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A Generalized-Moment Method for Control-Affine Ensemble Systems

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***Abstract*— Controlling dynamic ensemble systems is an essential yet challenging step to enable diverse applications in science and engineering. In this paper, we present a generalized moment method that gives rise to a moment representation of the control-affine ensemble system. The induced moment system is equipped with a banded structure that is beneficial to conducting systems-theoretic analysis and control design for ensemble systems. In addition, we introduce a Lie algebraic technique for exact bilinearization of nonlinear ensemble systems. This transformation provides a unified paradigm for studying highly intricate nonlinear ensemble systems through the associated bilinear moment systems. To demonstrate the applicability of the proposed method, we present numerical examples involving the control of nonlinear ensemble systems.**

I. INTRODUCTION

Dynamic ensembles are population systems consisting of collections of structurally similar dynamic units. Such systems are prevalent in nature and engineering fields, and the ability to finely control their collective behavior is critical to enable cutting-edge applications, such as nuclear magnetic resonance [1], swarm behavior control [2], systems neuroscience [3], [4], and robotics [5]. The fundamental challenge in these applications lies in their under-actuated and large-scale nature, i.e., using a single broadcast input to address the entire population. To tackle this challenge, moment-based methods have been proposed [6]–[9], which established a dynamic connection between an ensemble system and its moment system. These developments facilitate intricate systems-theoretic analysis and control designs for ensemble systems. However, they were limited to some classes of linear and bilinear ensemble systems and compelled the need of devising new methods for accommodating general ensemble systems.

In this paper, we present a generalized-moment method for addressing ensemble control problems. We start our analysis with linear ensembles governed by time-invariant parameter-dependent vector fields, in which we establish a systematic approach to construct the moment transform resulting in structured moment systems. Adopting this construction, we introduce a Lie algebraic technique providing conditions for exact bilinearization of nonlinear ensemble systems. This bilinearization provides a unified paradigm for studying highly complex nonlinear ensemble systems through the associated bilinear moment systems. We present numerical examples, involving the control of nonlinear ensemble systems, to demonstrate the applicability of the proposed generalized-moment method.

The paper is organized as follows. In Section II, we introduce the notion of ensemble moments and present a systematic approach to construct a generalized moment system associated with an ensemble system defined on a Hilbert space. In Section III, we derive a sufficient condition for exact bilinearization of nonlinear ensemble systems. In Section IV, we present examples to demonstrate the application of the developed generalized-moment method to ensemble control design.

II. GENERALIZATION MOMENT METHODS

In this paper, we focus on studying the control-affine ensemble system, indexed by the parameter β, of the form

$$\frac{d}{dt}x(t,\beta) = f(\beta, x(t,\beta)) + \sum_{j=1}^{m} u_j(t) g_j(\beta, x(t,\beta)), \quad (1)$$

where $x(t,\cdot)$ is the state variable defined on a separable Hilbert space $\mathcal{H}$, $\beta \in K \subset \mathbb{R}$ lies in a compact set K, $u \in L^\infty([0,T],\mathbb{R}^m)$ is the control input, and f and g_j are smooth vector fields defined on $\mathcal{H}$. To tackle this nonlinear ensemble system for the purpose of control design, we present a moment-based method by introducing generalized-moments defined with respect to a basis of $\mathcal{H}$. Leveraging the proposed moment method associated with the nonlinear ensembles, we transform the nonlinear ensemble system into a more tractable system and utilize the new system for ensemble control design.

A. Ensemble-moments under different bases

In a separable Hilbert space $\mathcal{H}$, the ensemble state $x(t,\beta)$ can be represented as formal linear combination of a countable basis $\mathcal{B}_\psi = \{\psi_k(\beta)\}_{k=0}^{\infty}$, i.e., $x(t,\beta) = \sum_{k=0}^{\infty} m_k(t)\psi_k(\beta)$ [10]. By utilizing this representation, the ensemble system as in (1) can be transformed into an infinite-dimensional system free of parameter β, which is referred to as the *moment system*.

Definition 1: **(Generalized ensemble-moments)** Given an ensemble system as in (1) with the state variable $x(t,\cdot) = (x_1(t,\cdot),\ldots,x_n(t,\cdot))^\intercal$, we define the *generalized k^{th} ensemble-moment* $m_k(t) = (m_{k1}(t),\ldots,m_{kn}(t))^\intercal$ with respect to a basis $\mathcal{B}_\phi = \{\phi_k(\beta)\}_{k=0}^{\infty}$ of $\mathcal{H}$ as

$$m_{ki}(t) = \langle \phi_k(\cdot), x_i(t,\cdot)\rangle, \quad (2)$$

where the expression $\langle u, v\rangle$ denotes the inner product of u and v. The ensemble-moment sequence is thus an infinite sequence denoted by $m(t) = (m_0^\intercal(t), m_1^\intercal(t),\ldots)^\intercal$.

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Remark 1: For instance, when $\mathcal{H} = L^2(K, \mathbb{R}^n)$, the inner product is defined as the integral over the subset K, i.e. $\langle u, v \rangle = \int_K u^\intercal \cdot v \, d\mu$. This definition can be further extended if we consider the generalized inner product, $m_{ki}(t) = \langle \psi_k(\cdot), x_i(t, \cdot) \rangle_w = \int_K \psi_k(\cdot) x_i(t, \cdot) w(\cdot) \, d\mu$, where $w(\cdot)$ is a weight function on the subset K.

Orthogonal bases are commonly used in a Hilbert space to approximate a given element. Thus, we consider an orthogonal basis $\mathcal{B}_p = \{p_k(\beta)\}$ and the associated weighted ensemble-moments, i.e. $m_k(t) = \langle p_k(\cdot), x(t, \cdot) \rangle_w$.

Definition 2: **(Orthogonal Polynomial Sequence)** A sequence of polynomials $\{p_k(\beta)\}_{k=0}^{\infty}$ is said to be *an orthogonal polynomial sequence (OPS)* with respect to a (non-negative) weight function $w(\beta)$ if

$$\langle p_m(\cdot), p_n(\cdot) \rangle_w = K_n \delta_{mn}, \tag{3}$$

where $K_n \neq 0$ and δ_{mn} is the Kronecker's delta function. If $K_n = 1$, then $\{p_k(\beta)\}$ is said to be an *orthonormal* polynomial sequence.

Remark 2: From the theory of orthogonal polynomials, an OPS is defined satisfying $\Psi_w [p_m(\cdot) p_n(\cdot)] = K_n \delta_{mn}$, where Ψ_w is a linear functional satisfying $\Psi_w[\beta^k] = \mu_k$, and μ_k is a given sequence called *moment* [11]. A necessary and sufficient condition for an OPS to exist for Ψ_w is if the linear functional Ψ_w is *quasi-definite*, i.e., the determinant of moment Hankel matrix is non-zero for all orders. Moreover, if Ψ_w is *positive-definite*[1], then by the Riesz representation theorem, we can define an inner product with a proper weight function $w(\beta)$, which is consistent with the definition of orthogonality stated in Definition 2 in an inner product space.

The generalized moments defined in (2) can be directly applied to ensemble dynamics with nonlinear parameterization. In the following, we will start with analyzing linear ensemble dynamics with nonlinear parameterization to illustrate the main idea of utilizing generalized moments to transform the ensemble system as in (1) to an associated moment system where ensemble analysis and ensemble control design become tractable.

B. Moment dynamics of linear ensemble systems with nonlinear parameterization

To fix ideas, we consider a linear ensemble system with nonlinear parameterization, given by

$$\frac{d}{dt} x(t, \beta) = f(\beta) A(t) x(t, \beta) + g(\beta) B(t) u(t), \tag{4}$$

where $\beta \in K \subset \mathbb{R}$, $f(\cdot)$, $g(\cdot) \in C^\infty(K)$ with f being injective; $x(t, \cdot) \in L^2(K, \mathbb{R}^n)$, $u \in L^\infty([0, T], \mathbb{R}^m)$, $A(t) \in \mathbb{R}^{n \times n}$, and $B(t) \in \mathbb{R}^{n \times m}$. We establish an orthogonal basis starting by choosing a set of functions $\mathcal{B}_f$ that consists of all the monomials f^k, i.e., $\mathcal{B}_f = \{f^0(\beta), f^1(\beta), f^2(\beta), \ldots\}$. By invoking Stone-Weierstrass theorem, we observe that $\mathcal{B}_f$ is a dense set in the function space $L^2(K, \mathbb{R})$, since $\mathcal{B}_f$ separates points in K. Then, by applying the Gram-Schmidt orthogonality procedure with respect to $\eta = f(\beta)$ (see Appendix V-A), we obtain the corresponding orthogonal basis $\mathcal{B}_{\psi|_f} = \{\psi_0, \psi_1, \ldots\}$ with respect to the weight functional Ψ_w. Furthermore, the constructed OPS satisfies the three terms recursive relation, i.e. there exist real-valued sequences $\{a_k\}, \{b_k\}, \{c_k\}$ such that $\psi_k(\eta)$ satisfies $\eta \psi_k(\eta) = a_k \psi_{k+1}(\eta) + b_k \psi_k(\eta) + c_k \psi_{k-1}(\eta)$, for all $k = 0, 1, \ldots$ with $c_{k+1} > 0$ and $\psi_{-1}(\eta) = 0$ [11]. This recursive property results in a banded structure for the moment dynamic associated with the ensemble system (1), which is preferable for the ensemble control design.

Theorem 3: **(Moment system with banded structure)** Given the linear ensemble system in (4), where f is injective. Consider the k^{th} ensemble-moment defined as in (2) with respect to an orthogonal basis $\{\psi_k\}$, then the corresponding moment system is a linear system with the system matrix $\hat{A}$ in a banded form, give by

$$\frac{d}{dt} m(t) = \hat{A} m(t) + \hat{B} u(t), \tag{5}$$

where

$$\hat{A} = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & c_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \otimes A, \quad \hat{B} = \begin{pmatrix} \langle \psi_0, g \rangle_w \\ \langle \psi_1, g \rangle_w \\ \vdots \end{pmatrix} \otimes B,$$

in which $\otimes$ denotes the Kronecker product.

Proof: The time derivative of the kth ensemble-moment is given by

$$\begin{aligned} \frac{d}{dt} m_k(t) &= \langle \psi_k(\eta), \frac{d}{dt} x(t, \eta) \ \rangle_w \\ &= A \langle \eta \psi_k(\eta), x(t, \eta) \rangle_w + B u(t) \langle \psi_k(\eta), g(\eta) \rangle_w \\ &= A \Big(a_k m_{k+1}(t) + b_k m_k(t) + c_k m_{k-1}(t) \Big) \\ &\quad + B u(t) \langle \psi_k, g \rangle_w, \end{aligned}$$

which results in the banded structure shown above. ■

From Appendix V-A, the coefficients of the recursive relation to the constructed OPS $\mathcal{B}_{\psi|_f}$ are given by $a_k = 1$, $b_k = d_{k+1}$, $c_k = l_{k+1}$. If we further normalize the constructed OPS $\mathcal{B}_{\psi|_f}$ and denote as $\mathcal{B}_{\tilde{\psi}|_f}$, i.e., $\mathcal{B}_{\tilde{\psi}|_f} = \{\tilde{\psi}_k = \psi_k / \|\psi_k\|\}$, then, from Theorem 3, the ensemble-moments are given by

$$m_{ki} = \langle \tilde{\psi}_k(\cdot), x_i(t, \cdot) \rangle_w, \tag{6}$$

with the sequence $\{a_k\}, \{b_k\}, \{c_k\}$ in $\hat{A}$ given by

$$a_k = \tfrac{\|\psi_{k+1}\|}{\|\psi_k\|}, \quad b_k = d_{k+1}, \quad c_k = l_{k+1} \tfrac{\|\psi_{k+1}\|}{\|\psi_k\|}.$$

For instance, the Legendre polynomials can be viewed as the set $\mathcal{B}_{\psi|_{\beta^k}}$ obtained from Gram-Schmidt process to the monomials β^k defined on $[-1, 1]$ with respect to a constant weight function $w \equiv 1$.

As illustrated above, the chosen basis determines the structure of the moment dynamics associated with the ensemble system. For instance, if the function $g(\cdot)$ is a linear combination of the basis functions, the resulting $\hat{B}$ is a sparse matrix. We take the following example to illustrate this idea.

[1]We say Ψ_w is *positive-definite* if the moment sequence μ_k is real value and the determinant of moment Hankel matrix is positive for all orders.

Example 1: Let us consider an ensemble of the form,

$$\frac{d}{dt}x(t,\beta) = \cos(\beta)x(t,\beta) + u_1(t), \beta \in [0,\pi];$$

if we choose $\mathcal{B}_f = \{1, \cos(\beta), \cos^2(\beta), \ldots\}$, we obtain

$$\frac{d}{dt}m_k(t) = m_{k+1}(t) + u_1(t)\langle \cos^k(\beta), 1\rangle; \tag{7}$$

whereas using orthogonal basis $\mathcal{B}_{\psi|_f} = \{\cos(k\beta)\}_{k=0}^{\infty}$ constructed by the Gram–Schmidt procedure with $w(\beta) = \frac{1}{\sin(\beta)}$, yields a moment system with a banded structure,

$$\frac{d}{dt}m_k(t) = \sum_{i=k-1}^{k+1} c_i m_i(t) + u_1(t)\langle \psi_k, 1\rangle_w, \tag{8}$$

with $\langle \psi_k, 1\rangle_w$ non-zero only when $k = 0$ due to orthogonality, and c_i are constant coefficients. Notice that in this particular case the moment system under the Legendre basis $P_k(\beta)$ will not equip with a banded structure, since $\cos(\beta)P_k(\beta)$ cannot be expressed as a sum of finite terms under the Legendre basis.

Therefore, the representation of the moment dynamics depends on the choice of bases. More importantly, ensemble-moments defined with respect to an orthogonal basis yield a moment system with a sparse representation equipped with a banded system matrix $\hat{A}$ as shown in (5). This distinct structure benefits us in evaluating truncated error [12] and therefore advantage us to leverage the truncation of the moment dynamical system for control design.

C. Moment systems of bilinear ensembles

The application of the introduced moment method goes beyond linear ensembles and encompasses the realm of nonlinear ensembles that are prevalent in practice. One popular method to approach a nonlinear system is to approximate it through bilinear system and leverages the bilinear structure to facilitate analysis and control design [13]. In this section, we will adopt such "bilinearization" ideas to develop techniques by which the control-affine ensemble system in (1) can be transformed into a bilinear ensemble system. Before delving into the detailed development, we first illustrate that the introduced generalized moment transformation preserves the structure of the bilinear ensemble system.

Example 2 (Bloch ensemble system): We consider an ensemble of Bloch systems, which describes the time-evolution of the bulk magnetization of a sample of nuclear spins immersed in an external field, given by [8],

$$\frac{d}{dt}x(t,\beta) = \beta\Big[u_1 B_1 + u_2 B_2\Big]x(t,\beta). \tag{9}$$

where $x(t,\beta) \in \mathbb{R}^3$ for each $t \in [0,T]$ and $\beta \in [1-\delta, 1+\delta]$ representing the rf-inhomogeneity with $0 < \delta < 1$; B_1 and B_2 are the generators of rotation around the y- and the x-axis, i.e.,

$$B_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since the system (9) has linear parameterization β, the constructed orthogonal basis $\mathcal{B}_{\psi|_{\beta^k}}$ by following the previous discussion in II-B with a constant weight function $w \equiv 1$ is the Legendre polynomials. Therefore, we apply the moment transformation (6) under normalized Legendre basis to this reparametrized system with $\eta = (\beta - 1)/\delta = [-1,1]$, and we derive the kth moment dynamic given as

$$\begin{aligned}\frac{d}{dt}m_k(t) &= \langle \psi_k(\eta), \frac{d}{dt}x(t,\eta)\rangle \\ &= \Big[u_1B_1 + u_2B_2\Big]\langle(\delta\eta+1)\psi_k(\eta), x(t,\eta)\rangle \\ &= \delta\Big[u_1B_1 + u_2B_2\Big]\Big(a_k m_{k+1}(t) + b_k m_k(t) \\ &\quad + c_k m_{k-1}(t)\Big) + \Big[u_1B_1 + u_2B_2\Big]m_k(t),\end{aligned}$$

which results in the moment dynamical system

$$\frac{d}{dt}m(t) = \Big[u_1\hat{B}_1 + u_2\hat{B}_2\Big]m(t), \tag{10}$$

where $\hat{B}_i = \mathcal{S} \otimes B_i$ are bounded operators with banded structure, and $\mathcal{S}$ is the coefficient matrix derived as

$$\mathcal{S} = \delta\begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & c_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} + \mathcal{I}$$

with $a_{k-1} = c_k = k/\sqrt{4k^2-1}$, $b_k = 0$, $k \in \mathbb{Z}^+$ and $\mathcal{I}$ denoting the infinite dimension identity operator.

This example of bilinear Bloch equations showcases that the proposed generalized moment transformation not only preserves the bilinear structure but leads to a moment system free of parameter β. Moreover, a finite truncation of such infinite-dimensional bilinear system, i.e., a classical finite-dimensional bilinear system, renders us a tractable way to facilitate analysis and ensemble control design (see Section IV for further discussion).

III. Exact bilinearization of nonlinear ensembles

Despite the advantages the moment transformation provides, it is in general difficult to know the structure of the moment system corresponding to a general nonlinear ensemble system. The analysis for the control-affine ensemble system as in (1) will become feasible if it can be exactly bilinearized, i.e., transformed into a bilinear ensemble system. In the following, we provide a sufficient condition for exact bilinearization of nonlinear control-affine systems by introducing higher dimensional embedding and demonstrate this bilinearization can be effectively applied to nonlinear ensembles.

To fix ideas, we start by considering the following control-affine nonlinear system defined on $\mathbb{R}^n$, in the form of

$$\dot{x}(t) = g_0(x) + \sum_{i=1}^{m} u_i(t)g_i(x), \tag{11}$$

where $g_0, g_1, \ldots, g_m$ are smooth vector fields defined on $\mathbb{R}^n$. Let $g_i = \sum_{k=1}^n g_{ik}(x)\frac{\partial}{\partial x_k}$ and assume $g_{0k}(x)$ are not constant function for all k. We further define $\mathbb{H} = \{h \in C^\infty(\mathbb{R}^n, \mathbb{R}) : h(x) = e_k(x) \text{ or } g_{ik}(x);\ 0 \le i \le m,\ 1 \le k \le n\}$ and $\mathbb{B} = \{\eta \in C^\infty(\mathbb{R}^n, \mathbb{R}) : \eta(x) = \mathcal{L}_{g_{i_1}} \ldots \mathcal{L}_{g_{i_r}} h(x);\ 0 \le i_r \le m,\ 1 \le r < \infty\}$ where $e_k(x) = x_k$ and $\mathcal{L}_g h$ denotes the Lie derivative of (real-valued) function h along vector field g. Furthermore, let $\mathbb{V} = \{1, v_1, v_2, \ldots, v_l\} \subset \mathbb{H} \cup \mathbb{B}$ be the smallest set such that all the elements in $\mathbb{H} \cup \mathbb{B}$ is in span($\mathbb{V}$). Then, the sufficient condition for the nonlinear system (11) to be exactly bilinearizable is provided as follows.

Theorem 4: Consider the control-affine nonlinear system as in (11). If $\mathbb{V}$ is finite-dimensional of dimension $l+1$, then there exists an embedding $\Phi : \mathbb{R}^n \to \mathbb{R}^l$ such that $z(t) \doteq \Phi(x(t))$ is governed by a bilinear system on $\mathbb{R}^l$, given by,

$$\dot{z} = Az + \sum_{i=1}^m u_i(t) C_i + \sum_{i=1}^m u_i(t) B_i z,$$

for some $l \ge n$, where A, $B_i \in \mathbb{R}^{l \times l}$ and $C_i \in \mathbb{R}^l$ for all $i = 1, \ldots, m$.

Proof: The proof is constructive. First, by the definition of the set $\mathbb{V}$ in the theorem, each $v_j \in \mathbb{V}$ is either in $\mathbb{H}$ or $\mathbb{B}$. Also note that $x_1, \ldots, x_n$ are always in span($\mathbb{V}$), thus we let $v_j = x_j$ for $1 \le j \le n$. Moreover, every element in $\mathbb{H} \cup \mathbb{B}$ can be represented as a linear combination of $v_1, \ldots, v_l$. Therefore, we construct new states $z = (z_1^\intercal, z_2^\intercal, \ldots, z_l^\intercal)^\intercal$ such that $z_j = v_j$ for $1 \le j \le l$, then

$$\dot{z}_j = g_{0j}(x) + \sum_{i=1}^m u_i(t) g_{ij}(x),$$

for $1 \le j \le n$ and

$$\dot{z}_j = \mathcal{L}_{g_0} \mathcal{L}_{g_{j_0}} \ldots \mathcal{L}_{g_{j_r}} h(x) + \sum_{i=1}^m u_i(t) \mathcal{L}_{g_i} \mathcal{L}_{g_{j_0}} \ldots \mathcal{L}_{g_{j_r}} h(x),$$

for $n+1 \le j \le l$ where all the terms are in $\mathbb{B}$ and therefore are in span($\mathbb{V}$). That is,

$$\dot{z}_j = A_{0j} z + \sum_{i=1}^m u_i(t) C_{ij} + \sum_{i=1}^m u_i(t) B_{ij} z,$$

where $A_{0j}, B_{ij} \in \mathbb{R}^{1 \times l}$ and $C_{ij} \in \mathbb{R}$ for all $i = 1, \ldots, m$. ∎

Theorem 4 provides a transformation to convert a nonlinear dynamic into an exact bilinear form under given sufficient conditions. Furthermore, this exact bilinearization is also suitable when applied to nonlinear ensemble systems. To illustrate the bilinearization within the context of ensemble systems, we take the following two nonlinear ensembles as examples.

Example 3 (Example of exact bilinearization): Consider a nonlinear ensemble system in $\mathbb{R}^2$ described by

$$\frac{d}{dt} x(t, \beta) = \beta \begin{pmatrix} x_1 \\ x_2 - x_1^2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{12}$$

with $\beta \in [-1, 1]$. The vector fields of this system is given as $g_0 = (x_1, x_2 - x_1^2)^\intercal$, $g_1 = (1, 0)^\intercal$ and $g_2 = (0, 1)^\intercal$. Therefore, we have $\mathbb{H} = \{h : h(x) = \{x_1, x_2, x_1, x_2 - x_1^2, 1, 0, 0, 1\}\}$ and $\mathbb{B} = \{\eta : \eta(x) = \{x_1, x_2 - x_1^2, x_2 - 3x_1^2, \ldots\}\}$ with all the other terms in $\mathbb{B}$ are in span($\{1, x_1, x_2, x_1^2\}$). Clearly, there is no other set with smaller dimension that span $\mathbb{H} \cup \mathbb{B}$, thus we let $\mathbb{V} = \{1, x_1, x_2, x_1^2\}$ and define the transformation as $z(t) = \Phi(x(t)) = (x_1, x_2, x_1^2)^\intercal$ with the corresponding dynamic for the new state z given as

$$\frac{d}{dt} z(t, \beta) = \beta A z + C_1 u_1 + C_2 u_2 + u_1 B_1 z \tag{13}$$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$. The result showcases that after the transformation Φ, the original ensemble system is embedded to a higher dimensional bilinear ensemble as desired. Since the dynamic of the new ensemble state z is bilinear, the moment dynamic under normalized Legendre basis is derived by following the previous discussion in example 2, given as

$$\dot{m}(t) = \hat{A} m + \hat{C}_1 u_1 + \hat{C}_2 u_2 + u_1 \hat{B}_1 m \tag{14}$$

where $\hat{A} = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & c_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \otimes A$, $\hat{C}_i = \begin{pmatrix} \langle \psi_0, g \rangle \\ \langle \psi_1, g \rangle \\ \vdots \end{pmatrix} \otimes C_i$ for $i = 1, 2$, and $\hat{B}_1 = \mathcal{I} \otimes B_1$ with $a_{k-1} = c_k = k/\sqrt{4k^2 - 1}$, $b_k = 0$, $k \in \mathbb{Z}^+$, and $\mathcal{I}$ denotes the infinite dimension identity operator.

We would like to emphasize that although we provide a sufficient condition for the exact bilinearization of a nonlinear ensemble system, the bilinearization is not unique. We present the following real-world nonlinear unicycle ensembles [5] for illustration.

Example 4 (Unicycle ensembles): Consider a collection of unicycles described by

$$\frac{d}{dt} x(t, \beta) = \beta \left(u \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right), \tag{15}$$

where $x(t, \beta) \in \mathbb{R}^3$ and model perturbations $\beta \in [1 - \delta, 1 + \delta]$ for $0 \le \delta < 1$. Following the analogous construction as in previous example, we have $\mathbb{H} = \{h : h(x) = \{x_1, x_2, x_3, \cos x_3, \sin x_3, 0, 1\}\}$ and $\mathbb{B} = \{\eta : \eta(x) = \{\cos x_3, \sin x_3, -\sin x_3, -\cos x_3\}\}$. Thus, $\mathbb{V} = \{1, x_1, x_2, x_3, \cos x_3, \sin x_3\}$ is the set with the smallest dimension that span $\mathbb{H} \cup \mathbb{B}$. Defining the transformation as $z(t) = \Phi(x(t)) = (x_1, x_2, x_3, \cos x_3, \sin x_3)^\intercal$, then the corresponding dynamic for $z \in \mathbb{R}^5$ is given as

$$\frac{d}{dt} z(t, \beta) = \beta (u B_1 z + v B_2 z) + \beta C v \tag{16}$$

where $B_i = [b_{jk}]_i$ is given by $[b_{14}]_1 = [b_{25}]_1 = 1$ for B_1 and $[b_{45}]_2 = -1, [b_{54}]_2 = 1$ for B_2 and all other elements are 0 and $C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix}^\intercal$.

Observe that x_3 is determined by $\cos x_3$ and $\sin x_3$, hence z_3 in the new state is redundant to the constructed system. Therefore, we redefine the transformation as $\tilde{x} = (x_1, x_2, \cos(x_3), \sin(x_3))^\intercal$ such that the resulting dynamics for $\tilde{x}$ is bilinear

$$\frac{d}{dt}\tilde{x}(t,\beta) = \beta\left(uB_1\tilde{x} + vB_2\tilde{x}\right) \tag{17}$$

where $B_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Consequently, we obtain a bilinear moment dynamics as in (10) where $\hat{B}_1$, $\hat{B}_2$ are bounded linear operators with banded structure, given by

$$\hat{B}_1 = \begin{pmatrix} & \tilde{B}_{10} & & \\ \tilde{B}_{10} & & \tilde{B}_{11} & \\ & \tilde{B}_{11} & & \ddots \\ & & \ddots & \end{pmatrix} \otimes I_2, \ \tilde{B}_{1k} = \begin{pmatrix} 0 & c_k \\ 0 & 0 \end{pmatrix},$$

$$\hat{B}_2 = \begin{pmatrix} & \tilde{B}_{20} & & \\ \tilde{B}_{20} & & \tilde{B}_{21} & \\ & \tilde{B}_{21} & & \ddots \\ & & \ddots & \end{pmatrix} \otimes A, \ \tilde{B}_{2k} = \begin{pmatrix} 0 & 0 \\ 0 & c_k \end{pmatrix},$$

and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $c_k = k/\sqrt{4k^2 - 1}$.

These two examples demonstrate that under the sufficient condition provided in theorem 4, a nonlinear ensemble dynamical system can be converted into a bilinear form and therefore the associated moment system also adopts a bilinear structure. Moreover, under piece-wise constant controls, the obtained bilinear moment dynamic behaves similarly as its linear counterparts. This enables us to extend the idea of constructing feasible controls based on truncated moment dynamics to the bilinear moment dynamics.

IV. Ensemble Control Design via Truncated Moment System

Thanks to the orthogonal basis used for defining moment terms, we have a dual relation between the ensemble system and its associated moment system manifested by the isometry,

$$\|x(t,\cdot) - x_f(\cdot)\|_{L^2} = \|m(t) - m_f\|_{\ell^2}. \tag{18}$$

However, one obstacle for designing controls directly for the moment dynamical system (10) is its dimension. Since ℓ^2 is an infinite dimensional space, a direct method for obtaining the control renders intractable in most cases. Instead, we utilize the banded structure in the moment dynamics representation and devise an approach via the truncated moment system for ensemble control design. Consequently, we formulate an optimal control problem from the truncated moment dynamics of a specific truncation order N in the form,

$$\begin{aligned} \min_{u,v} \ & \|\bar{m}(T) - \bar{m}_f\|_2 \\ s.t. \ & \dot{\bar{m}}(t) = \hat{A}_N\bar{m}(t) + \hat{B}_{N0}u(t) + \sum_i v_i(t)\hat{B}_{Ni}\bar{m}(t) \\ & \bar{m}(0) = \bar{m}_0, \end{aligned} \tag{19}$$

where $u(t), v(t)$ are the controls. To illustrate the process, we exhibit two simulation results for nonlinear ensembles.

A. *Steering a nonlinear ensemble system*

Consider the nonlinear ensemble systems in (12) with $\beta \in [-1,1]$. The objective is to design a control that steers the entire ensemble system from the initial profiles $x(0,\beta) = (0,0)^\intercal$ to the final states $x_f(\beta) = (1,1)^\intercal$ at a given final time T. The obtained state trajectories and the designed controls are shown in figure 1. For this simulation, we set the final time $T = 1$ and design control through truncated moment system (14) with truncation order $N = 4$. To solve the optimal control problem described in (19), we use *fmincon* solver in Matlab to find the control with the default optimization algorithm *interior-point*. The error tolerance (function tolerance) and the step tolerance are both selected as the default values of 10^{-6} and 10^{-10}, respectively. In particular, we embed the original system $x \in \mathbb{R}^2$ to a higher dimensional system $z \in \mathbb{R}^3$ and derive the moment dynamic corresponding to the new state z. When designing the control, since state z_3 is related to z_1, we let the desired final state $z_f(\beta) = (1,1,*)^\intercal$ instead of $(1,1,1)^\intercal$ where $*$ indicates that the final state $z_3(t,\beta)$ is free.

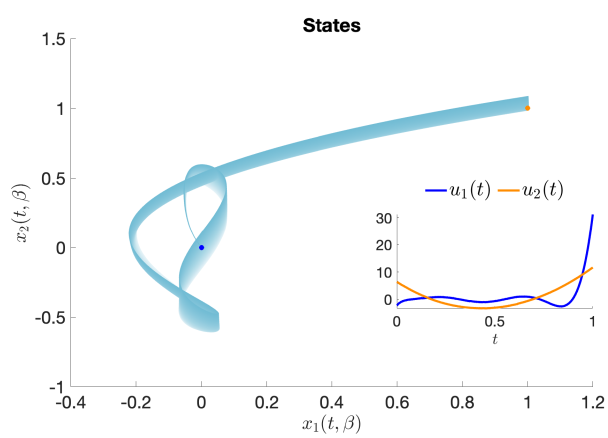

Fig. 1: Simulation result for nonlinear ensembles described in example 3 for 500 systems with the system parameter β uniformly spaced within $[-1,1]$. Controls are designed with the moment truncation order $N = 4$ and the state trajectories for all ensembles are shown in gradient light blue.

B. *Steering an ensemble of unicycle systems*

For the ensemble system in (15) where we set $\delta = 0.2$, i.e. model perturbations $\beta \in [0.8, 1.2]$, the control task aims at navigating the ensembles from the initial position to a predefined final position at a given time [5]. For instance, we set $x(0,\beta) = (0,0,0)^\intercal$ and $x_f(\beta) = (3,2,*)^\intercal$ for simulations with $*$ indicates that the angle variable $x_3(t,\beta)$ is free. The result is shown in figure 2, with the truncation order $N=4$ and simulation final time $T=2$. In comparison with the method proposed in [5], our ensemble control framework allows more flexibility in the obtained controls since the constraints imposed on the system controls can be conveniently incorporated in the optimization problem (19). Additionally, our formulation possesses the ability to introduce penalty terms into the objective function. This facilitates relaxing or constraining the solution as needed. For instance, a control energy term $\|u^\intercal u\|_2$ is added in the objective function in this example to avoid controls of large values, and the corresponding controls are solved by utilizing the aforementioned optimization solver.

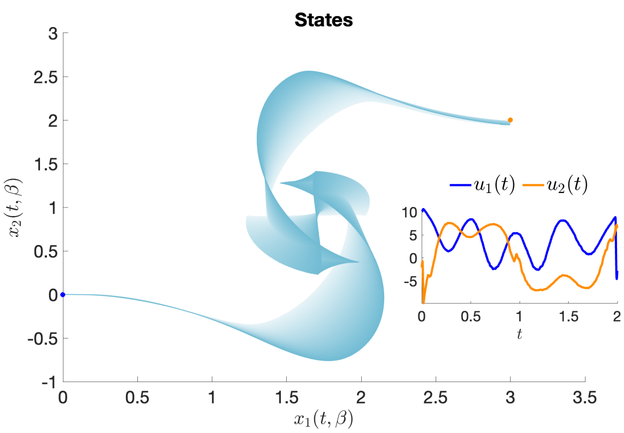

Fig. 2: Simulation result for ensemble of unicycles described in example 4 for 500 systems with the model perturbation $\beta \in [0.8, 1.2]$. Controls are designed with the moment truncation order $N=4$ and the state trajectories for all ensembles are shown in gradient light blue.

V. Conclusion

In this paper, we present a generalized moment-based method which facilitates systems-theoretic analysis and control design for dynamic ensemble systems. The main contributions of this work include the development of (i) a systematic approach to constructing ensemble-moments with respect to an orthogonal basis, which leads to moment systems equipped with banded structures; and (ii) an exact bilinearization technique for nonlinear ensemble systems based on constructing a higher-dimensional embedding using Lie algebra, which leads to bilinearized moment systems. The introduced generalized moment method provides a unified framework for exploiting intricate ensemble systems, facilitating control design through the use of their associated moment systems.

Appendix

A. Gram-Schmidt procedure for constructing an OPS

Lemma 5: The sequence of polynomials $\{\psi_k\}_{k=0}^{\infty}$ defined in the following way is an OPS with respect to Ψ_w

$$\begin{aligned} &\psi_0 = 1, \ \psi_1 = \eta - d_1 \\ &\psi_k = (\eta - d_k)\psi_{k-1} - l_k\psi_{k-2}, k \geq 2, \end{aligned} \tag{20}$$

where d_k and l_k are constant coefficients defined as

$$d_k = \frac{\Psi_w[\eta\psi_{k-1}^2]}{\Psi_w[\psi_{k-1}^2]}, \ l_k = \frac{\Psi_w[\eta\psi_{k-1}\psi_{k-2}]}{\Psi_w[\psi_{k-2}^2]}$$

Proof: The proof follows from Favard's theorem which states that a sequence defined by the recursive relation above is an OPS [11] (pp. 21-22). ■

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