(y, 0)\|_{L_2} < \delta$, a solution $\hat{\psi}(y, t)$ of the PDE (4) with (5) satisfies

$$\|\hat{\psi}(y, t)\|_{L_2} \leq K \|\hat{\psi}(y, 0)\|_{L_2} e^{-\gamma\, t}. \tag{6}$$

We also introduce a definition of a "normal-mode" (or "eigenvalue") - stability of the PDE (4):

Definition 2: The PDE (4) with boundary conditions (5) is said to be an "eigenvalue-stable" if there exist constants δ, K, $\gamma > 0$ such that for any solution of (4) of the form $\hat{\psi}(y, t) = \tilde{\psi}(y) \exp(-i\, \omega\, t)$, with $f(\pm 1) = f_y(\pm 1) = 0$, $\omega \in \mathbb{C}$, and $\|\tilde{\psi}(y)\|_{L_2} < \delta$, we have

$$\|\hat{\psi}(y, t)\|_{L_2} \leq K \|\tilde{\psi}(y)\|_{L_2} e^{-\gamma\, t}. \tag{7}$$

The following theorem proves that the eigenvalue stability is a weaker condition than a Lyapunov-based exponential stability.

Theorem 1: If the PDE (4) is stable according to the *Definition 1*, it is also stable according to the *Definition 2*. Conversely, if (4) is unstable according to the *Definition 2*, it is also unstable according to the *Definition 1*.

Proof: Suppose the PDE (4) is stable according to the *Definition 1*. Consider a solution to this PDE in the form $\hat{\psi}(y, t) = \tilde{\psi}(y) \exp(-i\, \omega\, t)$. According to the *Definition 1*, the Equation (6) is valid, therefore (7) is valid, and the PDE is stable according to the *Definition 2*.

Now suppose (4) is unstable according to the *Definition 2*. Then for some solution $\hat{\psi}(y, t) = \tilde{\psi}(y) \exp(-i\, \omega\, t)$ of this PDE, the condition (7) is violated. For the same solution, the condition (6) is also violated, and the system is unstable according to the *Definition 1*.

■

Theorem 2 shows that the eigenvalue stability is weaker (necessary but not sufficient) condition for the exponential stability of the system in the Lyapunov sense. In fact, a region of stability in the sense of Lyapunov is contained within the region of the eigenvalue stability, which is summarized in the following corollary.

Corollary 1: Denote $S_{Lyap}(k, Re)$ as the region of Lyapunov exponential stability (with respect to the PDE parameters k and Re) according to the *Definition 1*, and $S_{eig}(k, Re)$ as the region of eigenvalue stability according to the *Definition 2*. Then we have that

$$S_{Lyap}(k, Re) \subseteq S_{eig}(k, Re). \tag{8}$$

IV. STABILITY ANALYSIS

A. EIGENVALUE STABILITY

The eigenvalue stability analysis proceeds with substituting the assumed form of the solution $\hat{\psi}(y, t) = \tilde{\psi}(y) \exp(-i\, \omega\, t)$ into Eq. (4), upon which the equation transforms into a well-known Orr-Sommerfeld equation [3], [4]

$$\left[k\, U \left(\frac{d^2}{d y^2} - k^2 \right) - k \frac{d^2 U}{d y^2} + \frac{i}{Re} \left(\frac{d^2}{d y^2} - k^2 \right)^2 \right] \tilde{\psi} = \omega \left(\frac{d^2}{d y^2} - k^2 \right) \tilde{\psi}. \tag{9}$$

The PDE eigenvalue problem (9), with ω being the eigenvalue, is then solved numerically, typically via discretizing the wall-normal coordinate y with Chebyshev methods [5]. The resulting generalized matrix eigenvalue problem is then solved with iterative approaches. The eigenvalue stability methods are well established, and their detailed description can be found elsewhere [1], [2], [3], [4].

B. LYAPUNOV STABILITY

This section presents a novel approach developed in this paper to analyze stability of the linearized Navier-Stokes equation without invoking a normal-mode assumption on the solution form, by keeping a continuous formulation of the PDE (4) both in time and in a vertical coordinate y. Such an analysis is enabled through a transformation of the original PDE into an equivalent partial integral equation (PIE) representation and analyzing the stability of the PIE with the techniques based on linear partial inequality (LPI) approaches developed in previous papers [14]–[16].

1) REPRESENTATION OF A PDE AS A PARTIAL-INTEGRAL EQUATION: We now consider a representation of Eq. (4) in a partial-integral equation (PIE) form. PIE representation allows one to transform the boundary conditions into the equation dynamics, which, in turn, makes the formulation amenable to a stability analysis in a continuous framework [14]. For that, we first rewrite Eq. (4) in its state-space form

$$\hat{\Delta}^2\dot{\hat{\psi}} = ik\frac{\partial^2 U}{\partial y^2}\hat{\psi} - ik\,U\hat{\Delta}^2\hat{\psi} + \frac{1}{Re}\hat{\Delta}^4\hat{\psi}, \tag{10}$$

where we use the notation $\dot{\hat{\psi}} = \partial\,\hat{\psi}/\partial\,t$ for compactness.

Since Eq. (10) is in a complex form, while the PIE framework, including the corresponding open-source software PIETOOLS for manipulating PIEs [16], was previously developed for real-valued functions, we let $\hat{\psi} = \hat{\psi}_R + i\,\hat{\psi}_I$, and decompose Eq. (10) into a coupled system of equations for the real and imaginary components as

$$\begin{bmatrix}-\frac{1}{k^2} & 0\\ 0 & -\frac{1}{k^2}\end{bmatrix}\begin{bmatrix}\dot{\hat{\psi}}_{R\,yy}\\ \dot{\hat{\psi}}_{I\,yy}\end{bmatrix} + \begin{bmatrix}\dot{\hat{\psi}}_R\\ \dot{\hat{\psi}}_I\end{bmatrix} = \begin{bmatrix}-\frac{1}{k^2 Re} & 0\\ 0 & -\frac{1}{k^2 Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_{R\,yyyy}\\ \hat{\psi}_{I\,yyyy}\end{bmatrix} + \begin{bmatrix}\frac{2}{Re} & -\frac{U}{k}\\ \frac{U}{k} & \frac{2}{Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_{R\,yy}\\ \hat{\psi}_{I\,yy}\end{bmatrix} + \begin{bmatrix}-\frac{k^2}{Re} & \frac{1}{k}U_{yy} + k\,U\\ -\frac{1}{k}U_{yy} - k\,U & -\frac{k^2}{Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_R\\ \hat{\psi}_I\end{bmatrix}, \tag{11}$$

where $[]_{yy}$ and $[]_{yyyy}$ denote 2^{nd} and 4^{th} partial derivatives with respect to y. We define the set of boundary constraints as

$$B(\hat{\psi}_R, \hat{\psi}_I) : \hat{\psi}_R|_{\pm 1} = \hat{\psi}_{R\,y}|_{\pm 1} = \hat{\psi}_I|_{\pm 1} = \hat{\psi}_{I\,y}|_{\pm 1} = 0 \tag{12}$$

A solution to Eq. (11) satisfying the boundary constraints (12), $\hat{\psi}(y,t) = [\hat{\psi}_R,\, \hat{\psi}_I]^T \in H_4[-1,1] \cap B(\hat{\psi}_R, \hat{\psi}_I)$, where $H_4[-1,1]$ is the Sobolev space of functions with square-integrable derivatives up to 4^{th} order, will be denoted as a PDE state.

We seek a representation of the above system (11), with the boundary conditions (12), as a PIE. To formulate an equation as a PIE, we first need to define a fundamental (PIE) state $z(y,t)$, typically expressed as a vector of the highest spatial derivatives of the states entering the PDE, $z(y,t) = [z_R, z_I]^T = [\hat{\psi}_{R\,yyyy},\, \hat{\psi}_{I\,yyyy}]^T$ in the current case. Note that, by definition, the fundamental state $z(y,t) \in L_2[-1,1]$, where $L_2[-1,1]$ is a space of square-integrable functions on the domain $y \in [-1,1]$. The next step is to define a map between the PDE state $\hat{\psi}(y,t) = [\hat{\psi}_R,\, \hat{\psi}_I]^T$ (satisfying the boundary conditions) and the fundamental state $z(y,t)$ as

$$\hat{\psi}(y,t) = \mathcal{T} z(y,t). \tag{13}$$

This can be accomplished by multiple application of a fundamental theorem of calculus while taking into account the corresponding boundary conditions [14]. It can be shown that the operator $\mathcal{T}$ resulting from such a map can be written in a form of a partial-integral (PI) operator as $\mathcal{T} = \mathcal{T}_{\{R_0,R_1,R_2\}}$ defined as follows.

Definition 3: A partial integral operator $\mathcal{P} = \mathcal{P}_{\{R_0,R_1,R_2\}}$ is defined as a three-component operator acting on a fundamental state as

$$\mathcal{P}z(y,t) = \mathcal{P}_{\{R_0,R_1,R_2\}}z(y,t) = R_0(y)z(y,t) + \int_{-1}^{y} R_1(y,s)z(s,t)\,ds + \int_{y}^{1} R_2(y,s)z(s,t)\,ds, \tag{14}$$

where $\{R_0(y), R_1(y,s), R_2(y,s)\}$ are the matrices with the entries that are polynomials in the variables y and s [14].

Considering the left-hand side of Eq. (11), we also need to define an auxiliary map between the second-derivative state $\hat{\phi}_{yy}(y,t) = [\hat{\phi}_{R\,yy}, \hat{\phi}_{I\,yy}]^T$ and the fundamental state as $\hat{\phi}_{yy}(y,t) = \mathcal{T}_2\, z(y,t)$. In a general case, the maps $\mathcal{T}, \mathcal{T}_2$ are domain and boundary-conditions specific [14], [16]. For the current case of $y \in [-1,1]$ with homogeneous Dirichlet and Neumann boundary conditions on $\hat{\psi}_R$ and $\hat{\psi}_I$, Eq. (12), we find that

$$\begin{aligned}\mathcal{T}_{\{R_1\}} = \big[&-\frac{1}{24}y^3s^3 + \frac{1}{8}(y^3s - y^2s^2 + ys^3) + \frac{1}{12}(y^3 - s^3)\\ &+ \frac{1}{4}(ys^2 - y^2s) - \frac{1}{8}(y^2 - y\,s + s^2) + \frac{1}{24}\big]\, I,\\ \mathcal{T}_{\{R_2\}} = \big[&-\frac{1}{24}y^3s^3 - \frac{1}{8}(y^3s - y^2s^2 + ys^3) + \frac{1}{12}(y^3 - s^3)\\ &+ \frac{1}{4}(ys^2 - y^2s) - \frac{1}{8}(y^2 - ys + s^2) + \frac{1}{24}\big]\, I,\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{T}_{2\{R_1\}} &= \big[-\frac{1}{4}ys^3 + \frac{3}{4}ys - \frac{1}{4}s^2 + \frac{1}{2}y - \frac{1}{2}s - \frac{1}{4}\big]\, I,\\ \mathcal{T}_{2\{R_2\}} &= \big[-\frac{1}{4}ys^3 + \frac{3}{4}ys - \frac{1}{4}s^2 - \frac{1}{2}y + \frac{1}{2}s - \frac{1}{4}\big]\, I,\end{aligned} \tag{16}$$

$\mathcal{T}_{\{R_0\}} = \mathcal{T}_{2\,\{R_0\}} = 0$, with I being a 2×2 identity matrix. Substituting the corresponding mappings with the operators $\mathcal{T}$, $\mathcal{T}_2$ defined by Eqs. (15), (16) into Eq. (11), we obtain its equivalent PIE representation as

$$\mathcal{M}\,\dot{z} = \mathcal{A}\,z, \tag{17}$$

where the solution vector $z(y,t) = [z_R, z_I]^T \in L_2[-1,1]$, and thus is free of boundary conditions. The operators $\mathcal{M}$, $\mathcal{A}$ in Eq. (17) are given by

$$\mathcal{M} = -\frac{1}{k^2}\mathcal{T}_2 + \mathcal{T}, \tag{18}$$

$$\mathcal{A} = \left[-\frac{1}{k^2 Re} + \frac{2}{Re}\mathcal{T}_2 - \frac{k^2}{Re}\mathcal{T}\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \left[\frac{U}{k}\mathcal{T}_2 - \frac{1}{k}U_{yy}\mathcal{T} - k\,U\mathcal{T}\right]\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{19}$$

It can be seen that $\mathcal{M}$ is a PI operator by construction, while $\mathcal{A}$ is also a PI operator, as long as the mean velocity profile $U(y)$ is polynomial in y. In the current analysis, $U(y)$ is given by $U(y) = 1-y^2$, corresponding to a Poiseuille flow between two parallel plates, normalized with the centerline velocity U_c.

2) STABILITY ANALYSIS USING LINEAR PARTIAL INEQUALITIES: Similarly to the definition of the exponential stability of the PDE system (4) stated in the *Definition 1*, we can define the exponential stability of the PIE system (17).

Definition 4: The PIE system (17) is exponentially stable in L_2 if there exist constants δ, K, $\gamma > 0$ such that for any $\|z(y,0)\|_{L_2} < \delta$, a solution $z(y,t)$ of the PIE satisfies

$$\|\mathcal{M}z(y,t)\|_{L_2} \leq K\|\mathcal{M}z(y,0)\|_{L_2}e^{-\gamma t}. \tag{20}$$

Exponential stability of PIE is tested by defining and verifying a feasibility of a linear partial inequality (LPI) encapsulated in the following theorem.

Theorem 2: Suppose there exist δ, $\beta, \sigma > 0$, and a self-adjoint coercive PI operator $\mathcal{P}_{\{R_0,R_1,R_2\}}$ such that $\mathcal{P} = \mathcal{P}^*$, $\langle z, \mathcal{P}z\rangle_{L_2} \geq \beta\|z\|^2_{L_2}$, and

$$\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A} < -\sigma\mathcal{M}^*\mathcal{M}, \tag{21}$$

where $\mathcal{M}$, $\mathcal{A}$ are as defined in Eqs. (18), (19). Then any solution of the PIE system (17) with $\|z(y,0)\|_{L_2} < \delta$ satisfies

$$\|\mathcal{M}z(y,t)\|_{L_2} \leq \left(\frac{\xi_M}{\beta}\right)^{1/2}\|\mathcal{M}z(y,0)\|_{L_2}e^{-\sigma/(2\xi_M)t}, \tag{22}$$

where $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$.

Proof: Suppose $z(y,t)$ solves the PIE system (17) for some $z(y,0)$ satisfying $\|z(y,0)\|_{L_2} < \delta$. Consider the candidate Lyuapunov function defined as

$$V(z) = \langle \mathcal{M}z, \mathcal{P}\mathcal{M}\,z\rangle_{L_2} \geq \beta\|\mathcal{M}\,z\|^2_{L_2}. \tag{23}$$

The derivative of V along the solution trajectory $z(y,t)$ is

$$\begin{aligned}\dot{V}(z) = \langle \mathcal{M}\dot{z}, \mathcal{P}\mathcal{M}z\rangle_{L_2} + \langle \mathcal{M}z, \mathcal{P}\mathcal{M}\,\dot{z}\rangle_{L_2} = \\ \langle \mathcal{A}z, \mathcal{P}\mathcal{M}\,z\rangle_{L_2} + \langle \mathcal{M}z, \mathcal{P}\mathcal{A}\,z\rangle_{L_2} = \\ \langle z, (\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A})\,z\rangle_{L_2} \leq -\sigma\|\mathcal{M}z\|^2_{L_2}.\end{aligned} \tag{24}$$

Applying Gronwall-Bellman lemma to (23), (24), and using $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$, proves the theorem. ■

We now prove the main result of the current paper, namely the equivalence of the PIE stability condition tested by Theorem 2, and the stability of the original 2D LNS PDE system.

Theorem 3: Suppose there exist $\kappa, \beta, \sigma > 0$, and a self-adjoint coercive PI operator $\mathcal{P}_{\{R_0,R_1,R_2\}}$ such that $\mathcal{P} = \mathcal{P}^*$, $\langle z, \mathcal{P}z\rangle_{L_2} \geq \beta\|z\|^2_{L_2}$, and

$$\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A} < -\sigma\mathcal{M}^*\mathcal{M}, \tag{25}$$

where $\mathcal{M}$, $\mathcal{A}$ are as defined in Eqs. (18), (19). Then there exists a constant $C > 0$, such that any solution of the PDE system (4) with boundary conditions (5) and $\|\hat{\psi}(y,0)\|_{L_2} < \kappa$ satisfies

$$\|\hat{\psi}(y,t)\|_{L_2} \leq C\|\hat{\psi}(y,0)\|_{L_2}e^{-\sigma/(2\xi)t}. \tag{26}$$

Proof: Denote $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$, $\xi_T = \|\mathcal{T}\|_{\mathcal{L}(L_2)}$, $\xi_{T_2} = \|\mathcal{T}_2\|_{\mathcal{L}(L_2)}$. From Theorem (2), we have that, as long as $\|z(y,0)\|_{L_2} < \delta$,

$$\|\mathcal{M}z(y,t)\|_{L_2} \leq \left(\frac{\xi_M}{\beta}\right)^{1/2}\|\mathcal{M}z(y,0)\|_{L_2}e^{-\sigma/(2\xi)t}. \tag{27}$$

Considering that $\hat{\psi}(y,t) = \mathcal{T}z(y,t)$, and $\mathcal{T} = \mathcal{M} + \mathcal{T}_2/k^2$, we have

$$\begin{aligned}\|\hat{\psi}(y,t)\|_{L_2} &\leq \|\mathcal{M}z(y,t)\|_{L_2} + \frac{1}{k^2}\|\mathcal{T}_2\,z(y,t)\|_{L_2} \\ &\leq \|\mathcal{M}z(y,t)\|_{L_2} + \frac{\xi_{T_2}}{k^2}\,\|z(y,t)\|_{L_2}.\end{aligned} \tag{28}$$

Furthermore, we can write

$$\|\mathcal{M}z(y,t)\|_{L_2} \leq \xi_M\|z(y,t)\|_{L_2} = C_M\|z(y,t)\|_{L_2}, \tag{29}$$

where $C_M > 0$ is some constant such that $C_M \leq \xi_M$, from where we have that

$$\|z(y,t)\|_{L_2} = \frac{\|\mathcal{M}z(y,t)\|_{L_2}}{C_M}. \tag{30}$$

Continuing with Eq. (28), we have

$$\begin{aligned}\|\hat{\psi}(y,t)\|_{L_2} &\leq \|\mathcal{M}z(y,t)\|_{L_2}\left(1 + \frac{1}{k^2}\frac{\xi_{T_2}}{C_M}\right) \\ &\leq \left(\frac{\xi_M}{\beta}\right)^{1/2}\left(1 + \frac{1}{k^2}\frac{\xi_{T_2}}{C_M}\right)\|\mathcal{M}z(y,0)\|_{L_2}e^{-\sigma/(2\xi)t}.\end{aligned} \tag{31}$$

Considering an estimate similar to Eq. (30) for the $\mathcal{T}$ operator, we have that

$$\|z(y,t)\|_{L_2} = \frac{\|\mathcal{T}z(y,t)\|_{L_2}}{C_T}, \tag{32}$$

with $0 < C_T \leq \xi_T$. Applying Eq. (32) to $\hat{\psi}(y,0) = \mathcal{T}z(y,0)$ and substituting into Eq. (31), we have that, as long as $\|\hat{\psi}(y,0)\|_{L_2} < \kappa$, with $\kappa = C_T\,\delta$,

$$\|\hat{\psi}(y,t)\|_{L_2} \leq \left(\frac{\xi_M}{\beta}\right)^{1/2}\left(1 + \frac{1}{k^2}\frac{\xi_{T_2}}{C_M}\right)\left(\frac{\xi_M}{C_T}\right)\|\hat{\psi}(y,0)\|_{L_2}e^{-\sigma/(2\xi)t}, \tag{33}$$

which proves the theorem. ■

V. RESULTS

A. VERIFICATION OF THE PIE SYSTEM

We verify our representation of the 2D LNS equation as a PIE through the Method of Manufactured Solutions (MMS). With MMS, we construct an analytical solution to the PDE (11), with the boundary conditions (12), by first specifying the form of the solution as

$$\hat{\psi}_R(y,t) = f(y)e^{\alpha t},\ \hat{\psi}_I(y,t) = g(y)e^{\alpha t}, \tag{34}$$

and substituting this form of the solution into Eq. (11) to yield

$$\begin{bmatrix}-\frac{1}{k^2} & 0\\ 0 & -\frac{1}{k^2}\end{bmatrix}\begin{bmatrix}\dot{\hat{\psi}}_{R\,yy}\\ \dot{\hat{\psi}}_{I\,yy}\end{bmatrix}+\begin{bmatrix}\dot{\hat{\psi}}_{R}\\ \dot{\hat{\psi}}_{I}\end{bmatrix}=$$
$$\begin{bmatrix}-\frac{1}{k^2 Re} & 0\\ 0 & -\frac{1}{k^2 Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_{R\,yyyy}\\ \hat{\psi}_{I\,yyyy}\end{bmatrix}+\begin{bmatrix}\frac{2}{Re} & -\frac{U}{k}\\ \frac{U}{k} & \frac{2}{Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_{R\,yy}\\ \hat{\psi}_{I\,yy}\end{bmatrix}+$$
$$\begin{bmatrix}-\frac{k^2}{Re} & \frac{1}{k}U_{yy}+k\,U\\ -\frac{1}{k}U_{yy}-k\,U & -\frac{k^2}{Re}\end{bmatrix}\begin{bmatrix}\hat{\psi}_{R}\\ \hat{\psi}_{I}\end{bmatrix}+\begin{bmatrix}Q_R(y,t)\\ Q_I(y,t)\end{bmatrix}, \quad (35)$$

with the forcing terms $Q_R(y,t)$, $Q_I(y,t)$ expressed as

$$Q_R(y,t) =[\frac{1}{k^2 Re} f_{yyyy} - (\frac{\alpha}{k^2}+\frac{2}{Re}) f_{yy} + (\alpha+\frac{k^2}{Re}) f - (\frac{U_{yy}}{k}+kU) g + \frac{U}{k} g_{yy}] e^{\alpha t}, \quad (36)$$

$$Q_I(y,t) =[\frac{1}{k^2 Re} g_{yyyy} - (\frac{\alpha}{k^2}+\frac{2}{Re}) g_{yy} + (\alpha+\frac{k^2}{Re}) g + (\frac{U_{yy}}{k}+kU) f - \frac{U}{k} f_{yy}] e^{\alpha t}. \quad (37)$$

The functions $f(y)$, $g(y)$ in (34) are chosen as the polynomials satisfying the boundary conditions (12) as

$$f(y) = (-\frac{1}{2}y^5 - 2y^4 + y^3 + 4y^2 - \frac{1}{2}y - 2),$$
$$g(y) = (-4y^5 - \frac{3}{2}y^4 + 8y^3 + 3y^2 - 4y - \frac{3}{2}). \quad (38)$$

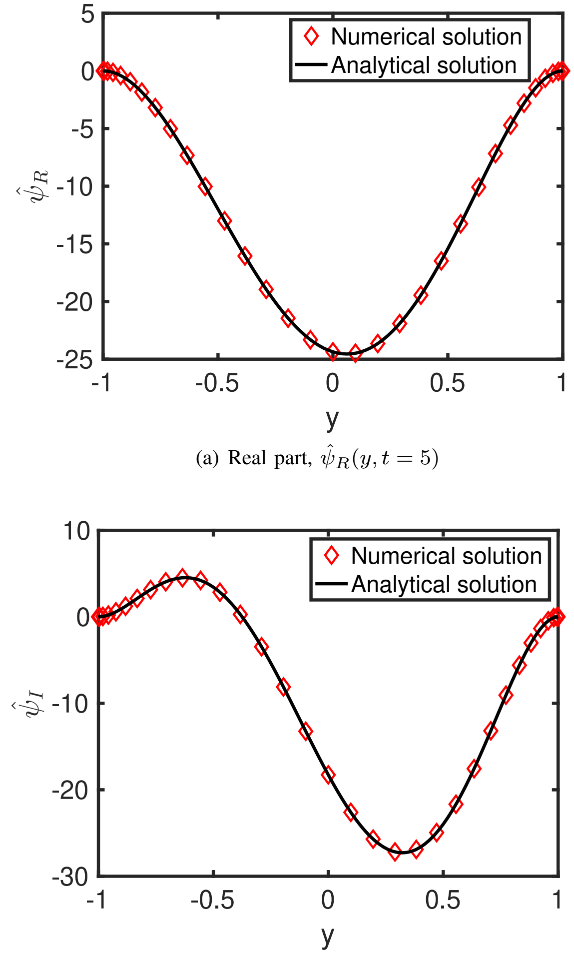
The PDE equation (35) with the boundary conditions (12) is transformed into a PIE as

$$\mathcal{M}\,\dot{z} = \mathcal{A}\,z + Q(y,t), \quad (39)$$

where $Q(y,t) = [Q_R(y,t), Q_I(y,t)]^T$, with $\mathcal{M}$, $\mathcal{A}$, $Q_R(y,t)$ and $Q_I(y,t)$ given by Eqs. (18), (19), (36) and (37), respectively. The analytical solution to the PIE equation constructed with MMS is given by

$$z_R(y,t) = f_{yyyyy} e^{\alpha t},\ z_I(y,t) = g_{yyyyy} e^{\alpha t}. \quad (40)$$

PIE system (39) with the initial conditions $z_R(y,0) = f_{yyyyy}, z_I(y,0) = g_{yyyyy}$ was numerically solved by a recently developed computational methodology for solving partial-integral equation systems implemented in the open-source numerical solver PIESIM [17], which is a part of the PIE analysis software PIETOOLS [16]. In PIESIM, the PIE state variables $z(y,t)$, together with the forcing functions $Q(y,t)$, are decomposed into a series of Chebyshev polynomials $z(y,t) = \sum_{i=1}^N a_k(t) T_k(y)$, $Q(y,t) = \sum_{i=1}^N q_k(t) T_k(y)$ with $a_k(t), q_k(t)$ being the vector-valued Chebyshev coefficients, and $T_k(y)$ are the Chebyshev polynomials of the first kind [18]. The actions of the PI operators $\mathcal{M}$, $\mathcal{A}$ on the Chebyshev polynomials $T_k(y)$ is evaluated analytically using recursive relations for multiplication and integration of Chebyshev polynomials [18]. This allows one to obtain a system of the ODE equations for the Chebyshev coefficients, which can be integrated in time analytically or using traditional time-stepping techniques [17]. Once the PIE solution $z(y,t)$ is obtained, the PDE solution $\hat{\psi}(y,t)$ is reconstructed via a PIE-to-PDE map $\hat{\psi}(y,t) = \mathcal{T} z(y,t)$, discretized in Chebyshev space using the same techniques as the ones employed for the PIE system.



(a) Real part, $\hat{\psi}_R(y, t = 5)$

(b) Imaginary part, $\hat{\psi}_I(y, t = 5)$

Fig. 1: Verification of the PIE system for 2D LNS with PIESIM [17].

Verification of the 2D LNS equation solution in a PIE form using the second-order backward differentiation scheme for time advancement with the time step $\Delta t = 10^{-3}$, $N = 32$, $U(y) = (1-y^2)$ and $(k, Re, \alpha) = (1, 180, 0.5)$ is presented in Fig. 1 at $t = 5$.

B. *STABILITY ANALYSIS USING LPIS*

We perform stability analysis of the two-dimensional linearized Navier-Stokes equations system in its continuous spatio-temporal formulation by testing feasibility of the LPI condition stated in Theorem 2. The feasibility test is accomplished via an open-source MATLAB-based software PIETOOLS developed for analysis and manipulation of the PIE equations [16]. In PIETOOLS, the feasibility problem is formulated as a convex optimization problem, which enforces a positivity of a PI operator parameterized by polynomial functions [14], [16]. Once formulated, a convex optimization problem is solved via a semi-definite programming solver SeDuMi of the package YALMIP [19].

Figure 3 documents the results of the stability analysis of the 2D LNS equation for a plane Poiseuille flow in the sense of Lyapunov as compared to the eigenvalue-based method [2], [5]. The neutral stability curve (that is the curve separating a region of stability from the region of instability) is plotted for each method. The results are consistent with the *Corollary 1* which indicates that the stability region of the system in the sense of Lyapunov is contained within the region identified by the eigenvalue test.

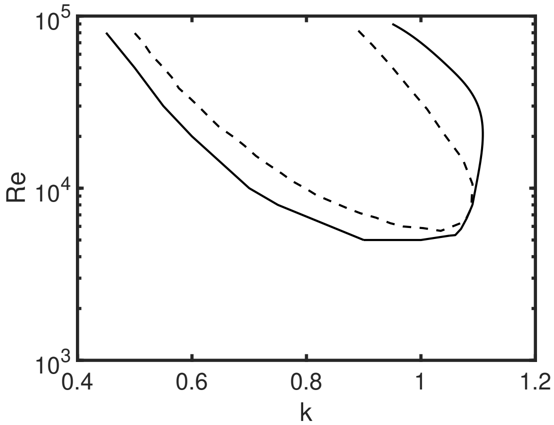


Fig. 2: Stability of 2D LNS equation via LPIs (solid line) compared to an eigenvalue stability for a plane Poiseuille flow (dashed line). Unstable regions are inside the curves. Plot is in agreement with *Corollary 1*: $S_{Lyap}(k, Re) \subseteq S_{eig}(k, Re)$.

VI. CONCLUSIONS

The current paper presents a methodology for a linear stability analysis of the fluid flow between two parallel plates capable of identifying the region of exponential stability in the sense of Lyapunov based on a continuous form of the governing equations. Navier-Stokes equations in a two-dimensional formulation are first linearized around the mean velocity profile and then Fourier-transformed in the streamwise direction to arrive at, for each streamwise wave-number, a linear fourth-order PDE equation in time and a wall-normal coordinate. As opposed to a conventional eigenvalue-based analysis that is restricted to certain forms of the solution, stability of the 2D LNS PDE equation is performed in a continuous setting, employing the results from the optimal control theory. For that, the PDE together with the boundary conditions is first transformed into an equivalent partial integral equation (PIE) representation. Within the PIE, the boundary conditions are implicitly embedded into the form of the partial integral operator, thus making the PIE solution reside in an L_2 space, free of boundary conditions. The PIE form makes it possible to apply a stability test in an infinite-dimensional setting by testing a feasibility of a linear partial inequality. We have proved that the PIE stability implies the stability of the underlying PDE system, and vice versa. The presented stability analysis is compared with the eigenvalue-based method for a plane Poiseuille flow. As expected, it is shown that the region of exponential stability of the system as identified by the LPI test is narrower than the one predicted by approximation methods based on an eigenvalue analysis. In fact, the eigenvalue test provides a necessary condition, why the LPI test provides a sufficient condition for stability. Further work will include a synthesis of the infinite-dimensional stabilizing controllers for the given problem, enabled by a reformulation of the 2D LNS PDE problem as a partial integral equation. In fact, a stabilizing controller synthesis can be posed as an LPI feasibility problem [15], and the developed PDE to PIE transformation for the parallel shear flow equations will serve as a departure point to pursue these efforts in the near future.

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