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# Identifiability of the Linear Threshold Model

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***Abstract*— In the linear threshold model, each individual has a time-invariant threshold and an initial action A or B. At each time step one or more individuals become active to revise their action depending on their own threshold and the population proportion of each action. The resulting decision-making dynamics can be predicted and controlled, provided that the thresholds of individuals are known. In practice, however, the thresholds are unknown and often only the evolution of the total number of individuals who have chosen one action is known. The question then is whether the thresholds are identifiable given this quantity over time. We find necessary and sufficient conditions for threshold identifiability of the linear threshold model under synchronous and asynchronous decision-making. The results open the door for reliable estimation of the thresholds, and in turn, prediction and control of the decision-making dynamics using real data.**

## I. Introduction

The *linear threshold model* [1] is used in explaining the cascading behavior of decision dynamics within binary decision settings. This model has been applied to various phenomena, such as spread of rumors, adoption of technologies, and diffusion of information [2]–[5].

In this model, each individual has a threshold that determines his decision to adopt an action based on the proportion of the population who has already adopted it. In the context of evolutionary game theory, the *best-response update rule* also capture this behavior, provided that the utility function is a linear function of the number of adaptors [6].

In the literature the linear threshold model was adopted to maximize the spread [7]–[9] and also to minimize the spread of undesirable entities, such as computer viruses and malicious rumors [10], [11]. Several works focused on characterizing and analyzing the equilibrium of the linear threshold decision dynamics [12], [13]. Further, recent attention was paid in controlling the dynamics by providing incentives and optimal seeding [14]–[18].

We need an accurate estimation of the threshold distributions, to be able to accurately predict the decision-making dynamics over time. Most studies assumed that the thresholds are the same for each individual or uniformly distributed over the population. This may not be true in the real-world scenario as the sensitivity to social influence varies from person to person. There are several works on estimating thresholds [19]–[21].

Nevertheless, identifiability comes first. It may not be possible to uniquely obtain the thresholds given the observed output, i.e., the individuals' decisions over time. If so, then one may estimate the "wrong" value for the thresholds, because several threshold distributions can result in the same observed output; namely, the threshold distribution is not *identifiable*. To the best of our knowledge, no studies have investigated the identifiability of the linear threshold model.

Different approaches and notions have been suggested in the literature for identifiability, such as *output equality* and *algebraic identifiability* both in the local and global sense. There is a rich literature for the identifiability of continuous space systems for both continuous-time [22]–[24] and discrete-time cases [25], [26]. However, the system of linear threshold dynamics belongs to the discrete space, thus, we use the discrete space extension of the output equality approach presented in [27] to analyze the identifiability.

We formulate the linear threshold decision-making dynamics for synchronous and asynchronous updates separately and analyze their local, structural, and global identifiability– Section II. For synchronous systems, we find that the system is identifiable if and only if the output trajectory monotonically increases (resp. decreases) and meets all thresholds except for possibly the highest (resp. lowest) threshold –Theorem 1. If a threshold is not met, the number of individuals with that threshold is not identifiable–Lemma 1. The conditions for the asynchronous case are more restrictive as the necessary and sufficient condition for the identifiability of an individual's threshold requires the individual to become active at the time steps when the population proportion of those who have selected action A in the population equals his threshold and $\frac{1}{n}$ less than to his threshold–Lemma 3 and Theorem 2.

## II. Problem formulation

Consider a well-mixed population of $n$ decision-making individuals who choose either of the actions A or B over time $t = 1, 2, \ldots$. Each individual has a time-invariant *temper* $\tau^i \in \{0, 1, \ldots, n\}$ and an initial action. They revise their decision based on others' actions and their own tempers. The temper divided by the population size, i.e., $\tau^i/n$, which belongs to $[0, 1]$, is the *threshold* of individual $i$.

An individual "tends" to choose action A at time $t+1$ if and only if a number of individuals greater than or equal to his temper have chosen A in the population at time $t$. The intention translates into action if the individual is active at time $t$; otherwise, he sticks to his action at time $t$. More specifically, the *linear threshold update rule* for individual $i$ who is active at time $t$ is

$$x^i(t+1) = \begin{cases} \mathtt{A}, & \text{if } A(t) \geq \tau^i, \\ \mathtt{B}, & \text{if } A(t) < \tau^i. \end{cases} \quad (1)$$

where $x^i(t) \in \{\mathtt{A}, \mathtt{B}\}$ is the decision of individual $i$ at time $t$ and $A(t)$ denotes the number of individuals in the population who have chosen action A at time $t$.

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Sort the individuals in the ascending order of their tempers and label them by $1, 2, \ldots, n$, yielding $\tau^1 \leq \tau^2 \leq \ldots \tau^n$. Then we have the *detailed temper vector* $\bar{\boldsymbol{\tau}} = (\tau^i)_{i=1}^n$ that characterizes the heterogeneity of the population and belongs to the space $\bar{\boldsymbol{\Theta}} = \{0, 1, \ldots, n\}^n$. Define $\bar{\boldsymbol{x}}$, the *state* of the population, as the stack of all individuals' actions, $\bar{\boldsymbol{x}}(t) \triangleq (x^1(t), ..., x^n(t))$, with the state space $\bar{\boldsymbol{\mathcal{X}}} = \{\mathtt{A}, \mathtt{B}\}^n$.

At every time $t$, a set of individuals become active to revise their decisions. Define the *activation sequence* $\langle \boldsymbol{\mathcal{U}}_t \rangle_{t=0}^{\infty}$, where $\boldsymbol{\mathcal{U}}_t \subseteq \{1, \ldots, n\}$, $\boldsymbol{\mathcal{U}}_t \neq \emptyset$, is the set of individuals active at time $t$. The activation sequence is *synchronous* if the whole population becomes active at each time, i.e., $|\boldsymbol{\mathcal{U}}_t| = n$ for all $t$, and is *asynchronous* if exactly one individual becomes active at a time. In the asynchronous case, we simplify the notation $\boldsymbol{\mathcal{U}}_t$ to $u_t$, where $u_t$ is the single active individual at time $t$.

Update rule (1) together with the activation sequence govern the evolution of the state $\bar{\boldsymbol{x}}$, resulting in the *linear threshold dynamics*. The goal is to determine whether the thresholds can be obtained uniquely given some measurable quantity (i.e., output) of the dynamics; namely, whether the thresholds are *identifiable*. In what follows, we provide the analytical form of the dynamics in Definitions 1 and 2.

### A. The asynchronous system

In asynchronous system, the activation sequence is considered as the input and the output is the total number of $\mathtt{A}$-selected individuals, i.e., those who have chosen action $\mathtt{A}$. We use $A(\bar{\boldsymbol{x}})$, with an abuse of notation, to denote the number of $\mathtt{A}$-selected individuals in the state $\bar{\boldsymbol{x}}$. The detailed temper vector $\bar{\boldsymbol{\tau}}$ is considered as the parameter. Inspired by [28], we define the asynchronous system in the following compact way. Define step function $\mathbb{1}(p, q)$ which equals 1 if $p \geq q$ and 0 otherwise. Then individual $i$ tends to select action $\mathtt{A}$ at time $t$ if and only if $\mathbb{1}(A(\bar{\boldsymbol{x}}), \tau^i) = 1$. Define $\boldsymbol{e}^i$ as the $i^{\text{th}}$ column of the $n \times n$ identity matrix. Assign numeric values 0 and 1 to the actions; namely, $\mathtt{A} := 1$ and $\mathtt{B} := 0$. Let $\boldsymbol{\mathcal{U}} = \{1, \ldots, n\}$.

*Definition 1 (Asynchronous system):* The *asynchronous decision-making system* is defined by

$$\begin{cases} \bar{\boldsymbol{x}}(t+1) = \bar{\boldsymbol{x}}(t) + (\mathbb{1}(A(\bar{\boldsymbol{x}}), \tau^i) - x^i(t))\boldsymbol{e}^i, \quad \bar{\boldsymbol{x}}_0 = \bar{\boldsymbol{x}}(0) \\ i = u_t, \\ y(t) = \sum_{j=1}^n x^j(t), \end{cases} \tag{2}$$

where $\bar{\boldsymbol{x}}(t) \in \bar{\boldsymbol{\mathcal{X}}}$ is the state, $\bar{\boldsymbol{\tau}} \in \bar{\boldsymbol{\Theta}}$ is the parameter, $\langle u_t \rangle_{t=0}^{\infty}, u_t \in \boldsymbol{\mathcal{U}}$, is the input, $\bar{\boldsymbol{x}}_0$ is the initial state, $y(t)$ is the output which belongs to the space $\boldsymbol{\mathcal{Y}} = \{0, \ldots, n\}$.

*Example 1:* Consider the asynchronous system with $n = 3$, the temper distribution $\bar{\boldsymbol{\tau}} = (0, 1, 2)$ implying that individual $i$ has a temper $\tau^i = i - 1$, and initial state $\bar{\boldsymbol{x}}_0 = (\mathtt{B}, \mathtt{B}, \mathtt{A})$. Under the activation sequence $\langle u_t \rangle_{t=0}^{\infty} = \langle 3, 2, 1, 3, 2, 1, \ldots \rangle$, the population state evolves as in Table I which is also illustrated in Fig. 1.

### B. The synchronous system

The dynamics are simpler in the synchronous case as the system becomes autonomous. Moreover, although the

TABLE I: Transition of the states in Example 1. The active individual at each time is indicated by red.

| Time $t$ | State $\bar{\boldsymbol{x}}(t)$ | Output $y(t)$ |
|---|---|---|
| 0 | $(\mathtt{B}, \mathtt{B}, \mathtt{A})$ | 1 |
| 1 | $(\mathtt{B}, \mathtt{B}, \mathtt{B})$ | 0 |
| 2 | $(\mathtt{B}, \mathtt{B}, \mathtt{B})$ | 0 |
| 3 | $(\mathtt{A}, \mathtt{B}, \mathtt{B})$ | 1 |
| 4 | $(\mathtt{A}, \mathtt{B}, \mathtt{B})$ | 1 |
| 5 | $(\mathtt{A}, \mathtt{A}, \mathtt{B})$ | 2 |
| 6 | $(\mathtt{A}, \mathtt{A}, \mathtt{B})$ | 2 |
| 7 | $(\mathtt{A}, \mathtt{A}, \mathtt{A})$ | 3 |
| 8 | $(\mathtt{A}, \mathtt{A}, \mathtt{A})$ | 3 |

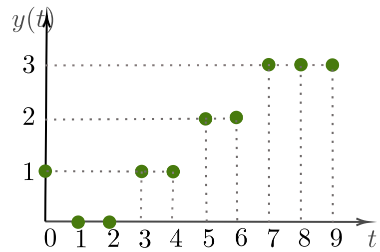


Fig. 1: Output of the asynchronous system with $\bar{\boldsymbol{\tau}} = (0, 1, 2)$ and initial state $\bar{\boldsymbol{x}}_0 = (\mathtt{B}, \mathtt{B}, \mathtt{A})$ given in Example 1.

detailed temper vector $\bar{\boldsymbol{\tau}}$ is again enough to determine the solution trajectory given any initial condition, it is unnecessary, because all individuals with the same tempers act in the same way. This motivates us to categorize equal tempers into the same *type*. Let $\mathrm{p} \in \mathbb{N}$ denote the total number of types and label the types in the ascending order as $1, 2, \ldots, \mathrm{p}$, where $\mathrm{p} \geq 2$. Then by denoting the temper of type-$p$ by $\tau_p$, we have that $\tau_1 < \tau_2, \ldots < \tau_{\mathrm{p}}$. Define the *categorized temper vector* as $\boldsymbol{\tau} = (\tau_p)_{p=1}^{\mathrm{p}}$. Stack the frequencies of each type to obtain the *temper distribution* $\boldsymbol{n}_{\boldsymbol{\tau}} \triangleq (n_1, ..., n_{\mathrm{p}})$, where $n_p$ is the number of *type-$p$* individuals which belongs to the set $\boldsymbol{\Theta} = \{\boldsymbol{z} \in \mathbb{Z}_{\geq 0}^{\mathrm{p}} \mid \sum_{p=1}^{\mathrm{p}} z_p = n\}$. Correspondingly, define the *categorized state* $\boldsymbol{x}(t) = (A_1(t), \ldots, A_{\mathrm{p}}(t))$ where $A_p$ is the number of $\mathtt{A}$-selected type-$p$ individuals, i.e., those with temper $\tau_p$ who have selected action $\mathtt{A}$. The state space equals $\boldsymbol{\mathcal{X}} = \{\boldsymbol{z} \in \mathbb{Z}_{\geq 0}^{\mathrm{p}} \mid z_p \leq n_p\}$. It can be shown that the categorized temper vector $\boldsymbol{\tau}$ with frequencies $\boldsymbol{n}_{\boldsymbol{\tau}}$ is sufficient to determine the dynamics for a given initial condition. Hence, we can now define the synchronous system in the following form by considering $\boldsymbol{n}_{\boldsymbol{\tau}}$ as the parameter.

*Definition 2 (Synchronous system):* Given the categorized temper vector $\boldsymbol{\tau} \in \{0, \ldots, n\}^{\mathrm{p}}$, the autonomous *synchronous decision-making system* is defined by

$$\begin{cases} \boldsymbol{x}(t+1) = \sum_{p=1}^{\mathrm{p}} n_p \mathbb{1}(A(\boldsymbol{x}), \tau_p), \quad \boldsymbol{x}_0 = \boldsymbol{x}(0) \\ y(t) = \sum_{p=1}^{\mathrm{p}} n_p g(\boldsymbol{x}(t); \boldsymbol{n}_{\boldsymbol{\tau}}), \end{cases} \tag{3}$$

where $\boldsymbol{x}(t) \in \boldsymbol{\mathcal{X}}$, $n_p$'s are the parameters that are stacked to form the parameter vector $\boldsymbol{n}_{\boldsymbol{\tau}}$, with the parameter space $\boldsymbol{\Theta}$, $\boldsymbol{x}_0$ is the initial state, $y(t)$ is the output which belongs to the space $\boldsymbol{\mathcal{Y}} = \{0, \ldots, n\}$.

*Example 2:* Consider the synchronous system with $n = 6$, $\boldsymbol{\tau} = (0, 2, 4)$ and $\boldsymbol{n}_{\boldsymbol{\tau}} = (2, 1, 3)$. Starting with initial state $\boldsymbol{x}_0 = (0, 0, 0)$, the population state evolves through $\boldsymbol{x}(1) =$

$(2,0,0)$ and $\boldsymbol{x}(t)=(2,1,0)$ for all $t\geq 2$. Type-3 individuals never switch to A as they never meet their temper.

We first provide the necessary identifiability definitions to determine the identifiability of systems.

## III. Preliminaries: Identifiability

Starting with the same initial condition and under the same input sequence, if two different parameter values yield identical output trajectories, then the system is not identifiable. This can be analyzed both globally and locally; global identifiability guarantees the uniqueness in the whole parameter space while in local identifiability, the space is restricted to a subset of the space. We use the notions of identifiability defined in [27] for discrete space systems, but rewrite them here in terms of the decision-making systems. The output sequences of the systems in Definition 1 and Definition 2 are denoted by $\langle y(t,\bar{\boldsymbol{x}}_0,\langle u_t\rangle_0^{T-1};\bar{\boldsymbol{\tau}})\rangle_0^T$ and $\langle y(t,\bar{\boldsymbol{x}}_0;\bar{\boldsymbol{\tau}})\rangle_0^T$ respectively.

*Definition 3 ( [27] definition 1):* The asynchronous system in Definition 1 is **locally (strongly) $\bar{\boldsymbol{x}}_0$-identifiable** at $\bar{\boldsymbol{\tau}}\in\bar{\boldsymbol{\Theta}}$ through the input sequence $\langle u_t\rangle_0^{T-1}\in\{1,\dots,n\}^{T-1}$ for some $T>0$, if there exists a neighborhood $\bar{\boldsymbol{\Theta}}'\subset\bar{\boldsymbol{\Theta}}$ of $\bar{\boldsymbol{\tau}}$, such that for any $\bar{\boldsymbol{\tau}},\bar{\boldsymbol{\tau}}'\in\bar{\boldsymbol{\Theta}}'$,

$$\begin{aligned}&\bar{\boldsymbol{\tau}}\neq\bar{\boldsymbol{\tau}}'\Rightarrow\\&\langle y(t,\bar{\boldsymbol{x}}_0,\langle u_t\rangle_{t=0}^{T-1};\bar{\boldsymbol{\tau}})\rangle_0^T\neq\langle y(t,\bar{\boldsymbol{x}}_0,\langle u_t\rangle_{t=0}^{T-1};\bar{\boldsymbol{\tau}}')\rangle_0^T.\end{aligned}\tag{4}$$

Definition 3 is for the identifiability at $\bar{\boldsymbol{\tau}}$ for a given $\bar{\boldsymbol{x}}_0$. The notion of *structural identifiability* requires subsets of initial conditions $\bar{\boldsymbol{\mathcal{X}}}'\subseteq\boldsymbol{\mathcal{X}}$, parameters $\bar{\boldsymbol{\Theta}}'\subseteq\bar{\boldsymbol{\Theta}}$, and inputs $\boldsymbol{\mathcal{U}}'\subseteq\boldsymbol{\mathcal{U}}^T$, defined as $\boldsymbol{\mathcal{U}}\times\ldots\times\boldsymbol{\mathcal{U}}$ for $T>0$ times, where the system is identifiable for almost all the points in these subsets except for those of a measure zero [22], [25], [29]. This is, however, not applicable to our setup as the size of these spaces is finite.

*Definition 4 ( [27]):* The asynchronous system in Definition 1 is **locally structurally identifiable** if there exist a $T>0$ and subsets $\bar{\boldsymbol{\Theta}}'\subset\bar{\boldsymbol{\Theta}}$, $\bar{\boldsymbol{\mathcal{X}}}'\subseteq\bar{\boldsymbol{\mathcal{X}}}$, and $\boldsymbol{\mathcal{U}}'\subseteq\boldsymbol{\mathcal{U}}^T$, such that the system is locally (strongly) $\bar{\boldsymbol{x}}_0$-identifiable at $\bar{\boldsymbol{\tau}}$ through the input sequence $\langle u_t\rangle_{t=0}^{T-1}$ for all $\bar{\boldsymbol{\tau}}\in\bar{\boldsymbol{\Theta}}'$, $\bar{\boldsymbol{x}}_0\in\bar{\boldsymbol{\mathcal{X}}}'$, and $\langle u_t\rangle_{t=0}^{T-1}\in\boldsymbol{\mathcal{U}}'$.

Finally, we provide the definition of global identifiability.

*Definition 5 ( [27]):* The asynchronous system in Definition 1 is **globally identifiable** at $\bar{\boldsymbol{\tau}}$ if there exist a $T>0$ and an input sequence $\langle \boldsymbol{u}_t\rangle_{t=0}^{T-1}$ such that for all $\bar{\boldsymbol{\tau}}'\in\bar{\boldsymbol{\Theta}}$ and all $\bar{\boldsymbol{x}}_0\in\bar{\boldsymbol{\mathcal{X}}}$, (4) holds.

Definitions for the synchronous system can be obtained similarly by removing the input sequence. All definitions presented so far are for the identifiability of the system, that is whether all parameters of the system are identifiable. However, even if the whole system is not identifiable, some parameters may be identifiable. To analyze the identifiability of a parameter, similar definitions as Definitions 3 to 5 can be used with the difference of replacing the whole parameter vector $\bar{\boldsymbol{\tau}}$ with a single parameter $\tau^i$ for some $i\in\{1,\dots,n\}$.

## IV. Identifiability of Synchronous System

In synchronous updating, if the total number of A-selected individuals at time $t$ is $A(t)$, then at time $(t+1)$ every individual who has a temper less than or equal to $A(t)$ will update to A and others will switch to B resulting in $\boldsymbol{x}(t{+}1)=(n_1,\dots,n_p,0,\dots,0)$ for some $p\in\{1,\dots,\mathtt{p}\}$. Then the output of the system at $t+1$ is $y(t+1)=\sum_{k=1}^{p}n_k$.

Define the cumulative distribution function of the tempers by $F(p,\boldsymbol{n_\tau})=\sum_{k=1}^{p}n_k$, where $p\in\{1,\dots,\mathtt{p}\}$. It can be shown that $F(p,\boldsymbol{n_\tau})$ is a piece-wise constant, non-decreasing, and right-continuous function with discontinuities at all points $k\in\{1,\dots,\mathtt{p}\}$. The system output is the same as the cumulative tempers at some $p\in\{1,\dots,\mathtt{p}\}$. Let $\hat{\boldsymbol{\mathcal{X}}}=\{\boldsymbol{z}\in\mathbb{Z}_{\geq 0}^{\mathtt{p}}\mid z_p<n_p\}$, that is the set of states where the individuals of none of the types have all selected A. Define $\hat{\boldsymbol{\Theta}}=\{\boldsymbol{z}\in\mathbb{Z}_{\geq 1}^{\mathtt{p}}|\sum_{p=1}^{\mathtt{p}}z_p=n\}$ and $y(\boldsymbol{x}_0)=y(0)$, that is the output at time zero. The following is the main result of this subsection.

*Theorem 1:* Consider the synchronous system in Definition 2 with initial condition $\boldsymbol{x}_0\in\hat{\boldsymbol{\mathcal{X}}}$. The system is locally $\boldsymbol{x}_0$-identifiable at $\boldsymbol{n_\tau}\in\hat{\boldsymbol{\Theta}}$ if and only if one of the following two cases hold:

*Case 1.* $\boldsymbol{x}_0$ and $\boldsymbol{n_\tau}$ satisfy

$$y(\boldsymbol{x}_0)\in[\tau_1,\tau_2),\tag{5}$$

$$(\forall p\in\{1,\dots,\mathtt{p}-2\})\quad F(p,\boldsymbol{n_\tau})\in[\tau_{p+1},\tau_{p+2}).\tag{6}$$

*Case 2.* $\boldsymbol{x}_0$ and $\boldsymbol{n_\tau}$ satisfy

$$y(\boldsymbol{x}_0)\in[\tau_{\mathtt{p}-1},\tau_{\mathtt{p}}),\tag{7}$$

$$(\forall p\in\{2,\dots,\mathtt{p}-1\})\quad F(p,\boldsymbol{n_\tau})\in[\tau_{p-1},\tau_p).\tag{8}$$

Moreover, if (5) and (6) hold, then

$$\begin{cases}n_1=y(1),\\ n_p=y(p)-y(p-1),\quad p=2,\dots,\mathtt{p}-1,\\ n_{\mathtt{p}}=n-y(\mathtt{p}-1),\end{cases}\tag{9}$$

and if (7) and (8) hold, then

$$\begin{cases}n_1=y(\mathtt{p}-1),\\ n_p=y(p-1)-y(p),\quad p=2,\dots,\mathtt{p}-1,\\ n_{\mathtt{p}}=n-y(1).\end{cases}\tag{10}$$

We provide all the proofs in the appendix. The intuition behind Theorem 1 is that the identifiability of a synchronous system is ensured when exactly one type of individuals change their actions at a time and every type of individual changes their strategy at some point. The system output should start at a value between the least temper and the one after and gradually increase to achieve this (or the other way around as in the second case of the theorem). This intuition is illustrated by the following example.

*Example 3 (revisiting Example 2):* Consider the output trajectory $\langle y(t)\rangle_{t=0}^{\infty}=\langle 0,2,3,3,\dots\rangle$ depicted by green in Fig. 2. By observing the output, parameter $\boldsymbol{n_\tau}=(2,1,3)$ can be uniquely obtained as $n_1=y(1)=2$, $n_2=y(2)-y(1)=1$ and $n_3=n-y(2)-y(2)=6-3=3$. Consider another output trajectory $\langle y(t)\rangle_{t=0}^{\infty}=\langle 0,4,6,6,\dots\rangle$ with the same initial condition given by blue in Fig. 2. It implies that $n_1=y(1)=4$, thus the parameter $n_1$ is identifiable. However, $n_2+n_3=y(2)=2$ as both type-2 and 3 individuals met their tempers at time $t=1$. Hence, parameters $n_2$ and $n_3$ are not identifiable as all three temper distributions $(4,1,1)$, $(4,2,0)$, and $(4,0,2)$ can result in the same output.

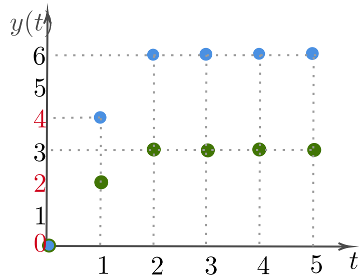


Fig. 2: Output of the synchronous system with $n = 6$, $\boldsymbol{n_\tau} = (2, 1, 3)$, and $\boldsymbol{\tau} = (0, 2, 4)$ given in Example 3.

Example 3 also illustrates how one parameter can be identifiable but not the whole system. The following lemma provides the condition for the identifiability of one parameter.

*Lemma 1:* Consider the synchronous system in Definition 2 with initial condition $\boldsymbol{x}_0 \in \hat{\boldsymbol{\mathcal{X}}}$. Parameter $n_p, p \in \{2, \dots, \mathrm{p}-1\}$, is locally $\boldsymbol{x}_0$-identifiable at $n_p^*$, where $n_p^* \geq 1$, if and only if there exists some time $T \geq 0$ such that at least one of the following two conditions are met:

$$\begin{cases} y(T) \in [\tau_{p-1}, \tau_p), \\ y(T+1) \in [\tau_p, \tau_{p+1}), \end{cases} \tag{11}$$

or

$$\begin{cases} y(T) \in [\tau_p, \tau_{p+1}), \\ y(T+1) \in [\tau_{p-1}, \tau_p). \end{cases} \tag{12}$$

Additionally, under (11) and (12), $n_p^* = |y(T+2) - y(T+1)|$.

According to Lemma 1, to know the number of individuals with temper $\tau_p$, the total number of A-selected individuals should consecutively enter the intervals $[\tau_{p-1}, \tau_p)$ and $[\tau_p, \tau_{p+1})$ in either direction.

*Remark 1:* The categorized tempers $\tau_1, \dots, \tau_p$ are assumed to be known in Theorem 1 and Lemma 1, because the only parameter in the synchronous system in Definition 2 is the temper distribution $\boldsymbol{n_\tau}$. By observing the monotonically increasing output of the system with $\boldsymbol{n_\tau}$ (Case 1 of Theorem 1), it is unclear whether after a jump in the output, say from $y(T) \in [\tau_{p-1}, \tau_p)$ the new value $y(T+1)$ falls in the interval $[\tau_p, \tau_{p+1})$ or beyond. If it falls in the interval $[\tau_{p+1}, n)$, then we never know about the existence of the temper $\tau_p$ and may incorrectly assume that the temper after $\tau_{p-1}$ is $\tau_{p+1}$. This is not the case when the number of types p is known a priori as then Conditions (5) and (6) are met if the number of jumps in $y$ are $\mathrm{p} - 2$ (if $y$ does not reach the total population size $n$ at its final jump; otherwise, the conditions are met if the number of jumps in $y$ are $\mathrm{p} - 1$).

We end this section with the following result.

*Corollary 1:* The synchronous system in Definition 2 is locally structurally identifiable but not globally identifiable.

## V. Identifiability of Asynchronous System

The asynchronous dynamics depend on both the tempers and the activation sequence. The following result determines the identifiability of the system.

*Theorem 2:* Asynchronous system Definition 1 is locally $\bar{\boldsymbol{x}}_0$-identifiable at $\bar{\boldsymbol{\tau}}$ (through some input) if and only if every individual becomes active and chooses different actions at two consecutive output values.

According to the theorem, one needs to observe the output trajectory and activation sequence to see whether every individual becomes active at some output value $y^*$ and also becomes active at $y^* \pm 1$ and chooses a different action than that at $y^*$. The parametric identifiability requires the same for just the individual whose temper is the parameter of interest and is rigorously stated in Lemma 3 in the appendix. The following example illustrates these results.

*Example 4:* Consider an asynchronous system with $n = 3$, $\bar{\boldsymbol{\tau}} = (0, 1, 2)$, and $\langle u_t \rangle_{t=0}^{\infty} = \langle 3, 2, 1, 3, 2, 1, \dots \rangle$. Now consider another temper distribution $\bar{\boldsymbol{\tau}}' = (0, 1, 1)$. Starting with the state $\bar{\boldsymbol{x}}_0 = \{\mathtt{B}, \mathtt{B}, \mathtt{B}\}$, the evolution of the population state for both temper distributions is given in Table II. Clearly, at time $t = 4$, the systems have two different population states resulting in different outputs. This is in line with Lemma 3 as individual 3 is active at time $t_1 = 3$ when the output is equal to his temper, $y(3) = 1$ under $\bar{\boldsymbol{\tau}}'$.

TABLE II: Transition of the states for $\bar{\boldsymbol{\tau}}$ and $\bar{\boldsymbol{\tau}}'$ over time.

| Time | State for $\boldsymbol{\tau}$ | Output for $\boldsymbol{\tau}$ | State for $\boldsymbol{\tau}'$ | Output for $\boldsymbol{\tau}'$ |
|---|---|---|---|---|
| 0 | (B, B, B) | 0 | (B, B, B) | 0 |
| 1 | (B, B, B) | 0 | (B, B, B) | 0 |
| 2 | (B, B, B) | 0 | (B, B, B) | 0 |
| 3 | (A, B, B) | 1 | (A, B, B) | 1 |
| 4 | (A, B, B) | 1 | (A, B, A) | 2 |
| 5 | (A, A, B) | 2 | (A, A, A) | 3 |

However, for what detailed temper distribution $\bar{\boldsymbol{\tau}}$ and activation sequence, the conditions in Theorem 2 and Lemma 3 are met? This appears to be a complex problem. Let us start with the simple case when $\bar{\boldsymbol{\tau}} = (0, 1, \dots, n-1)$. Then the system is identifiable only for a non-self-fulfilling initial condition as explained the following. Define the total number of A-selected individuals at state $\bar{\boldsymbol{x}}$ who have a temper less than or equal to $i$ by $F_A(i, \bar{\boldsymbol{x}}) = \sum_{k=1}^{i} x^k$. Individual $i \in \{1, \dots, n\}$ is *self-fulfilling* at state $\bar{\boldsymbol{x}}$ if $F_A(i, \bar{\boldsymbol{x}}) \geq \tau^i$ for $\tau^i > 0$ and 0 otherwise. Namely, a self-fulfilling individual can never switch his action to B. A population state $\bar{\boldsymbol{x}}$ is *self-fulfilling* if at least one of the individuals is so, and otherwise the state is *not self-fulfilling*. Being not self-fulling ensures that all individuals, except for those with a threshold of zero, can switch from A to B under an appropriate activation sequence. The states of all-A and all-B are examples of self-fulfilling and not self-fulling states.

*Proposition 1:* The asynchronous system in Definition 1 is locally $\bar{\boldsymbol{x}}_0$-identifiable at $\bar{\boldsymbol{\tau}} = (0, 1, \dots, n-1)$ (through some input) if and only if $\bar{\boldsymbol{x}}_0$ is not self-fulfilling.

Proposition 1 leads to another result. When all individuals initially choose action B–as in many real-world scenarios such as the spread of innovation or diseases the only identifiable detailed threshold vector is $(0, 1, \dots, n-1)$.

*Proposition 2:* The asynchronous system in Definition 1 is locally $\bar{\boldsymbol{x}}_0 = (\mathtt{B}, \dots, \mathtt{B})$-identifiable at $\bar{\boldsymbol{\tau}}$ (through some input) if and only if $\bar{\boldsymbol{\tau}} = (0, 1, \dots, n-1)$.

*Corollary 2:* The asynchronous system in Definition 1 is locally structurally but not globally identifiable.

## VI. Concluding remarks

We analyzed the identifiability of the linear threshold model for under synchronous and asynchronous dynamics. We derived the necessary and sufficient conditions for the identifiability of temper distributions, which implies the identifiability of threshold distributions. The thresholds are not globally identifiable, however, they are locally structurally identifiable, meaning there exists a subset of the parameter space and a set of initial states where the system is identifiable only if it starts from those initial states and the threshold distribution belongs to that parameter subset.

In the context of the spread of innovation, the results imply that identifiability is guaranteed if and only if the number of adaptors can be observed from the beginning, that is, from zero adaptors. By observing how the number of adaptors changes over time, the threshold distribution can be obtained: the length of a jump in the number of adaptors equals the number of individuals of one same type. However, if the number of adaptors increases in a way that it bypasses certain threshold levels, identifying those thresholds becomes impossible, although other thresholds can still be identified. Furthermore, while the exact threshold values may not be uniquely determined, tight upper and lower bounds exist.

The results for the asynchronous case are somewhat discouraging in the sense that identifying the threshold of each individual requires them being active at two consecutive output values and choosing different actions at each value.

The results open the door to confidently estimate the thresholds in linear threshold dynamics. Knowing the distribution of thresholds can be insightful by itself, and it also allows for the control of the dynamics, bringing us closer to real-world implementation of existing control algorithms. Extending the results to populations consisting of imitators, those who imitate the highest earners in terms of payoffs in a game theory framework [30], [31] is subject to future research.

## Appendix

*Proof:* [Proof of Lemma 1] (sufficiency) If (12) holds, then $y(T+1) = F(p, \boldsymbol{n_\tau})$, as the temper of all types $1, \dots, p-1$ are met at time $T$ and no other type's temper is met. Consequently, because of the second condition in (12), $y(T+2) = F(p-1, \boldsymbol{n_\tau})$. Therefore, $y(T+1) - y(T+2) = n_p^*$. Hence, for any temper distribution $\boldsymbol{n'_\tau}$ different from $\boldsymbol{n_\tau}$ at $n_p^*$, i.e., $n'_p \neq n_p^*$, the outputs at time $T+1$ or $T+2$ will be different. The proof for (11) is similar. (necessity) Assume on the contrary that neither (12) nor (11) holds. Then one of the following cases holds: *Case 1. For all $T \geq 0$, either $y(T) < \tau_{p-1}$ or $y(T) \geq \tau_{p+1}$.* Then the same system with a different parameter defined by $\boldsymbol{n'_\tau} = (n_1, \dots, n_{p-2}, n_{p-1} - 1, n_p^* + 1, n_{p+1}, \dots, n_{\mathrm{p}})$ will exhibit the same output trajectory. This can be shown as follows. Since $\boldsymbol{x}_0 \in \hat{\boldsymbol{\mathcal{X}}}$, $x_{p-1}(0) < n_{p-1}$. Thus, reducing $n_{p-1}$ by one does not affect the initial condition in the new system and it keeps $\boldsymbol{n'_\tau}$ feasible as $\boldsymbol{n_\tau} \in \hat{\boldsymbol{\Theta}}$. On the other hand, at every time $t \geq 1$ in the original system, either all individuals of both types $p-1$ and $p$ update simultaneously to $\mathtt{B}$ or $\mathtt{A}$. In either case, $y(t)$ changes in the same way in both systems as $x_{p-1}(t) + x_p(t)$ either equals 0 or $n_{p-1} + n_p^*$. *Case 2.* One of the following two cases:

$$\begin{cases} (\forall T \geq 0)\ y(T) \in [\tau_p, \tau_{p+1}) \Rightarrow y(T+1) \geq \tau_p, \text{ or} \\ (\forall T \geq 0)\ y(T) \in [\tau_p, \tau_{p+1}) \Rightarrow y(T+1) < \tau_{p-1}, \end{cases} \tag{13}$$

together with one of the following two cases are in force:

$$\begin{cases} (\forall T \geq 0)\ y(T) \in [\tau_{p-1}, \tau_p) \Rightarrow y(T+1) \geq \tau_{p+1}, \text{ or} \\ (\forall T \geq 0)\ y(T) \in [\tau_{p-1}, \tau_p) \Rightarrow y(T+1) < \tau_p. \end{cases} \tag{14}$$

Using Lemma 2, it can be shown that (13) and (14) imply

$$(\forall T \geq 0, k \geq 1)\ y(T) \in [\tau_p, \tau_{p+1}) \Rightarrow y(T+k) \notin [\tau_{p-1}, \tau_p), \tag{15}$$

$$(\forall T \geq 0, k \geq 1)\ y(T) \in [\tau_{p-1}, \tau_p) \Rightarrow y(T+k) \notin [\tau_p, \tau_{p+1}). \tag{16}$$

Now (15) and (16) imply that either the output never enters any of the intervals $[\tau_{p-1}, \tau_p)$ and $[\tau_p, \tau_{p+1})$, which is the same as Case 1, or the output enters only one of these intervals and never enters the other. Should the output never enter the interval $[\tau_{p-1}, \tau_p)$, type-$p$ individuals would always choose the same action as type-$(p-1)$ individuals. Hence, the system with the aforementioned parameter $\boldsymbol{n'_\tau}$ will exhibit the same output trajectory. Should the output never enter the interval $[\tau_p, \tau_{p+1})$, type-$p$ individuals would always choose the same action as type-$(p+1)$ individuals. Thus, the system with the parameter $\boldsymbol{n''_\tau} = (n_1, \dots, n_{p-1}, n_p^* - 1, n_{p+1} + 1, n_{p+2} \dots, n_{\mathrm{p}})$ will again exhibit the same output trajectory. Hence, the system becomes unidentifiable. ∎

*Lemma 2:* Consider the synchronous system (2). Consider an arbitrary time $T \geq 1$ and let $p$ be the type that satisfies $y(T) \in [\tau_p, \tau_{p+1})$. If $y(T+1) \geq \tau_p$, then $y(T+k) \geq \tau_p$ for all $k \geq 1$ and if $y(T+1) < \tau_p$, then $y(T+k) < \tau_{p+1}$ for all $k \geq 1$.

The proof follows induction on $k$.

*Proof:* [Proof 1 of Theorem 1] (sufficiency) If Case 1 holds, then the population state evolves through $\boldsymbol{x}(1) = (n_1, 0, \dots, 0)$, $\boldsymbol{x}(2) = (n_1, n_2, 0, \dots, 0)$, …, $\boldsymbol{x}(\mathrm{p}-1) = (n_1, n_2, \dots, n_{\mathrm{p}-1}, 0)$. That is, for each $p \in \{1, \dots, \mathrm{p}-1\}$, there exist a time $T \geq 0$ that satisfies (11). For $p = \mathrm{p}$, $n_{\mathrm{p}}$ simply equals $n - y(\mathrm{p}-1)$. This proves the first case as well as (9). The second case and (10) can be proven similarly. (necessity) Define the notation $\mathcal{I}_p = [\tau_p, \tau_{p+1})$ for $p = 1, \dots, \mathrm{p}$. Define a directed graph whose nodes are $\mathcal{I}_p$ and there is a link from $\mathcal{I}_p$ to $\mathcal{I}_k$ if and only if for some time $T \geq 0$, $y(T) \in \mathcal{I}_p$ and $y(T+1) \in \mathcal{I}_k$. Clearly, the out-degree of each node in this graph is 1. On the other hand, in view of Lemma 1, for each parameter $n_p, p = 2, \dots, \mathrm{p}-1$ to be identifiable, the output should either go from $\mathcal{I}_{p-1}$ to $\mathcal{I}_p$ or vice versa. Hence, the graph must be in the form of a directed path: either $\mathcal{I}_2 \to \mathcal{I}_3 \to \dots \mathcal{I}_{\mathrm{p}-1}$ or $\mathcal{I}_{\mathrm{p}-1} \to \mathcal{I}_{\mathrm{p}-2} \to \dots \mathcal{I}_2$. On the other hand, in the first case, once the output enters $\mathcal{I}_2$, it will not enter $\mathcal{I}_1$ afterwards due to Lemma 2. Moreover, the output must reach $\mathcal{I}_1$ at some point, as otherwise, the temper distribution $\boldsymbol{n'_\tau} = \boldsymbol{n_\tau} + (1, -1, 0, \dots, 0)$ exhibits the same output trajectory. Hence, the graph must be $\mathcal{I}_1 \to \mathcal{I}_2 \to \mathcal{I}_3 \to \dots \mathcal{I}_{\mathrm{p}-1}$. Now, again due to Lemma 2, the graph either leaves $\mathcal{I}_{\mathrm{p}}$ as a singleton or reaches it from $\mathcal{I}_p$. This proves Case 1 and that of Case 2 is similar. ∎

*Proof:* [Proof of Corollary 1] The proof of the structural identifiability follows, Theorem 1 by choosing the subset $\boldsymbol{\Theta'} = \hat{\boldsymbol{\Theta}}$ that satisfy (6) and the subset $\boldsymbol{\mathcal{X}}' = \hat{\boldsymbol{\mathcal{X}}}$ that satisfy (5). The proof of the global unidentifiability is trivial. ∎

Proof of Theorem 2 is an immediate result of Lemma 3.

*Lemma 3:* Consider the asynchronous system Definition 1 with initial condition $\bar{\boldsymbol{x}}_0$. Parameter $\tau^i, i \in \{1, \ldots, n\}$, is locally $\bar{\boldsymbol{x}}_0$-identifiable at $\tau^* \geq 1$ (through some input) if and only if there exist times $t_1$ and $t_2$ such that *(i)* $y(t_1) = \tau^* - 1$ and individual $i$ becomes active at $t_1$ and chooses action $\mathtt{B}$, and *(ii)* $y(t_2) = \tau^*$ and individual $i$ becomes active at $t_2$ and chooses action $\mathtt{A}$.

*Proof:* [Proof of Proposition 1] (sufficiency) Define $\langle X \rangle^k$ as the sequence of $\langle X, \ldots, X \rangle$ where $X$ is repeated $k$ times, $k \geq 1$. Under the activation sequence $\langle n, n-1, \ldots, 2 \rangle$, the output reaches the state of $(\mathtt{B}, \ldots, \mathtt{B})$. Then, under the activation sequence $\langle 1, \langle n, n-1, \ldots, 1 \rangle^n \rangle$, the output trajectory becomes $\langle 0, \langle 1 \rangle^{n-1}, \langle 2 \rangle^{n-1}, \ldots, \langle n \rangle^{n-1} \rangle$ and each individual $i$ becomes active at each output value $0, 1, \ldots, n$, completing the proof according to Lemma 3. (necessity) If an individual $i$ is self-fulfilling at the initial state $\bar{\boldsymbol{x}}_0$, then by induction over time $t$, it can be shown that individuals $1, \ldots, i$ will never choose $\mathtt{B}$, which violates Lemma 3. ∎

*Proof:* [Proof of Proposition 2] The sufficiency is the same as that in the proof of Proposition 1. For the necessity, if $\bar{\boldsymbol{\tau}} \neq (0, 1, \ldots, n-1)$, there are more than one individual with the same temper. Let $i, j$ be two individuals with the same temper $\tau$. Starting from $\bar{\boldsymbol{x}}_0$, individuals can only switch from $\mathtt{B}$ to $\mathtt{A}$, hence, the output never reaches back to $\tau$, after it increases to $\tau + 1$. Hence, both individuals $i$ and $j$ cannot be active when the output equals $\tau$. ∎


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