2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Altruism Improves Congestion in Series-Parallel Nonatomic Congestion Games

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***Abstract*— Self-interested routing policies from individual users in a system can collectively lead to poor aggregate congestion in routing networks. The introduction of altruistic agents (referred to as *altruists*), whose goal is to minimize other agents' routing time in addition to their own, can seemingly improve aggregate congestion. However, in some network routing problems, it is known that altruists can actually worsen congestion compared to that which would arise if all agents had simply behaved selfishly. This paper provides a thorough investigation into the necessary conditions for altruists to be guaranteed to improve total congestion. In particular, we study the class of series-parallel nonatomic congestion games, where one sub-population is selfish and the other is altruistic. We find that a game is guaranteed to have improved congestion in the presence of altruists (regardless of their population size) compared to if all agents route selfishly, provided the path set for the network is *symmetric* (all agents can access all paths), and the path set cannot exhibit Braess's paradox (a phenomenon we refer to as a *Braess-resistant* path set). Our results appear to be the most complete characterization of when behavior that is designed to improve total congestion (which we refer to as altruism) is guaranteed to do so.**

I. INTRODUCTION

As society and technology become increasingly connected, there is growing demand to understand and improve the coordination between human behavior and technological operations within sociotechnical systems [1]. The growing presence of autonomous vehicles provides another opportunity for system designers to optimize traffic routing in transportation networks, making it a canonical, (and significant) example of a sociotechnical system. It is well-known that when agents choose routes solely to minimize their own travel time, suboptimal congestion can emerge [2]. Game theory offers promising concepts to address this problem, providing well-studied methods to compare the inefficiencies resulting from the behaviors of self-interested agents with that of the optimal aggregate congestion for a network [3].

Centralized routing control is one method to reduce network inefficiencies; however, it is often infeasible, so many studies have aimed to improve network efficiency in a less centralized manner. Techniques designed to influence routing behavior have been studied such as direct fleet routing strategies [4], information distribution [5], and monetary incentives such as tolls and subsidies [6]. Additionally, a system designer for a fleet of autonomous vehicles may design the fleet's routing policies to consider their own contribution to congestion, a concept we refer to as *altruism* [7], [8]. The system designer then needs to understand when the presence of altruistic agents improve aggregate congestion relative to all-selfish traffic?

A common measure of the inefficiency that arises from an all-selfish population is the *price of anarchy*; the ratio between the worst-case congestion in a system where all agents route to minimize their own commute time (modeled by a *Nash equilibrium*) and the congestion that can be achieved if a system designer centrally coordinates all agents' actions for the overall benefit of the system [3]. While a system designer for an autonomous fleet may be interested in the price of anarchy, she may also be interested in whether using an altruistic routing policy improves congestion compared to a selfishly routed fleet. The inefficiency that may arise from altruism is formalized with the *perversity index*; the ratio of the worst-case congestion that arises when a heterogeneous population of agents routes to minimize the latency they experience at Nash equilibrium, compared to the congestion that arises if all agents route selfishly [9]. If the perversity index is less than 1 (e.g., when the entire population is altruistic [10]), altruism improves congestion relative to selfishness. If the perversity index is greater than 1, altruists may actually worsen congestion compared to if they route selfishly – a phenomenon referred to as *perversity*. That is, partially-adopted altruism has the potential to degrade congestion relative to an all-selfish population [7].

The key motivation for our work is to fully characterize the necessary network requirements so altruists are guaranteed to not worsen congestion. That is, what constraints must be put on a network so that the perversity index is less than 1 for a heterogeneous population of altruistic and selfish agents? Previous work has provided the network topology that produces efficient equilibria [11], and has shown that altruism cannot degrade congestion on serially-linearly-independent networks [12]. Our work extends previous results, and shows that altruism cannot degrade series-parallel networks provided the path set adheres to two requirements: it is symmetric and the path set does not exhibit Braess's paradox. Theorem 3.1 shows that altruism is certain to improve aggregate congestion in person-hours, provided the network and path set restrictions mentioned above. That is, if selfish agents become altruistic, they are assured to improve aggregate congestion; likewise, altruists are guaranteed to worsen aggregate congestion if they start routing selfishly.

II. MODEL AND RELATED WORK

A. Routing Problem

We consider a routing problem on a network (V, E), for vertex set V and edge set E. Each edge $e \in E$ connects

*This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-23-1-0171 and by the National Science Foundation under award number ECCS-2013779.

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two distinct vertices, and we say that an edge e is *incident* with vertex $v \in V$ if v is a vertex of e. A sub-path σ is an alternating sequence of distinct vertices and edges, beginning and ending with vertices, where each edge $e \in \sigma$ is incident with the vertex preceding and the vertex succeeding e. The vertices that are incident with σ are the first and last vertex of σ. A path p comprises a set of edges or sub-paths connecting common origin o to common destination t. We write $\mathcal{R} \subseteq 2^E$ to denote the set of *all* feasible paths connecting o to t, and is referred to as the set of *routes*, and $\mathcal{P} \subseteq \mathcal{R}$ to denote the set of *paths* accessible to agents. We restrict network topology to *series-parallel*; a series-parallel network consists of

1) a single edge,
2) two series-parallel networks connected in series, or
3) two series-parallel networks connected in parallel [11].

A unit mass of traffic is routed from o to t and is composed of two types, an *altruistic* sub-population and a *selfish* sub-population. Altruistic agents comprise mass r^{a}, and selfish users make up mass r^{s}, so that $r^{\mathrm{a}} + r^{\mathrm{s}} = 1$. Each type $\theta \in \{\mathrm{a}, \mathrm{s}\}$ of traffic can access an arbitrary subset of paths $\mathcal{P}^\theta \subseteq \mathcal{P}$, where x_p^θ denotes the flow of agents of type θ using path $p \in \mathcal{P}^\theta$. A *feasible flow* for type θ assigns r^θ mass of traffic to paths in $\mathcal{P}^\theta$, denoted by $x^\theta \in \mathbb{R}_{\geq 0}^{|\mathcal{P}^\theta|}$, such that $\sum_{p \in \mathcal{P}} x_p^\theta = r^\theta$. A *network flow* is a combined allocation of altruistic and selfish agents to paths, denoted $x \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}$, such that $x_p := \sum_{\theta : p \in \mathcal{P}^\theta} x_p^\theta$ for all $p \in \mathcal{P}$, where x^θ is feasible for each respective type. Provided network flow x, the flow on edge $e \in E$ is given by $x_e = \sum_{p : e \in p} x_p$, where flow of type $\theta = \{\mathrm{a}, \mathrm{s}\}$ on edge e is denoted by x_e^θ. For each $e \in E$, commute time is expressed as a function of traffic flow and is associated with a latency function $\ell_e : [0, 1] \to [0, \infty)$, where $\ell_e(x_e)$ denotes the actual cost (in person-hours) it takes to use edge e with flow x_e. We assume the latency function for each edge is a non-decreasing, convex posynomial. So, for every $e \in E$,

$$\ell_e(x_e) = \sum_{i=0}^{d} a_{e,i} x_e^i, \tag{1}$$

where $a_{e,i} \in \mathbb{R}_{\geq 0}$ and degree $d \in \mathbb{N}$. For any path or sub-path ρ, we define the latency of ρ to be the sum of the latencies of each edge $e \in \rho$:

$$\ell_\rho(x) := \sum_{e \in \rho} \ell_e(x_e). \tag{2}$$

The aggregate congestion that arises from a given flow is represented by the *total latency*, given by

$$\mathcal{L}(x) = \sum_{e \in E} x_e \ell_e(x_e) = \sum_{p \in \mathcal{P}} x_p \ell_p(x). \tag{3}$$

Intuitively, total latency represents the total person-hours that a population spends travelling from origin o to destination t.

An instance of a *routing problem* is fully specified by the tuple $G = (V, E, \{\ell_e\}_{e \in E}, \mathcal{P}, r^{\mathrm{a}})$, and we write $\mathcal{G}(d, r^{\mathrm{a}})$ to denote the set of all routing problems on series-parallel networks with posynomial latency functions of degree at most d and altruistic population r^{a}.

B. Heterogeneous Routing Game

To understand how altruists effect congestion within the context of a heterogeneous population, we model the routing problem as a nonatomic congestion game. Each type of traffic is a continuum of agents, where the cost each agent experiences is determined by their type. Given flow x, the cost selfish agents experience for using path $p \in \mathcal{P}^{\mathrm{s}}$ is the actual latency of the path:

$$\ell_p^{\mathrm{s}}(x) := \sum_{e \in p} \ell_e(x_e), \tag{4}$$

where $\ell_p^{\mathrm{s}}(x)$ is often denoted $\ell_p(x)$. Intuitively, (4) assumes selfish agents uniformly focus on minimizing their own commute time only.

The cost experienced by altruists considers both their commute time, as well as their contribution to congestion along the path. Hence, the cost an altruistic agent experiences for using path $p \in \mathcal{P}^{\mathrm{a}}$ is the marginal cost of the path:

$$\ell_p^{\mathrm{a}}(x) := \sum_{e \in p} \left[\ell_e(x_e) + x_e \ell_e'(x_e)\right], \tag{5}$$

where ℓ' denotes the flow derivative of ℓ, and $\ell_p^{\mathrm{a}}(x)$ is often denoted $\ell_p^{\mathrm{mc}}(x)$. For a formal treatment of altruism, see [13].

All agents travel from origin o to destination t using the minimum-cost path from those available in their path set. We call flow x a *Nash flow* if all agents are individually using minimum-cost paths relative to the choices of others. That is, for each type $\theta \in \{\mathrm{a}, \mathrm{s}\}$, there exists a feasible x^θ such that the following condition is satisfied:

$$\forall p, p' \in \mathcal{P}^\theta, x_p^\theta > 0 \Longrightarrow \ell_p^\theta(x) \leq \ell_{p'}^\theta(x). \tag{6}$$

It is well known that a Nash flow exists for any heterogeneous nonatomic congestion game [14]. For each $\theta \in \{\mathrm{a}, \mathrm{s}\}$, and Nash flow x, we denote the common latency of minimum cost paths as $\Lambda^\theta(x)$ (i.e., $\Lambda^\theta(x) = \ell_p^\theta(x)$ for any $x_p^\theta > 0$).

C. Related Work

The inefficiency of selfish routing in congestion games has been extensively studied [15], with a focus on the cost at equilibrium as a function of network topology [16], and the degree of cost functions [17]. Moreover, it is known that in all networks with affine cost functions and a homogeneously altruistic population ($r^{\mathrm{a}} = 1$), the total latency is guaranteed to be better than when the population is homogeneously selfish [18]. It is also known that altruism can produce unbounded improvements over selfishness in parallel networks [19]. In parallel networks with symmetric path sets ($\mathcal{P}^{\mathrm{a}} = \mathcal{P}^{\mathrm{s}} = \mathcal{P}$), heterogeneous altruism lowers the price of anarchy compared to homogeneous selfishness [13], and the perversity index is unity in these networks [20]. It is known that in serially-linearly-independent networks, altruism is guaranteed to improve network efficiency [12]. The equilibrium cost for users is also monotone with respect to overall population size (even if the path set is not symmetric), provided all users have the same cost functions [21].

These results do not extend to more general networks as altruism can cause unbounded harm if the pathsets are not symmetric ($\mathcal{P}^{\rm a} \neq \mathcal{P}$) [7]. Similarly, altruism can cause unbounded harm if the path set is not Braess-resistant (exhibits Braess's paradox) [9]. However, partial altruism can be used to mitigate the risk of harm caused by selfishness and heterogeneous altruism [8].

III. Contribution: Altruism is Guaranteed to Improve Congestion in symmetric and Braess-resistant Networks

It is well-known that altruism can cause inefficiency with respect to the total latency of a network. These inefficiencies often arise because of restrictions on the path set and path access for agents. Hence, we seek a set of conditions that guarantees the presence of altruists will improve total latency.

Our first definition provides terminology for when altruistic and selfish agents have the same path sets, a necessary condition for altruism to improve overall system welfare.

Definition 1: $\mathcal{P}$ is *symmetric* if $\mathcal{P}^{\rm a} = \mathcal{P}^{\rm s} = \mathcal{P}$.

Next, path sets on series-parallel networks can be designed to exhibit Braess's paradox. Our next definition clarifies the intention of a series-parallel network, and precludes path sets that produce Braess's paradox, another necessary condition for altruism to improve overall system welfare.

Definition 2: $\mathcal{P}$ is *Braess-resistant* if $\mathcal{P} = \mathcal{R}$.

Path sets that fail either condition can be shown to have perversity; examples are given in Section V.

Our main result shows that any series-parallel network with a symmetric and Braess-resistant path set has improved total latency in the presence of altruism; likewise, total latency deteriorates if any altruists become selfish.

Theorem 3.1: Let $G = (V, E, \{\ell_e\}_{e\in E}, \mathcal{P}, r^{\rm a})$ be a routing game on a series-parallel network, let $\varepsilon \in (0, r^{\rm a}]$, and define $\tilde{G} = (V, E, \{\ell_e\}_{e\in E}, \mathcal{P}, r^{\rm a} - \varepsilon)$. Let x be a Nash flow for G and $\tilde{x}$ a Nash flow for $\tilde{G}$. If the path set for G (and thus $\tilde{G}$) is both symmetric and Braess-resistant, then

$$\mathcal{L}(x) \leq \mathcal{L}(\tilde{x}). \tag{7}$$

The proof is completed in Section IV; we provide intuition for the result, and briefly discuss its consequences here. The importance of Theorem 3.1 is twofold: it is quite simple, and has a wide breadth of applications. For any game G with altruistic population $r^{\rm a}$, if the path set for G satisfies Definitions 1 and 2, then if the mass of altruists decreases even slightly, the resulting total congestion is guaranteed to weakly worsen. Since the result implies an increase in altruists will weakly improve total congestion, a system planner can use this result to design the topology of a road network, ensuring that future tolls, fleet managers of autonomous vehicles, and independent altruists all contribute to improving network efficiency. Next, we present the supporting material for Theorem 3.1, then provide its proof.

IV. Proof of Theorem 3.1

The proof of Theorem 3.1 is completed in three steps:

1) Lemma 4.1 is an extension of [11, Lemma 2]. Under mild assumptions on total origin-destination flow, if one sub-population's mass increases, then some path used by agents of that sub-population experiences an increase in flow on every edge.
2) Corollary 4.2 specifies Lemma 4.1 for our context: where an increase in selfish population coincides with a decrease in altruistic population. It demonstrates that selfish agents use a path such that each edge in the path increases from x to $\tilde{x}$, and altruists use a path such that each edge in the path decreases from x to $\tilde{x}$.
3) Lemma 4.3 shows that the flow for paths used by selfish agents increases from x to $\tilde{x}$, and the flow for paths used by altruists decreases from x to $\tilde{x}$.

We state the supporting material here, and proceed with their proofs in the Appendix.

Lemma 4.1: Let $\bar{G} = (V, E, \{\ell_e\}_{e\in E}, \mathcal{P}, r^{\rm a} + \mu)$ for $\mu \in [-r^{\rm a}, r^{\rm s}]$ (the same game as G, except for differing altruistic and selfish masses), and $\theta \in \{{\rm a}, {\rm s}\}$. Assume G (and thus $\bar{G}$) has a symmetric and Braess-resistant path set. If $\hat{x}$ and $\bar{x}$ are Nash flows ($\hat{x}$ for G and $\bar{x}$ for $\bar{G}$) satisfying

$$||\hat{x}||_1 \geq ||\bar{x}||_1 \tag{8a}$$

$$||\hat{x}^\theta||_1 \geq ||\bar{x}^\theta||_1, \text{ and} \tag{8b}$$

$$||\hat{x}^\theta||_1 > 0, \tag{8c}$$

then there exists a path $p \in \mathcal{P}$ such that for each edge $e \in p$, $\hat{x}_e \geq \bar{x}_e$ and $\hat{x}^\theta_e > 0$.

Intuitively, Lemma 4.1 says that if either the altruistic or selfish population increases, then there is a path used by the increased sub-population such that each edge in the path increases flow. Additionally, even though the Nash flows have equal masses of traffic traversing the network in our application, (8a) shows that this also holds if the total mass of traffic has a strict order in the same direction as the sub-population under consideration. The results of Lemma 4.1 are made explicit with the following corollary.

Corollary 4.2: Assume x is a Nash flow for G and $\tilde{x}$ is a Nash flow $\tilde{G}$, and $\mathcal{P}$ is symmetric and Braess-resistant. Since $||x||_1 = ||\tilde{x}||_1$, $||x^{\rm s}||_1 \leq ||\tilde{x}^{\rm s}||_1$ and $||x^{\rm a}||_1 \geq ||\tilde{x}^{\rm a}||_1$, there exist minimum-cost paths (denoted $p_{\rm s}$ and $p_{\rm a}$), such that for each edge $e \in p_{\rm s}$, and for each edge $e' \in p_{\rm a}$,

$$x_e \leq \tilde{x}_e \text{ and } \tilde{x}^{\rm s}_e > 0, \tag{9}$$

$$x_{e'} \geq \tilde{x}_{e'} \text{ and } x^{\rm a}_{e'} > 0. \tag{10}$$

Intuitively, it follows that the total origin-destination flow for x and $\tilde{x}$ is equal. Thus, since $r^{\rm s} < r^{\rm s} + \varepsilon$, there exists a path in $\tilde{x}$ used by selfish agents such that each edge along that path is non-decreasing in flow from x, which we denote $p_{\rm s}$. Similarly, since $r^{\rm a} > r^{\rm a} - \varepsilon$, there exists a path in x used by altruists such that each edge along that path is non-increasing in flow in $\tilde{x}$, which we denote $p_{\rm a}$.

Lemma 4.3: Let x and $\tilde{x}$ be a Nash flow for G and $\tilde{G}$, respectively, and assume $\mathcal{P}$ is symmetric and Braess-resistant. Then, for any paths $p, q \in \mathcal{P}$, the following hold:

$$x^{\rm s}_p > 0 \Longrightarrow x_p \leq \tilde{x}_p, \tag{11}$$

$$\tilde{x}^{\rm a}_q > 0 \Longrightarrow x_q \geq \tilde{x}_q. \tag{12}$$

Lemma 4.3 demonstrates that the path flows for paths used by selfish agents in x are guaranteed to be non-decreasing in $\tilde{x}$, and the path flows for paths used by altruists in $\tilde{x}$ are guaranteed to be non-increasing from x.

Proof of Theorem 3.1: Recall, G and $\tilde{G}$ are identical games, except $\tilde{G}$ corresponds to changing $\varepsilon \in (0, r^{\mathrm{a}}]$ altruists to selfish agents. That is, $r^{\mathrm{a}} - \varepsilon$ agents are altruistic and commute according to the marginal-cost of a path in $\tilde{G}$, and $r^{\mathrm{s}} + \varepsilon$ are selfish and commute according to the actual cost associated with a path in $\tilde{G}$. Now, since $\mathcal{L}$ is convex:

$$\begin{aligned}\mathcal{L}(\tilde{x}) - \mathcal{L}(x) &\geq \sum_{p\in\mathcal{P}} \ell_p^{\mathrm{mc}}(x)\cdot(\tilde{x}_p - x_p)\\ &= \sum_{p\in\mathcal{P}^+} \ell_p^{\mathrm{mc}}(x)\cdot\delta_p + \sum_{q\in\mathcal{P}^-} \ell_q^{\mathrm{mc}}(x)\cdot\delta_q,\end{aligned}$$

where $\delta_p \coloneqq \tilde{x}_p - x_p$ for $p \in \mathcal{P}$. Further, $\mathcal{P}^+$ is the set of paths such that $\delta_p > 0$ for all p, and $\mathcal{P}^-$ is the set of paths such that $\delta_q < 0$ for all q; paths that do not change flow from x to $\tilde{x}$ can be ignored. The contrapositive of (11), from Lemma 4.3, shows that if a path flow decreases from x to $\tilde{x}$ (i.e., $\delta_q < 0$), then altruists use that path in x. Now, we use p_{a}, the minimum cost path for altruists in x (as defined in (10) of Corollary 4.2), to continue the bound:

$$\begin{aligned}\mathcal{L}(\tilde{x}) - \mathcal{L}(x) &\geq \sum_{p\in\mathcal{P}^+} \ell_p^{\mathrm{mc}}(x)\cdot\delta_p + \sum_{q\in\mathcal{P}^-} \ell_q^{\mathrm{mc}}(x)\cdot\delta_q\\ &\geq \ell_{p_{\mathrm{a}}}^{\mathrm{mc}}(x)\sum_{p\in\mathcal{P}^+}\delta_p + \ell_{p_{\mathrm{a}}}^{\mathrm{mc}}(x)\sum_{q\in\mathcal{P}^-}\delta_q \qquad (13)\\ &= \left(\ell_{p_{\mathrm{a}}}^{\mathrm{mc}}(x) - \ell_{p_{\mathrm{a}}}^{\mathrm{mc}}(x)\right)\sum_{p\in\mathcal{P}^+}\delta_p \qquad (14)\\ &= 0,\end{aligned}$$

where (13) follows from the fact that $\ell_{p_{\mathrm{a}}}^{\mathrm{mc}}(x) \leq \ell_p^{\mathrm{mc}}(x)$ for all $p \in \mathcal{P}^+$, and $\ell_{p_{\mathrm{a}}}^{\mathrm{mc}}(x) = \ell_q^{\mathrm{mc}}(x)$ since q is used by altruists in x for all $q \in \mathcal{P}^-$. Additionally, (14) follows from the fact that $\sum_{p\in\mathcal{P}^+}\delta_p = -\sum_{q\in\mathcal{P}^-}\delta_q$. Hence, $\mathcal{L}(x) \leq \mathcal{L}(\tilde{x})$. ■

V. Path Sets that are not Symmetric and Braess-resistant are Susceptible to Perversity

Here, we present a short discussion on the inefficiencies that arise if a path set is not Braess-resistant or not symmetric. It is well-known that networks which are not series-parallel can experience increased total congestion in the presence of altruism. Hence, we maintain the series-parallel assumption, and provide networks that may exhibit worsened total congestion in the presence of altruism.

A. Networks which are not Braess-resistant

Consider sending 2 units of traffic across a series-parallel network consisting of five edges, where the path set is $\mathcal{P} = \{(e_1, e_3), (e_1, e_4), (e_2, e_4), e_5\}$; notice that (e_2, e_3) is not a path available to agents, hence $\mathcal{P}$ is not Braess-resistant. The edge latencies are $\ell_{e_1}(x_{e_1}) = x_{e_1}$, $\ell_{e_2}(x_{e_2}) = \ell_{e_3}(x_{e_3}) = 1$, $\ell_{e_4}(x_{e_4}) = x_{e_4}$, and $\ell_{e_5}(x_{e_5}) = 3$. If all traffic is selfish, then sending 1 unit of traffic on path (e_1, e_3), and 1 unit of traffic on (e_2, e_4) is a Nash flow (denoted $\tilde{x}$) with common latency $\Lambda^{\mathrm{s}}(\tilde{x}) = 2$ (paths (e_1, e_4) and e_5 have latencies of 2 and 3, respectively). If half the population becomes altruistic (i.e. $r^{\mathrm{a}} = r^{\mathrm{s}} = 1$), then sending selfish agents on path (e_1, e_4) and altruists on path e_5 is a Nash flow (denoted x) with common latencies $\Lambda^{\mathrm{s}}(x) = 2$ and $\Lambda^{\mathrm{a}}(x) = 3$, respectively. Straightforward computation shows that $\mathcal{L}(\tilde{x}) = 4$, and $\mathcal{L}(x) = 5$. This example is shown graphically in Figure 1a.

B. Networks which are not Symmetric

Consider sending 2 units of traffic across a parallel network consisting of three edges, where $\mathcal{P} = \{e_1, e_2, e_3\}$, $\mathcal{P}^{\mathrm{a}} = \{e_1, e_2\}$ and $\mathcal{P}^{\mathrm{s}} = \mathcal{P}$; notice e_3 is not available to altruists, thus $\mathcal{P}$ is not symmetric. The edge latencies are $\ell_{e_1}(x_{e_1}) = 1 + d$, $\ell_{e_2}(x_{e_2}) = x^d$, and $\ell_{e_3}(x_{e_3}) = 1$. When all agents are selfish, $\tilde{x} = [0, 1, 1]^\top$ is a Nash flow. Now, if half of traffic becomes altruistic (i.e. $r^{\mathrm{a}} = r^{\mathrm{s}} = 1$), $x = [r^{\mathrm{a}}, r^{\mathrm{s}}, 0]^\top$ is a Nash flow. Finally, straightforward computation shows that $\mathcal{L}(\tilde{x}) = 2$, and $\mathcal{L}(x) = 2 + d$. This example is shown graphically in Figure 1b.

VI. Conclusions

We have defined a specific set of conditions for the network that ensures altruistic routing in heterogeneous traffic improves overall congestion compared to all-selfish routing. Future work will focus on extending to heterogeneous populations that consist of partially altruistic sub-populations, so the effects of multiple partially altruistic sub-populations on total congestion can be better understood.

Appendix

If two sub-paths are in parallel with each other and have equal and constant latency functions, we call them *redundant-parallel sub-paths.* Because the inclusion of these paths does not lead to altruists degrading perversity, we assume networks do not contain redundant-parallel sub-paths in all proofs.

Proof of Lemma 4.1: Recall, G and $\bar{G}$ have the same path set, $\mathcal{P}$; we slightly abuse notation and also refer to the network (which is also equivalent for G and $\bar{G}$) upon which $\mathcal{P}$ is defined, by $\mathcal{P}$ itself. The proof proceeds by induction on the number of edges in $\mathcal{P}$. If $\mathcal{P}$ consists of a single edge, the result is trivial, so assume $\mathcal{P}$ has two or more edges. By the induction hypothesis, the lemma holds for any two Nash flows that satisfy (8) in every series-parallel, symmetric, and Braess-resistant path set with fewer edges than $\mathcal{P}$. It is known that $\mathcal{P}$ can be constructed by connecting two series-parallel networks, $\mathcal{P}'$ and $\mathcal{P}''$, in series or in parallel. First, consider $\mathcal{P}'$ and $\mathcal{P}''$ are connected in series, so the destination node of $\mathcal{P}'$ is the origin node of $\mathcal{P}''$. Now, it is clear that paths in $\mathcal{P}'$ are sub-paths of paths in $\mathcal{P}$. Hence, every Nash flow $\hat{x}$ of G induces a Nash flow $\hat{x}'$ of G' (the game G on $\mathcal{P}'$), given by $\hat{x}' = (\hat{x}_{p'})_{p'\in\mathcal{P}'}$. Likewise, every Nash flow $\bar{x}$ of $\bar{G}$ induces a Nash flow $\bar{x}'$ of $\bar{G}'$ (the game $\bar{G}$ on $\mathcal{P}'$), given by $\bar{x}' = (\bar{x}_{p'})_{p'\in\mathcal{P}'}$. Now, the flow x'_e is the total flow on all the paths in P' containing edge e, that is

$$x'_e = \sum_{p'\in\mathcal{P}', p'\ni e} x_{p'}. \qquad (15)$$

Thus, (15) implies $x'_e = x_e$, so if $\hat{x}$ and $\bar{x}$ are Nash flows satisfying (8) for G and $\bar{G}$, then the induced Nash flows

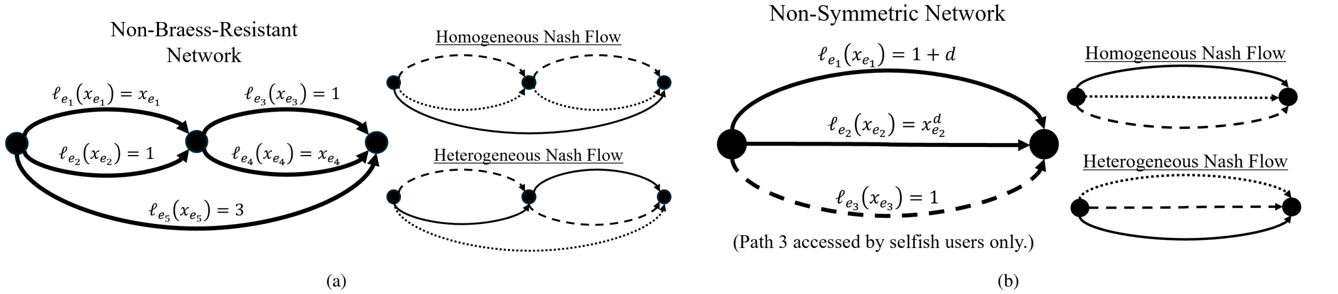

Fig. 1: Graphical representation of the examples from Section V; in each example, two units of traffic are routed. In figure 1a, the path set is constructed so agents using e_2 cannot use e_3 (hence, no Braess-resistance). If all traffic is selfish, one unit uses (e_1, e_3) and the other unit uses (e_2, e_4) at Nash flow, resulting in a total latency of $\mathcal{L}(\tilde{x}) = 4$. If half the population becomes altruistic, selfish agents use (e_1, e_4) and altruists use e_5 in a Nash flow, resulting in total latency $\mathcal{L}(x) = 5$. In figure 1b, altruists have access to the top two edges while selfish agents have access to all three (hence, not symmetric). If all traffic is selfish and at Nash flow, the two units of traffic are split between the bottom two edges, resulting in total latency $\mathcal{L}(\tilde{x}) = 2$. If half the population becomes altruistic, altruists may use the top edge in a Nash flow, and the total latency becomes $\mathcal{L}(x) = 2 + d \geq 2 = \mathcal{L}(\tilde{x})$.

$\hat{x}'$ and $\bar{x}'$ satisfy (8) for G' and $\bar{G}'$, as well. Thus, by the induction hypothesis, there is a path $p' \in \mathcal{P}'$ such that $\hat{x}'_e \geq \bar{x}'_e$, and $\hat{x}'^{\theta}_e > 0$ for all $e \in p'$. Now, let $\hat{x}''$ be the induced Nash flow for G'' (G on $\mathcal{P}''$), and $\bar{x}''$ be the induced Nash flow for $\bar{G}''$ ($\bar{G}$ on $\mathcal{P}''$). By similar arguments, there exists a path $p'' \in \mathcal{P}''$ such that $\hat{x}''_e \geq \bar{x}''_e$, and $\hat{x}''^{\theta}_e > 0$ for all $e \in p''$. Since $\mathcal{P}$ is Braess resistant, there is a path $p \in \mathcal{P}$ such that $p = p' + p''$. Hence, $\hat{x}_e \geq \bar{x}_e$, and $\hat{x}^{\theta}_e > 0$ for all $e \in p$, and since $\mathcal{P}$ is symmetric, p is a minimum cost path for agents belonging to θ. This proves the case that $\mathcal{P}$ is the result of connecting two series-parallel networks in series.

Next, consider $\mathcal{P}$ is the result of connecting $\mathcal{P}'$ and $\mathcal{P}''$ in parallel, so that o and t are also the origin and destination nodes of $\mathcal{P}'$ and $\mathcal{P}''$. By definition, $\mathcal{P}'$ and $\mathcal{P}''$ are disjoint, and $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$. Any Nash flow $\hat{x}$ of G induces Nash flows $\hat{x}'$ and $\hat{x}''$ for G' and G'' (the game G on $\mathcal{P}'$ and $\mathcal{P}''$, respectively), where $\hat{x}' = (\hat{x}_{p'})_{p' \in \mathcal{P}'}$ and $\hat{x}'' = (\hat{x}_{p''})_{p'' \in \mathcal{P}''}$. Similarly, any Nash flow $\bar{x}$ of $\bar{G}$ induces Nash flows $\bar{x}'$ and $\bar{x}''$ for $\bar{G}'$ and $\bar{G}''$ (the game $\bar{G}$ on $\mathcal{P}'$ and $\mathcal{P}''$, respectively). We claim either $\hat{x}'$ and $\bar{x}'$ satisfy the induction hypothesis for G' and $\bar{G}'$, or $\hat{x}''$ and $\bar{x}''$ satisfy the induction hypothesis for G'' and $\bar{G}''$. Suppose not, then it follows that $||\hat{x}'||_1 < ||\bar{x}'||_1$ or $||\hat{x}'^{\theta}||_1 < ||\bar{x}'^{\theta}||_1$, and $||\hat{x}''||_1 < ||\bar{x}''||_1$ or $||\hat{x}''^{\theta}||_1 < ||\bar{x}''^{\theta}||_1$ holds. Since

$$\begin{aligned} ||\hat{x}||_1 &= ||\hat{x}'||_1 + ||\hat{x}''||_1 \\ &= ||\bar{x}'||_1 + ||\bar{x}''||_1 \\ &= ||\bar{x}||_1, \end{aligned}$$

and

$$\begin{aligned} ||\hat{x}^{\theta}||_1 &= ||\hat{x}'^{\theta}||_1 + ||\hat{x}''^{\theta}||_1 \\ &\geq ||\bar{x}'^{\theta}||_1 + ||\bar{x}''^{\theta}||_1 \\ &= ||\bar{x}^{\theta}||_1, \end{aligned}$$

it is without loss of generality that we may assume $||\hat{x}'||_1 < ||\bar{x}'||_1$, then $||\hat{x}''||_1 > ||\bar{x}''||_1$. Hence, $||\hat{x}''^{\theta}||_1 < ||\bar{x}''^{\theta}||_1$, which in turn implies $||\hat{x}'^{\theta}||_1 > ||\bar{x}'^{\theta}||_1$. Now, fix $\psi \in \{\mathrm{a}, \mathrm{s}\}$ such that $\psi \neq \theta$. It follows that $||\hat{x}'^{\psi}||_1 < ||\bar{x}'^{\psi}||_1$ and $||\hat{x}''^{\psi}||_1 > ||\bar{x}''^{\psi}||_1$. Hence, $\hat{x}'$ and $\bar{x}'$ satisfy (8) on $\mathcal{P}'$ for ψ, where $G = \bar{G}'$ and $\bar{G} = G'$. Thus, by the induction hypothesis and the fact that $\hat{x}$ and $\bar{x}$ have the same flow on each edge as their induced flows, there exists $p' \in \mathcal{P}' \subset \mathcal{P}$ such that $\hat{x}_{e'} < \bar{x}_{e'}$ and $\bar{x}^{\psi}_{e'} > 0$ for all $e' \in p'$. Hence, $\ell^{\psi}_{p'}(\bar{x}) = \Lambda^{\psi}(\bar{x})$, and

$$\begin{aligned} \ell^{\psi}_{p'}(\hat{x}) &= \sum_{e' \in p'} \ell^{\psi}_{e'}(\hat{x}_{e'}) \\ &\leq \sum_{e' \in p'} \ell^{\psi}_{e'}(\bar{x}_{e'}) \\ &= \ell^{\psi}_{p'}(\bar{x}). \end{aligned}$$

Hence, it follows that $\Lambda^{\psi}(\hat{x}) \leq \Lambda^{\psi}(\bar{x})$. Similarly, $\hat{x}''$ and $\bar{x}''$ satisfy (8) on $\mathcal{P}''$ for ψ, where $G = G''$ and $\bar{G} = \bar{G}''$. So, by identical arguments, there exists $p'' \in \mathcal{P}'' \subset \mathcal{P}$ such that $\hat{x}_{e''} > \bar{x}_{e''}$ and $\hat{x}^{\psi}_{e''} > 0$ for all $e'' \in p''$. Hence, $\ell^{\psi}_{p''}(\bar{x}) \leq \ell^{\psi}_{p''}(\hat{x}) = \Lambda^{\psi}(\hat{x})$, and so $\Lambda^{\psi}(\hat{x}) \geq \Lambda^{\psi}(\bar{x})$. But then, either $\ell^{\psi}_{p'}$ and $\ell^{\psi}_{p''}$ are constant and equal, implying p' and p'' are redundant-parallel paths, or $\Lambda^{\psi}(\hat{x}) < \Lambda^{\psi}(\bar{x})$ and $\Lambda^{\psi}(\hat{x}) > \Lambda^{\psi}(\bar{x})$, both of which are contradictions. Thus, either $\hat{x}'$ and $\bar{x}'$ satisfy the induction hypothesis for G' and $\bar{G}'$, or $\hat{x}''$ and $\bar{x}''$ satisfy the induction hypothesis for G'' and $\bar{G}''$, proving the case that $\mathcal{P}$ is the result of connecting $\mathcal{P}'$ and $\mathcal{P}''$ in parallel. ■

Proof of Corollary 4.2: The existence of p_{s} and p_{a} follows immediately from Lemma 4.1, and by [11, Lemma 3], p_{s} is a minimum cost path for selfish agents in $\tilde{x}$, and p_{a} is a minimum cost path for altruists in x, as desired. ■

Proof of Lemma 4.3: Assume by contradiction that the claim is false. Then there exists a path $p_* \in \mathcal{P}^{\mathrm{s}}$ such that

$$x^{\mathrm{s}}_{p_*} > 0, \tag{16}$$

$$x_{p_*} > \tilde{x}_{p_*}. \tag{17}$$

Now, it is without loss of generality that there exists a sub-path $\sigma \in p_*$ such that $x_e > \tilde{x}_e$ for all $e \in \sigma$; otherwise, $x_e \leq \tilde{x}_e$ for all $e \in p_*$, so $x_{p_*} \leq \tilde{x}_{p_*}$, contradicting (17). Further, by Corollary 4.2, p_{s} and p_{a} exist and satisfy (11) and (12), respectively. Hence, it follows that

$$\ell_{p_{\mathrm{s}}}(x) \leq \ell_{p_{\mathrm{s}}}(\tilde{x}) = \Lambda^{\mathrm{s}}(\tilde{x}), \tag{18}$$

$$\ell^{\mathrm{mc}}_{p_{\mathrm{a}}}(\tilde{x}) \leq \ell^{\mathrm{mc}}_{p_{\mathrm{a}}}(x) = \Lambda^{\mathrm{a}}(x). \tag{19}$$

Case 1: First, assume $p_* = \sigma$. Then, since $x_e > \tilde{x}_e$ for all $e \in p_*$, and by (16), $\Lambda^{\mathrm{s}}(x) = \ell_{p_*}(x) \geq \ell_{p_*}(\tilde{x})$. Thus, by

(18), $\ell_{p_*}(x) \leq \ell_{p_s}(x) \leq \ell_{p_s}(\tilde{x}) \leq \ell_{p_*}(\tilde{x}) \leq \ell_{p_*}(x)$. Hence, either $\ell_{p_*}(x) < \ell_{p_*}(x)$, or ℓ_{p_*} and ℓ_{p_s} are constant and equal to one another, implying the network has redundant-parallel paths. Both of which are contradictions.

Case 2: Next, assume $p_* = \sigma \cup \sigma_2$, where $x_e \leq \tilde{x}_e$ for all $e \in \sigma_2$. Because the flow on σ_2 is non-decreasing, and the flow on σ is decreasing, there exists a sub-path σ_1 that is parallel to σ, where $x_e < \tilde{x}_e$ for all $e \in \sigma_1$. Now, because each edge $e \in \sigma_1$ is increasing flow from x to $\tilde{x}$, and $x_e = \sum_{p \in \mathcal{P}: p \ni e} x_p$, it follows that σ_1 can be selected so that agents of the same type use each edge of σ_1 in $\tilde{x}$, and are then using σ_2 in $\tilde{x}$. That is, the path $p' = \sigma_1 \cup \sigma_2$ is a minimum cost path for either altruists or selfish agents in $\tilde{x}$. First assume p' is a minimum cost path for selfish agents. Since σ and σ_1 are parallel, we can apply identical arguments from *Case 1* to show that either $\ell_\sigma(x) < \ell_\sigma(x)$, or ℓ_σ and ℓ_{σ_1} are constant and equal to one another, implying the network has redundant-parallel sub-paths, both contradictions. If p' is a minimum cost path for altruists in $\tilde{x}$, it follows that $\ell^{\rm mc}_{p'}(x) \leq \ell^{\rm mc}_{p'}(\tilde{x}) = \Lambda^{\rm a}(\tilde{x})$, since $x_e \leq \tilde{x}_e$ and $\tilde{x}^{\rm a}_e > 0$ for all $e \in p'$. Hence, by (19), $\ell^{\rm mc}_{p_{\rm a}}(x) \leq \ell^{\rm mc}_{p'}(x) \leq \ell^{\rm mc}_{p'}(\tilde{x}) \leq \ell^{\rm mc}_{p_{\rm a}}(\tilde{x}) \leq \ell^{\rm mc}_{p_{\rm a}}(x)$. Thus, either $\ell^{\rm mc}_{p_{\rm a}}(x) < \ell^{\rm mc}_{p_{\rm a}}(x)$, or $\ell^{\rm mc}_{p_{\rm a}}$ and $\ell^{\rm mc}_{p'}$ are redundant-parallel paths, both of which are contradictions. The case where σ is the second sub-path and σ_2 is the first sub-path can be done by switching o and t, and proceed in an identical manner to this case.

Case 3: Finally, let $\mathbb{E}$ be the set of even natural numbers, and consider the case that the sub-paths of p_* are constructed so that each sub-path consists of decreasing edges, or non-decreasing edges. That is, $p_* = \bigcup_{i=1:p_* \ni \sigma_i}^{n} \sigma_i$ for some $n \in \mathbb{N}$, where $x_e > \tilde{x}_e$ for $\{\sigma_i\}_{i \in \mathbb{E}}$, and $x_e \leq \tilde{x}_e$ for $\{\sigma_i\}_{i \notin \mathbb{E}}$. Since flow increases and then decreases from x to $\tilde{x}$ among subsequent sub-paths in p_*, selfish agents must be sharing each odd-numbered sub-path of p_* in $\tilde{x}$ with agents using a path other than p_*. Since the network is series-parallel, it can be shown that agents diverting away from each sub-path $\underline{\sigma}_i \in p_*$ for $i \in \mathbb{E}$ go onto a sub-path (denoted $\bar{\sigma}'_i$) that is parallel to $\underline{\sigma}_i$. Now, we claim only altruists use $\bar{\sigma}'_i$. Suppose not, and selfish agents use $\bar{\sigma}'_i$ for some $i \in \mathbb{E}$. Then, because $\underline{\sigma}_i$ and $\bar{\sigma}'_i$ are parallel, similar arguments from *Case 1* show that either $\ell_{\underline{\sigma}_i}(x) < \ell_{\underline{\sigma}_i}(x)$, or $\ell_{\underline{\sigma}_i}$ and $\ell_{\bar{\sigma}'_i}$ are constant and equal to one another, implying the network has redundant-parallel sub-paths. Both of which are contradictions. Hence, there exists a path p' such that each sub-path $\sigma'_i \in p'$ can be defined

$$\sigma'_i = \begin{cases} \sigma_i & i \in \mathbb{E} \\ \bar{\sigma}'_i & i \notin \mathbb{E} \end{cases},$$

for $i = 1, \ldots, n$. That is, the sub-paths of p' can be expressed such that each odd sub-path of p' is also an odd sub-path of p_*, and each even sub-path of p' is the sub-path parallel to a sub-path in p_* that agents divert onto. Hence, $x_e \leq \tilde{x}_e$ and $\tilde{x}^{\rm a}_e > 0$ for all $e \in p'$. Thus, it follows that $\ell^{\rm mc}_{p'}(x) \leq \ell^{\rm mc}_{p'}(\tilde{x}) = \Lambda^{\rm a}(\tilde{x})$. Since $\mathcal{P}$ is symmetric and Braess-resistant, p' is a minimum cost path for altruists in $\tilde{x}$. Similar arguments as *Case 2* show that either $\ell^{\rm mc}_{p_{\rm a}}(x) < \ell^{\rm mc}_{p_{\rm a}}(x)$, or $\ell^{\rm mc}_{p_{\rm a}}$ and $\ell^{\rm mc}_{p'}$ are redundant-parallel paths, each a contradiction. Identical reasoning shows the case where odd-numbered sub-paths of p_* decrease flow from x to $\tilde{x}$, and even-numbered sub-paths of p_* increase flow from x to $\tilde{x}$.

Similar arguments show that (12) holds. ■

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