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A Simple Bounded Controller for the Finite–Time Stabilization of the Heisenberg System

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Abstract—**In this paper, a novel controller design is proposed to stabilize the Heisenberg system (also known as the Nonholonomic Integrator or Brockett's Integrator) in a finite time. Although the controller design is based on the unit vector control for the Sliding–Mode Control theory, no sliding manifold design is required. Instead, some inherent properties of the Heisenberg system, *e.g.,* the skew symmetric/diagonal structure, are exploited to obtain a simple to tune and bounded controller that ensures the finite–time stabilization of the origin for any arbitrary initial condition outside the origin. Additionally, the resulting controller is globally bounded and, contrasting with other similar approaches for the Heisenberg system, this design allows the estimation of the settling–time function.**

Index Terms—**Heisenberg System, Nonholonomic Integrator, Nonlinear Control.**

I. Introduction

Nonholonomic systems have been studied extensively in the last decades, primarily due to the development of new applications involving mobile robots [1], current–fed induction motors [2], surface vessels [3], and other unconventional vehicles and robots (see [4] and [5]). Secondly, this class of systems does not fulfill the well–known Brockett's necessary condition for smooth state feedback stabilization [6]. So, the design of non–smooth and time–varying feedback controllers is not only interesting from the theoretical point of view but a requirement for the regulation of this class of systems.

The Heisenberg system (so–called because its vector fields generate the Heisenberg algebra, see [7] and [8]), also known as the nonholonomic integrator, is a nonlinear system diffeomorphic to many of physical models (some examples can be found in [9] and [10]). For this reason, the Heisenberg system and the canonical chained form [11] have been frequently used as a benchmark for controller design and stability analysis of nonholonomic systems (see, *e.g.,* [12], [13] and [14]).

Thus, the development of controllers that ensure the convergence of the tracking error to the origin in finite–time (FT) has been crucial as well (see, for instance, [15], [16] and [17]).

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One remarkable approach that has been widely used to develop FT controllers for nonholonomic systems is the sliding–mode control (SMC) approach, mainly because additionally to its non–smooth nature, it has the possibility of granting FT convergence and robustness of the closed–loop system. Therefore, it has been applied to solve the stabilization and the tracking problem for this class of systems (see, *e.g.,* [18], [19] and [20]).

More recent results that consider SMC approaches to solve the trajectory tracking problem include the use of integral SMC design (see [21] and [9]), the super–twisting algorithm [22], and the modified super–twisting algorithm [23].

Contrasting with the SMC approach, the proposed controller does not require the selection of a sliding manifold, but instead exploits the structure of the Heisenberg systems and how the interaction between its subsystems produces the movement in $\mathbb{R}^3$. The design, as it is shown later on, is intuitive and easy to tune. Some other main highlights of this approach can be summarized as follows:

- The obtained controller ensures FT convergence of the states of the Heisenberg system to the origin.
- The resulting control function is globally bounded.
- The simplicity of the controller structure facilitates its implementability, including the parameter selection and tuning, since it only requires the choice of two scalar parameters.
- The control scheme can be applied to many physical systems with minor adjustments.

The structure of the rest of this paper is the following. Some relevant definitions and important theoretical notions are included in Section III. The problem statement can be found in Section II. In Section IV, the proposed control design is presented. In Section V, a simulation to illustrate a possible implementation of the proposed control scheme is given. Finally, in Section VI, some concluding remarks and some comments on the potential extension of the main result can be found.

Notation. Denote $\mathbb{R}_+ = \{x \in \mathbb{R}\colon x > 0\}$, $\mathbb{R}_- = \{x \in \mathbb{R}\colon x < 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R}\colon x \geq 0\}$, where $\mathbb{R}$ is the set of all real numbers; $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^n$. Define the function $\lceil a \rfloor^{\gamma} = |a|^{\gamma}\text{sign}(a)$, for any $\gamma \in \mathbb{R}_{\geq 0}$ and any $a \in \mathbb{R}$. The term S^1 is the $1-$sphere. $SO(2)$ represents the special orthogonal group in $\mathbb{R}^2$, any element

$SO(2)$ represents a standard rotation in $\mathbb{R}^2$, defined for $\theta \in S^1$, by the matrix

$$\boldsymbol{R}(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

II. Problem Statement

The Heisenberg system dynamics is given by

$$\dot{z}(t) = Y^\top(t)\boldsymbol{J}X(t), \tag{1}$$
$$\dot{X}(t) = Y(t), \tag{2}$$

with the vector state $\mathcal{X} = \left[z,\ X^\top\right]^\top \in \mathbb{R}^3$, such that $z \in \mathbb{R}$, $X \in \mathbb{R}^2$, and the initial conditions $\mathcal{X}_0 = \left[z_0,\ X_0^\top\right]^\top$. The control input is $Y \in \mathbb{R}^2$ and $\boldsymbol{J} \in \mathbb{R}^{2\times 2}$ is the skew–symmetric matrix

$$\boldsymbol{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is important to recall that in order to stabilize the origin of system (1) – (2), it is necessary to provide a control input $Y(t)$ that avoids stabilizing the origin of the subsystem (2) before stabilizing the origin of (1). Otherwise, if $X(T) = 0$, and $z(T) = c \neq 0$, at any given time instant $T \geq 0$, then $z(t) = c$, for all $t > T$ such that $X(t) = 0$.

Considering this well–known constraint of the Heisenberg system, the aim of this paper is to design a bounded control input $Y(t)$ that stabilizes the origin of system (1)–(2) in a finite time providing a settling–time estimation.

III. Preliminaries

Consider the system

$$\dot{x} = f(t,x),\ t \in \mathbb{R}_{\geq 0},\ x(0) = x_0, \tag{3}$$

where $x \in \mathbb{R}^n$ is the state vector. The function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally bounded uniformly in t. For f locally measurable but discontinuous with respect to x, the solutions are understood in the sense of Filippov [24]. That is, $x(t, x_0)$ is a solution to (3) if it is absolutely continuous, and if it satisfies the differential inclusion

$$\dot{x} \in K[f](t,x) = \overline{\text{co}} \bigcap_{\varepsilon>0} \bigcap_{\mu N=0} f(t, B(x,\varepsilon)\backslash N),$$

where $\overline{\text{co}}(M)$ represents the convex closure of the set M, $B(x,\varepsilon)$ represents the ball centred at x with radius ε, μ is the Lebesgue measure. Note that the intersections are taken over all the sets N of measure zero, over all $\varepsilon > 0$. Let Ω be open neighborhood of the origin in $\mathbb{R}^n$, $\mathbf{0} \in \Omega$.

Definition 1. [25, 26]. *At the steady state $x = 0$, the system* (3) *is said to be:*

$a)$ Uniformly Stable (US) *if for any $\epsilon > 0$ there is $\delta(\epsilon)$ such that for any $x_0 \in \Omega$, if $||x_0|| \leq \delta(\epsilon)$ then $||x(t,x_0)|| \leq \epsilon$ for all $t \geq t_0$, for any $t_0 \in \mathbb{R}$;*

$b)$ Uniformly Finite–Time Stable (UFTS) *if it is US and finite–time converging from Ω, i.e. for any $x_0 \in \Omega$ there exists $0 \leq T_{x_0} < +\infty$ such that $x(t,t_0,x_0) = 0$ for all $t \geq t_0 + T_{x_0}$, for any $t_0 \in \mathbb{R}$. The function $T_0(x_0) = \inf\{T_{x_0} \geq 0 : x(t,x_0) = 0\ \forall t \geq t_0 + T_{x_0}\}$ is called the **settling–time** of the system* (3).
If $\Omega = \mathbb{R}^n$, then $x = 0$ is said to be globally US (GUS), or globally UFTS (GUFTS), respectively

IV. Controller Design

A. Mathematical Structure

In order to stabilize the system (1)–(2), the following control input is proposed

$$Y(t) = -\gamma \boldsymbol{R}\left(\omega(\mathcal{X})\right) \frac{X(t)}{\|X(t)\|}, \tag{4}$$

with the functions

$$\omega(\mathcal{X}) = \text{atan}\left(\Phi(\mathcal{X})\right),$$
$$\Phi(\mathcal{X}) = \frac{2\beta \left\lceil z \right\rfloor^{\frac{1}{2}}}{\|X\|},$$

some constants parameters $\gamma, \beta \in \mathbb{R}_{\geq 0}$, and the matrix $\boldsymbol{R} \in SO(2)$. Notice that it is possible to write (4) as

$$Y(t) = -\gamma \cos(\omega(\mathcal{X}))\frac{X(t)}{\|X(t)\|} - \gamma \sin(\omega(\mathcal{X}))\boldsymbol{J}\frac{X(t)}{\|X(t)\|}.$$

By substituting (4) in (1)–(2), and noticing that

$$\|\boldsymbol{J}X\| = \|X\|,$$

the corresponding closed–loop system dynamics are obtained as

$$\dot{z}(t) = -\gamma \sin(\omega)\|X(t)\|, \tag{5}$$
$$\dot{X}(t) = -\gamma \boldsymbol{R}(\omega)\frac{X(t)}{\|X(t)\|}. \tag{6}$$

Then, the following facts can be verified directly from the definitions of ω and Φ, for all $\mathcal{X} \in \mathbb{R}^3$,

$$\cos(\omega) = \frac{1}{\sqrt{1+\Phi^2}} > 0, \tag{7}$$
$$\text{sign}(\sin(\omega)) = \text{sign}(\Phi) = \text{sign}(z), \tag{8}$$
$$\Phi\cos(\omega) = \frac{\Phi}{\sqrt{1+\Phi^2}} = \sin(\omega). \tag{9}$$

Let $\mathcal{M}_x = \left\{\mathcal{X} \in \mathbb{R}^3 : \|X\| = 0\right\}$; and, for some constant $\overline{\beta} > 0$, define the set

$$\mathcal{D}_\beta = \left\{\mathcal{X} \in \mathbb{R}^3 \backslash \mathcal{M}_x : |\Phi(\mathcal{X})| < \overline{\beta}\right\},$$

and the constants $\omega_\beta = \text{atan}(\overline{\beta})$ and $C_\beta = \cos(\omega_\beta)$. The main result of this paper is given in the following Theorem.

Theorem 1. *The origin of system* (1)–(2)*, applying the control input* (4)*, with $\beta > |z_0|^{\frac{1}{2}}/\|X_0\|$ and $\gamma > 0$, is UFTS* *on $\Omega_\beta = \mathcal{D}_\beta \cup \{\mathbf{0}\}$, for $\overline{\beta} = 2\beta$. Moreover, the settling–time $T_{\mathcal{X}}(\mathcal{X}_0)$, is upper–bounded as*

$$T_{\mathcal{X}}(\mathcal{X}_0) \leq \frac{\|X_0\|}{\gamma C_\beta}.$$

The proof is omitted due to space limitations.

Remark 1. *The control input generated by* (4) *is bounded,* i.e.,

$$\|Y(t)\|^2 = \gamma^2 \frac{X^\top(t)}{\|X(t)\|} \boldsymbol{R}^2(\omega) \frac{X(t)}{\|X(t)\|} \leq \gamma^2,\ \forall t \geq 0.$$

This facilitates its implementabilty and increases the number of applications that could potentially benefit from this design.

Remark 2. *Although according to Theorem 1, the resulting closed–loop system* (5)–(6) *is only UFTS on $\mathcal{D}_\beta$; for any $\mathcal{X}_0 \in \mathbb{R}^3$, there always exists $\beta > |z_0|^{\frac{1}{2}}\|X_0\|^{-1}$. Then, for any $\mathcal{X}_0 \notin \mathcal{M}_x$, the origin of the closed–loop system is always stabilizable in a finite time. Also notice that the bound of $\|Y(t)\|$ does not depend on β. Therefore, large initial conditions do not pose a potential implementation issue.*

Remark 3. *The control input* (4) *is simple to tune. The implementation of the proposed controller* (4) *only requires the choice of two parameters. The parameter β is directly fixed by the initial conditions, while γ can be selected accordingly to the application requirements and constraints,* i.e., *available input amplitude and desired convergence time.*

It is also noteworthy that even though the controller structure is somewhat reminiscent of the unit vector control approach for SMC on multiple input systems, the manifold $\mathcal{M}_x$ is not a sliding manifold. In fact, $\mathcal{M}_x$ should not be reached before the origin of (1) is reached, because any point on $\mathcal{M}_x$ is an equilibrium, and renders the subsystem (1) uncontrollable. This is totally contrasting with the sliding manifold selection for the unit vector control approach.

B. On the Physical Meaning of the Controller

The design of the controller is based intuitively on how the Heisenberg algebra produces the movement in $\mathbb{R}^3$. A tangent velocity to the manifold $\|X\| = r$, produces a movement in z proportional to r and the magnitude of the velocity. A radial velocity (*i.e.,* the rate of change of the state respect to the origin) modifies $\|X\|$, but leaves z unaffected. Then, one can roughly picture the movement described by the Heisenberg system first as a screw, *i.e.,* in order to ascend or descend at any given point a rotation around a circle of radius r has to be performed; and then, as a change in the radius r.

This is what essentially the proposed controller (4) does. For $\omega = \frac{\pi}{2}$, $\|X\|$ remains constant, but z decreases/increases. For $\omega = 0$, $\|X\|$ decreases/increases, but z remains constant. For any $\omega \in (0,\ \pi/2)$ the change of z and X can be decomposed in these two cases. The quantity $|z_0|^{\frac{1}{2}}\|X_0\|^{-1}$ could be interpreted as a proportion of the numbers of "turns", around the circle of radius $\|X_0\|$, required to descend or ascend from z_0 to 0.

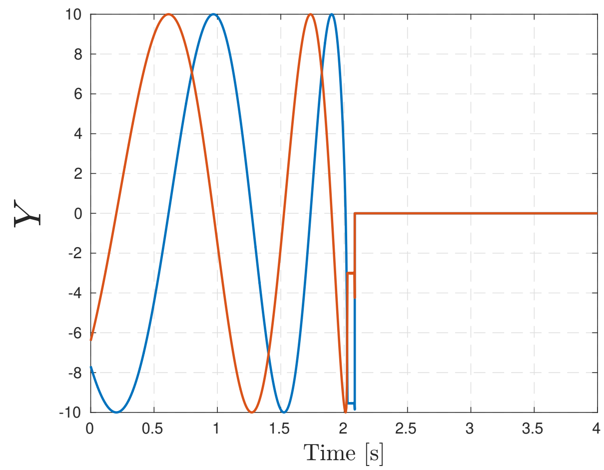

Figure 1. Control Input Y

C. Implementation Aspects

The control input (4) becomes discontinuous once the manifold $\mathcal{M}_x$ is reached. This could pose an implementation issue for certain applicatons. However, because the origin of (1)–(2) is an invariant set for $Y = 0$, it is possible to switch–off the controller input as soon as the manifold $\mathcal{M}_x$ is reached, to maintain the control input continuous; *i.e.,*

$$Y(t) = \begin{cases} -\gamma \boldsymbol{R}(\omega(\mathcal{X})) \frac{X(t)}{\|X(t)\|} & \text{if} \quad X \neq 0, \\ \mathbf{0} & \text{if} \quad X = 0. \end{cases} \tag{10}$$

V. Simulations

The simulations have been carried out in Matlab using the Euler discretization method with the integration step of 0.0001[s]. The control input is designed as proposed in Theorem (1), considering the following parameters $\gamma = 10,\ z_0 = 40,\ X_0 = [3/\sqrt{2}, -3/\sqrt{2}]^\top$ and $\beta = 2.5 > |z_0|^{\frac{1}{2}}/\|X_0\| = 2.1082$. This gives the settling–time estimation $T_{\mathcal{X}} \leq 3.7620$.

The control signal $Y(t)$, as proposed in (10), is shown in Figure 1. The trajectories of $|z|^{\frac{1}{2}}$ and $\|X\|$ are depicted in Figure 2. Finally, the system trajectories vs Time are presented in Figure 3, while Figure 4 presents the state trajectories in $\mathbb{R}^3$. The simulation results effectively illustrate the FT convergence to the origin of the proposed controller.

VI. Conclusions

In this paper, a novel globally bounded FT controller design for the Heisenberg system is presented. The resulting controller is straightforward to tune, and implement, while stabilizing the system origin for any initial condition in a finite time. Moreover, its structure is intuitive and easy to grasp, from both the mechanical and control theory points of view. A simulation is included to illustrate the results and complement the main ideas.

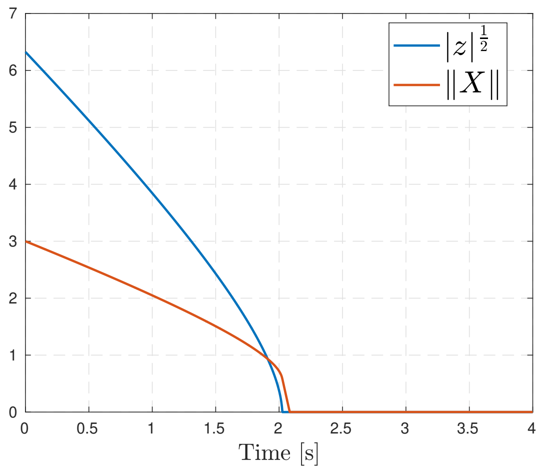

Figure 2. Trajectories of $|z|^{\frac{1}{2}}$ and $\|X\|$

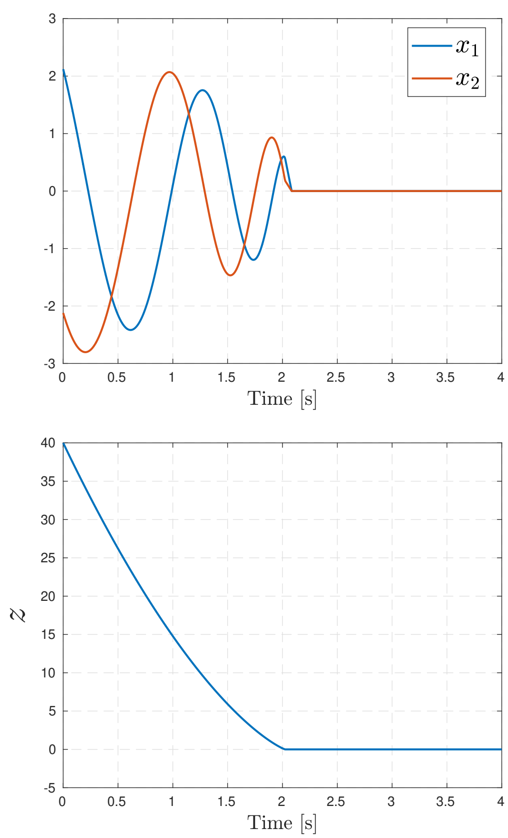

Figure 3. X and z vs Time

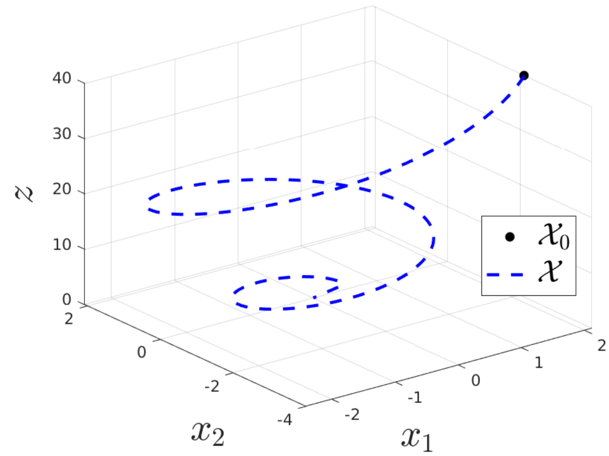

Figure 4. System trajectories

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