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On the Compressibility of State Snapshots for String Systems

Chenyan Zhu and Sandip Roy

Abstract— **Sparse representation of the states of strings or cascades of dynamical systems is examined. Specifically, a notion of compressibility for string system states is introduced, which captures whether and to what extent the state can be expressed sparsely in a fixed basis. For a highly simplified string system model (made up of scalar, linear, discrete-time objects), compressibility in the Laplacian spectrum and Gramian spectrum bases is characterized analytically. The main result of this analysis is that the energy in the state snapshot is captured in a diminishing fraction of the basis vectors, as the string is made long. Simulations are used to illustrate the formal results, and demonstrate state recovery from sparse, randomly-located samples.**

I. Introduction

Models of interconnected dynamical systems arranged in a string topology are apt for assessing vehicle platoons and dense highway traffic, among other applications [1], [2]. Similar models also arise in spatially-discretized approximations of one-dimensional wave and diffusion processes, such as heat flow in a bar or fluid flow in a pipe [3], [4]. These applications have motivated an extensive research effort on the external stability of strings of dynamical systems, under the heading of *string stability*. These studies aim to assess the propagation of disturbances introduced at one point on the string; broadly, stability is defined based on whether or not the disturbance amplifies unboundedly, as the number of dynamical systems forming the string is made large. A parallel track of work has considered disturbance propagation and attenuation in strings of dynamical systems using wave approximations. Beyond external stability and stabilization through control, a few recent studies have begun to consider additional control-theoretic notions for strings, such as controllability from local actuation.

Our interest in this study is to understand at a more basic level the patterns or configurations taken on by a string system, when it is actuated at one or a small number of locations along the string. Intuitively, one might expect the states of string systems to be strongly patterned provided that actuations are localized: for example, dense vehicle traffic on highways tends to organize in particular configurations, and fluid flow in channels is patterned even when subjected to stochastic disturbances. To study this, we introduce and examine a notion of *string compressibility*, which captures whether snapshots of the string's state can be expressed as a sparse linear combination of basis vectors in an appropriate basis, as the dimension of the string is increased. If it can be verified, string compressibility can enable a number of computational and design benefits, such as state recovery from sparse measurements via regularized optimization.

The focus of our effort here is to assess string compressibility for a canonical model in two bases: 1) the Fourier or Laplacian-spectrum basis and 2) the spectrum of the controllability Gramian of the string system. In this initial study, we deliberately focus on the simplest possible string system model – a string of discrete-time scalar linear subsystems with symmetric linear couplings, which is actuated at one potentially unknown location – to enable a complete analysis of compressibility. For this model, we are able to verify compressibility in both bases of interest, in the following sense: an arbitrarily small fraction of the basis vectors capture (almost) all of the energy in the state snapshot, provided that the string is sufficiently long. For the Gramian basis, an even sparser (sub-linear) representation is obtained, however at the cost of needing to know the location of the actuation. Via simulations, the compressibility result is illustrated and state recovery from sparse data is also explored.

The research described here connects to the rich literature on compressive sensing and graph signal processing, which originated in the signal-processing community [5]–[14]. Within this literature, there have been a track of work on compressing sensing for dynamical processes [10], [11]. While these efforts largely focus on the practical recovery of states from measurements, our work here and in a couple of complementary studies [9], [15] are focused on understanding whether and in what basis dynamical-system states can be compressed. In this study, we further seek to understand compressibility as the dimension of the string system is made large, in analogy with efforts on string stability and analyses of wave-like phenomena. The proofs are omitted here in the interest of space and can be found in the extended journal paper, see [16].

II. Problem Formulation

The goal of this study is to determine whether state snapshots of an (infinite-dimensional) string system subject to local disturbances is compressible in some fixed bases; and if so, determine the level of sparsity. To formulate this problem, we present the simple string-system model, define compressibility, and introduce bases for compression.

In general, a string system consists of n dynamical objects in a line, where each object's state dynamics is interdependent on those of its predecessor and successor (neighbors). In discrete time ($k = 0, 1, 2, \ldots$), a string system with objects $i = 1, \ldots, n$ would in general have vector states $x_i[k]$ with the following dynamical evolution: $x_i[k+1] = f_i(x_i[k], x_{i-1}[k], x_{i+1}[k], u_i[k])$ for $i = 2, \ldots, n-1$, while $x_1[k+1] = f_1(x_1[k], x_2[k], u_1[k])$ and $x_n[k+1] =$

$f_n(x_n[k], x_{n-1}[k], u_n[k])$. Analyses of string systems of this sort are mostly focused on the assessment of disturbance propagation and hence external stability as n becomes large, due to inputs $u_i[k]$ at one or a few locations.

In this article, we consider the simplest possible string system of this form, with scalar linear objects that have symmetric linear interactions with their neighbors, and an input applied at one (possibly unknown) object. Specifically, the dynamics of the objects in the string are given by: $x_i[k+1] = ax_{i+1}[k] + ax_{i-1}[k] + (1-2a)x_i[k] + b_i u_i[k]$, $x_1[k] = ax_2[k] + (1-a)x_1[k] + b_1 u_1[k]$, and $x_n[k] = ax_{n-1}[k] + (1-a)x_n[k] + b_n u_n[k]$. Here, the constant a has the range $a \in (0, 0.5)$ and exactly one of the constants $b_1, \ldots, b_n$ equals 1, while the others are 0 (i.e. a disturbance is applied at one object). The string system's dynamics can be readily written in vector form as:

$$\mathbf{x}[k] = A\mathbf{x}[k] + Bu[k] \quad (1)$$

where $\mathbf{x}[k] = \begin{bmatrix} x_1[k] \\ \vdots \\ x_n[k] \end{bmatrix} \in R^n$ is a state vector at time k, the system matrix A is the symmetric row-stochastic tridiagonal Toeplitz matrix

$$A = \begin{bmatrix} 1-a & a & & & & \\ a & 1-2a & a & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & a & 1-2a & a \\ & & & & a & 1-a \end{bmatrix}, \quad (2)$$

and B is a 0-1 indicator vector with only the zth entry equal to 1, indexing the (possibly unknown) input location. We typically will assume that the input $u[k]$ is a zero-mean unit-variance white noise process input, although other models for the input (as unknown, or having a different statistics) will also be considered. Although extremely simplistic, the model is approximative of some real-world processes, such as heat flow in a bar subject to a disturbance input, consensus among distributed processors, or position/phase alignment of well-damped interconnected masses in a line topology.

We define string compressibility based on whether or not, and to what extent, states of the string system are sparse in a fixed basis. Conceptually, the string system can be viewed as generating an ensemble of stochastic or unknown states, which depend on the input applied. The string system's dynamics is said to compressible if all or most of these states can be expressed in terms of a small number of basis vectors.

Formally, a state of a string system $\mathbf{x}[k]$ is said to be compressible in an orthogonal basis Φ, if it can be written as $\mathbf{x}[k] = \Phi\mathbf{s}[k]$, where the transformed state $\mathbf{s}[k]$ is K-sparse (i.e. has at most K non-zero/dominant entries). In many cases, it is unrealistic to expect exact K-sparsity in a basis of interest, and hence it is more natural to consider the energy in the signal that is contained in K basis vectors. To formalize this concept, we note that for an orthogonal basis, $\mathbf{s}[k]$ is uniquely determined from $\mathbf{x}[k]$ as $\mathbf{s}[k] = \Phi^{-1}\mathbf{x}[k]$. For convenience, we relabel the entries in $\mathbf{s}[k]$ in order of decreasing amplitude, i.e. as $\hat{s}_1[k], \ldots, \hat{s}_n[k]$, where $|\hat{s}_1[k]| \geq \ldots \geq |\hat{s}_n[k]|$. Then, the fraction of energy captured by the K largest-amplitude entries is considered as a measure of compressibility of the state at the sparsity level K. Finally, compressibility of the ensemble of states can be defined in a statistical sense, provided that a stochastic model is in force. Precisely, we use $F(K) = \frac{\sum_{i=1}^{K} E[\hat{s}_i^2[k]]}{\sum_{j=1}^{n} E[\hat{s}_j^2[k]]}$ as a measure of compressibility at a K-sparsity level, for the string dynamics as a whole (the ensemble of states). In a non-random setting, worst-case energy fractions among an ensemble of signals needs to be considered, as we will briefly discuss later in the paper. In the ensuing analysis, we will largely focus on whether a limited fraction of the basis vectors ($\frac{K}{n}$ small) is sufficient for $F(K)$ to approach 1, as n becomes large.

Remark: As we will formalize in Section III, for the bases considered, the dominant component $s_1[k]$ grows unboundedly with the time horizon k for a white-noise input. To appropriately assess compressibility, we will therefore slightly modify the measure F to exclude $s_1[k]$, i.e. we will consider $\frac{\sum_{i=2}^{K} E[\hat{s}_i^2[k]]}{\sum_{j=2}^{n} E[\hat{s}_j^2[k]]}$ as the measure.

The core of this study is to develop bounds on the number of basis vectors K required for almost exact compressibility for string systems, in the limit of large n. Compressibility in two bases is considered: the *Laplacian spectrum* or Fourier basis, and the *Gramian spectrum* basis. The Laplacian spectrum basis is defined from the Laplacian matrix L for an undirected line graph, which captures the string system's topology. Specifically, L is an $n \times n$ matrix where $L_{i,i+1} = L_{i+1,i} = -1$ for $i = 1, \ldots, n-1$, and all other off-diagonal entries are nil ($L_{ij} = 0$). The diagonal entries are chosen to make the row and column sums of L equal to zero, i.e. $L_{11} = L_{nn} = 1$, and $L_{ii} = 2$ for $i = 2, \ldots, n-1$. We order the (real, nonnegative) eigenvalues of L as $0 = \widehat{\lambda}_1 < \widehat{\lambda}_2 < \ldots < \widehat{\lambda}_n$, and use the notation $\mathbf{v}_1 = \mathbf{1}, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for the corresponding eigenvectors. The Laplacian spectrum basis is defined as $\Phi_L = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$. For large n, it is well-known that the eigenvectors of the Laplacian look like sinusoids at equally-spaced frequencies, and hence the basis is equivalently a Fourier basis.

Remark: The eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ need to be normalized in a consistent way, so that the components of the state snapshot in each basis direction can be fairly compared. For the formal analysis in the next section, we have normalized the eigenvectors so that the largest entry in each is 1, in which case the energy (two-norm) of each vector is of order n. Normalization to unit energy is also appropriate.

Remark: The Laplacian spectrum basis has no dependence on either the parameter a of the string system, or the location of the external input.

As an alternative, the spectrum of the string system's controllability Gramian is also considered as a basis for compression. The time controllability Gramian of the dynamics (1) is given by $G(0,k) = \sum_{\tau=0}^{k-1} A^\tau BB'(A')^\tau$ at time k. Since $G(0,k)$ is a positive semidefinite symmetric matrix, its eigenvalues are real and nonnegative. It is convenient to label the eigenvalues of $G(0,k)$ in increasing order, as $\overline{\lambda}_1(k) \leq \ldots \leq \overline{\lambda}_n(k)$. We again use the notation $\overline{\mathbf{v}}_1(k), \ldots, \overline{\mathbf{v}}_n(k)$ for an orthogonal set of eigenvectors corresponding to the eigenvalues, where again the eigenvectors have been normalized in a commensurate way. The Gramian spectrum basis is defined as $\Phi_G(k) = \begin{bmatrix} \overline{\mathbf{v}}_1(k) & \ldots \overline{\mathbf{v}}_n(k) \end{bmatrix}$. We consider the Gramian spectrum largely in the infinite horizon as well, i.e. $k \to \infty$. The spectrum notation and the Gramian basis is simplified at $k \to \infty$ as $\overline{\lambda}_i$, $\overline{\mathbf{v}}_i$ for $i = 1, \ldots, n$ and Φ_G, respectively.

The Gramian spectrum is an interesting basis for sparsifying string system state snapshots, as it distinguishes configurations that require little energy to achieve (and hence could be substantially represented in the state) from ones that are hard to achieve, given the input location. The eigenvectors for the Gramian have a wavelet-like shape, with a peak around the input location and an oscillatory falloff away from the input (Figure 1). In contrast with the Laplacian spectrum, the Gramian spectrum depends on the specifics of the string system, including the connection strength a and the input location.

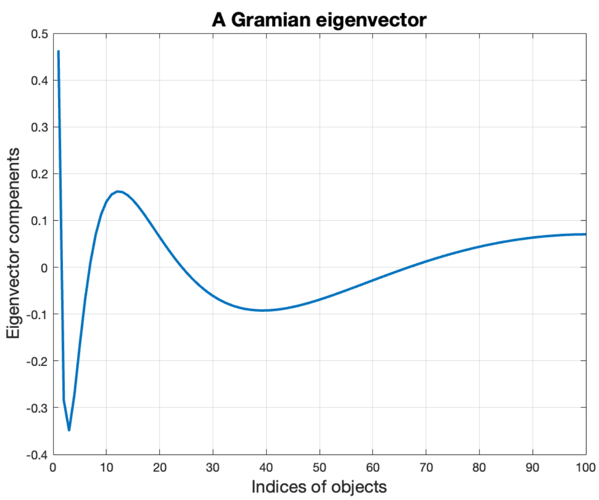

Fig. 1. An illustration of one Gramian eigenvector where an input is applied at location 1 (object 1).

III. Compressibility Analysis for the Laplacian Spectrum Basis

In this section, string compressibility in the Laplacian spectrum basis is characterized, given a white-noise input at an arbitrary location. As a preliminary step, we first recognize that the energy function $F(K)$ is hard to analyze directly, because it depends on the order statistics of $\mathbf{s}[k]$ (i.e., the ordering of the random entries in $\mathbf{s}[k]$ by amplitude). As a lower bound, we instead consider the energy in the K specific components (say $s_1[k], \ldots, s_K[k]$ without loss of generality) in a basis of interest. This energy is given by $F_s(K) = \frac{\sum_{i=1}^{K} E[s_i^2[k]]}{\sum_{j=1}^{n} E[s_j^2[k]]}$. The relationship between $F_s(K)$ and $F(K)$ is formalized in the following lemma:

Lemma 1: Consider the linear string system (1), in the case of a stochastic input $u[k]$. Consider compression of state snapshots at a time k. The energy fraction $F(K)$ is lower-bounded by $F_s(K)$, i.e. $F(K) \geq F_s(K)$.

The majorization in Lemma holds for any basis. For the Laplacian spectrum basis as defined in Section II, the first K basis vectors correspond to slower-varying sinusoids, which we expect to be primary components in the state snapshot (as we will verify in what follows). Therefore, a good bound is achieved by considering the first K entries rather than the K largest.

To bound the sparsity level that is sufficient to capture the state snapshot in the Laplacian-spectrum basis, we undertake a full algebraic analysis of the energy in the first k components of $\mathbf{s}[k]$. We do this using explicit expressions for the eigenvalues and eigenvectors of the state matrix A if the string system. Standard results on the eigenanalysis of tridiagonal Toeplitz matrices can be used to get explicit formulas for the eigenvalues of A. This eigenanalysis, and also the relationship to the eigenanalysis of L, is captured in the following:

Lemma 2: Consider the state matrix A of the string system. Denote eigenvalues of A as $1 = \lambda_1 > \ldots > \lambda_n$. Consider the eigenvectors matrix is shared by A and L, we denote the right eigenvector of A as same as Φ_L. Then eigenvalues and eigenvectors of A are the following:

(1). The ith eigenvalue is $\lambda_i = 1 - 2a + 2a\,cos\frac{(i-1)\pi}{n}$, for $i = 1, \ldots, n$. Furthermore, $\lambda_i \in (1-2a, 1]$.

(2) The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of the Laplacian L are also the right eigenvectors of A with each $\mathbf{v}_i$ corresponding to λ_i as defined above. The jth entry of the ith right eigenvector satisfies $v_{ji} = cos\frac{(j-1)(2i-1)\pi}{2n}$.

(3). The left eigenvector matrix $W = \Phi_L^{-1} = [w_{ij}]$ is given by:

$$w_{ij} = \begin{cases} \frac{1}{n}, & i = 1 \\ \frac{2}{n} cos\frac{(i-1)(2j-1)\pi}{2n}, & i \geq 2 \end{cases}, \tag{3}$$

where each row is a left eigenvector of A.

The results are derived easily from the analysis of the eigenvalue and eigenvector of the tridiagonal Toeplitz matrix, see [17].

In the next lemma, for the string system's state at a time k, the expected energy in each component in the Laplacian spectrum basis ($E[s_i^2[k]]$) is characterized. The result follows directly from a result in our previous work [9], hence details are omitted. We note the result holds only when the input is assumed to be a zero-mean unit-variance white noise process.

Lemma 3 ([9]): Consider the string system (1), in the case where the input is a zero-mean unit-variance white noise process input with zero-mean and unit variance. Consider expression of the time-k state $\mathbf{x}[k]$ in the Laplacian spectrum basis, i.e. $\mathbf{x}[k] = \Phi_L \mathbf{s}[k]$. Then, in the limit of large k, the expected energy $E[(s_i[k])^2]$ in each component approach:

$$E[s_i^2[k]] = \begin{cases} w_{iz}^2 k, & i = 1 \\ \frac{w_{iz}^2}{1-\lambda_i^2}, & i = 2, 3, \ldots, n, \end{cases} \tag{4}$$

where z is an input location, w_{iz} are the left eigenvectors of L as specified in Lemma 2.

Next, by substituting the exact expressions for the eigenvalues and eigenvector components and pursuing approximations for the large-n limit, the following expressions for the component energies are obtained:

Lemma 4: Consider the string system (1), with a stochastic input provided at location z. Now consider the expected energy in each component in the Laplacian eigenvector basis, i.e. $E(s_i^2[k])$, for sufficiently large k. Then, for sufficiently large n, $E[s_i^2[k]]$ approaches:

$$E[s_i^2[k]] = \begin{cases} \frac{k}{n^2}, & i=1 \\ \frac{4}{\pi}\frac{1}{2a}(\psi_1(i)+\psi_2(i)), & i=2,\ldots,n \end{cases} \quad (5)$$

where

$$\begin{cases} \psi_1(i) = \frac{\frac{\pi}{2n} cos^2 \frac{(i-1)(2z-1)\pi}{2n}}{sin^2 \frac{(i-1)\pi}{2n}} \\ \psi_2(i) = \frac{\frac{\pi}{2n} cos^2 \frac{(i-1)(2z-1)\pi}{2n}}{\frac{1}{2a} - sin^2 \frac{(i-1)\pi}{2n}} \end{cases} \quad (6)$$

Lemmas 3 and 4 show that the energy in the direction of the first eigenvector $\mathbf{v}_1 = \mathbf{1}$ asymptotes to a function that is growing linearly k, while the energy in the other directions approach a constant. The unbounded growth of the energy in the first eigenvector direction reflects that A has an eigenvalue at on the unit circle (at 1), and hence the stochastic input causes a slowly drifting offset in the states of all objects in the string. For this reason, the energy in the string state concentrates in the first basis vector in a formal sense. However, we are primarily concerned about whether or not the *shape* of the string state (not the offset) is captured in a small number of basis vectors. Hence, in the ensuing development, we exclude the component in the first basis vector direction in the analysis of compressibility. We therefore consider the following alternate energy-fraction measure for the compressibility at a K-sparsity level: $\widehat{F}(K) = \frac{\sum_{i=2}^{K} E[\hat{s}_i^2[k]]}{\sum_{j=2}^{n} E[\hat{s}_j^2[k]]}$.

As for the original energy-fraction measure, direct analysis of the modified energy-fraction measure is difficult because it requires consideration of order statistics. We therefore bound $\widehat{F}(K)$ in terms of the energy contained in basis components $2,\ldots,K$. Specifically, we define $\widehat{F}_s(K) = \frac{\sum_{i=2}^{K} E[s_i^2[k]]}{\sum_{j=2}^{n} E[s_j^2[k]]}$. Using the same logic as for the proof of Lemma 1, we immediately recover that $\widehat{F}(K) \geq \widehat{F}_s(K)$, i.e. $\widehat{F}_s(K)$ lower bounds the energy measure of interest $\widehat{F}(K)$.

We are now ready to present our main result, which shows that the the state snapshot's energy can be captured by an arbitrarily small fraction $\frac{K}{n}$ of basis vectors in the Laplacian spectrum basis.

Theorem 1: Consider the string system (1), in the case that it is driven by a 0-mean and unit variance white-noise input at location z. Consider the representation $\mathbf{s}[k]$ of a state snapshot $\mathbf{x}[k]$ in the Laplacian spectrum basis, for a sufficiently large time k. In particular, consider the sparsity level K required for an energy fraction $\widehat{F}(K) > \delta$ for a specified $\delta \in (0,1)$. For any $\delta \in (0,1)$, the desired energy fraction is achieved for $K > \epsilon n$ for any $\epsilon \in (0,1)$, provided that n is sufficiently large.

The theorem indicates that the string system's state is compressible using an arbitrarily small fraction of the Laplacian spectrum basis vectors, in the limit of a long string (large n). In short, the Laplacian spectrum or Fourier basis is able to sparsely represent string states, even without knowledge of the stochastic input location and model parameters.

IV. Compressibility in the Controllability Gramian Spectrum Basis

In this section, we study string compressibility in the controllability Gramian spectrum basis. The Gramian spectrum should be appealing as a means for compressing process state snapshots, because it can distinguish state configurations that are reachable with limited energy from ones that cannot be reached at a plausible energy level.

In the following development, we first give two simple results that give insight into the connection between the Gramian spectrum and compressibility, and hence provides a motivation for a closer assessment of sparsity in the Gramian spectrum basis. Finally, we discuss a specific characterization of sparsity levels that are sufficient to represent the state snapshot in the Gramian basis, when the dimension n of the string becomes large.

The first result is a characterization of the covariance of the string system's state snapshot in the Gramian spectrum basis:

Lemma 5: Consider the string system (1), in the case that the input is a zero-mean unit-variance additive Gaussian white noise process at location z. where the input location z is know, $u[k]$ is a 0-mean, 1-variance Gaussian white noise process. Consider expression of the string state at time k in terms of the Gramian spectrum basis (i.e. $\mathbf{x}[k] = \Phi_G(k)\mathbf{s}[k]$, where the spectrum $\Phi_G(k)$ is computed from the time Gramian $G(0,k) = \sum_{\tau=0}^{k-1} A^\tau BB'(A')^\tau$. Then the second moment of the transformed state $\mathbf{s}[k]$ in the Gramian spectrum basis is:

$$E[\mathbf{s}[k]\mathbf{s}'[k]] = \Lambda, \quad (7)$$

where Λ is a diagonal matrix with the eigenvalues of $G(0,k)$.

The above lemma shows that the Gramian spectrum whitens the state snapshot. Whitening filters are known to maximize compressibility, which gives an indication that the Gramian basis should allow sparse representation of the string state.

Of interest, there is a logic for compression in the Gramian spectrum basis, even in the case that the input signal is entirely arbitrary except for being energy-limited. Specifically, the following lemma shows a tight relationship between the maximum allowable component energy in different basis directions and the eigenvalues of the Gramian:

Lemma 6: Consider the string system (1) in the case where an input $u[k]$ is applied at location z on the string, which is energy bounded ($\sum_{i=0}^{k} u[i]^2 < \mu$). Then the components $s_i[k]$ of the string state in the controllability Gramian spectrum basis Φ_G in the infinity horizon satisfy $\sum_{i=1}^{n} \frac{s_i^2[k]}{\overline{\lambda_i}} < \mu$.

The lemma shows that components in basis vector directions must necessarily be small, if the corresponding eigenvalues $\overline{\lambda_i}$ of the Gramian are small.

For the Gramian spectrum basis in the infinity horizon, we also have been able to obtain a specific bound on the number of basis vectors K needed to capture (almost) all energy in the string state. To develop the result, we rely on the rich theory on the Gramian spectrum for symmetric matrices developed by T. Penzl and others [18]. The key insight of this work is that the eigenvalues of the Gramian fall off very quickly for systems with symmetric state matrices and a limited number of inputs. Applying the methodology to the string compressibility analysis requires three technical advances: 1) specializing the result to the particular state matrix form considered here, 2) understanding the scaling of the result as n is made large, and 3) correcting for the fact that the the state matrix has an eigenvalue on the unit circle. Applying this process, one finds that the any desired expected energy fraction $\widehat{F}(K) = \delta$, where $0 < \delta < 1$, can be achieved with a sparsity level K that is sublinear in n. Indeed, we posit and simulations suggest that $K = \mathcal{O}(n^{\alpha})$, for some $\alpha < 1$, basis vectors are sufficient. We expect to include a complete development .

In sum, the Gramian spectrum basis permits more effective compression of the string state than the Laplacian eigenvector basis, but at the cost of requiring knowledge of the string model's parameters and the input location.

V. Simulations

Simulations are developed which illustrate the compressibility of a string system's state (1) in the Laplacian spectrum basis and Gramian spectrum basis.

In Figure 2, we examine the compressibility fraction $\widehat{F}(K)$ as a function of the sparsity level K for the Laplacian spectrum basis, in a string with 2000 objects which is driven by a white noise input at location $z = 1$. The parameter in the state matrix is $a = 0.35$. The state snapshot is seen to be effectively compressed using 10 percent of the basis vectors.

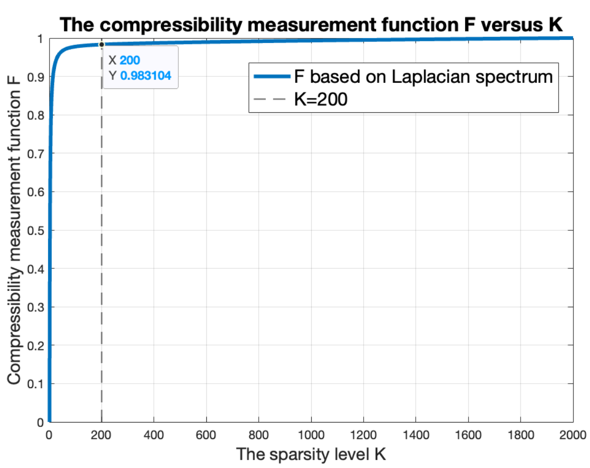

Fig. 2. Illustration of $F(K)$ versus the sparsity level K for state snapshot at time $k = 8 \times 10^5$ in the Laplacian spectrum.

Next, to illustrate the value of compressibility, recovery of the spread state from samples is demonstrated. Specifically, a vector y with 200 objects' states at randomly sampled locations is used for recovery. The sampling locations are shown in 5, and a state snapshot is also illustrated on the figure.

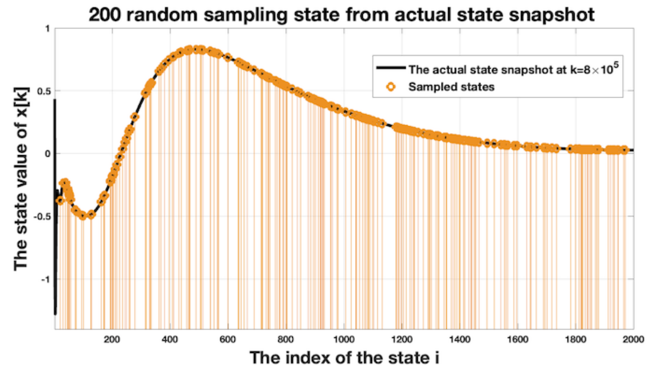

Fig. 3. Illustration of density and position of 200 randomly sampled object states from a state snapshot at $k = 8 \times 10^5$.

The recovery is based on the classical compressed sensing processes: given the sampling vector y and a known fixed basis (the Laplacian eigenvector basis in this case), a sparse solution with 200 sparsity level is generated by LASSO(least absolute shrinkage and selection operator) regression, and then the snapshot is recovered by $\mathbf{x}[k] = V\mathbf{s}[k]$. A recovered snapshot is shown in Figure 4. The snapshot is recovered well, with a mean square error (MSE) of 0.002.

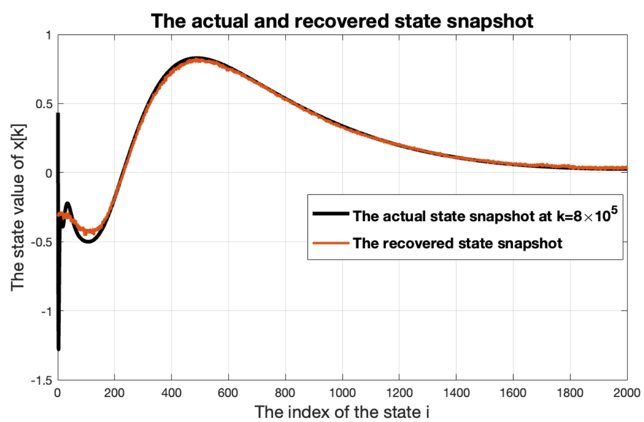

Fig. 4. A recovered state snapshot at time $k = 8 \times 10^5$ based in the Laplacian spectrum given 200 sampling state values.

Next, compressibility in the Gramian spectrum basis is illustrated. A string with $n = 100$ objects is considered, with $a = 0.2$ and the input location $z = 1$. The dependence of the energy fraction $F(K)$ on the sparsity is shown in Figure 5. In this case, even for a small system with $n = 100$ objects, a very small fraction of basis vectors (5-10) are needed to capture the energy in the string state.

State recovery using the Gramian spectrum basis is pursued. The states at 15 randomly-sampled locations are used for recovery Figure 6. The recovered state is shown in Figure 7.

The simulations demonstrate that both the Laplacian spectrum basis and the Gramian spectrum basis permit sparse

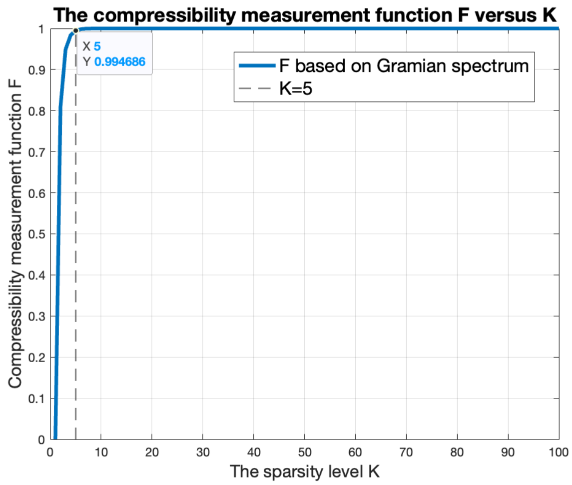

Fig. 5. Illustration of $F(K)$ versus the sparsity level K for state snapshot at $k = 30000$ in the Gramian spectrum.

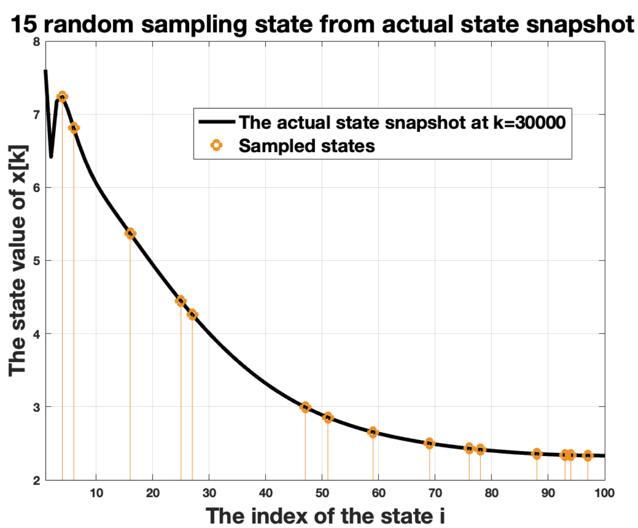

Fig. 6. Illustration of density and position of 15 randomly sampled states.

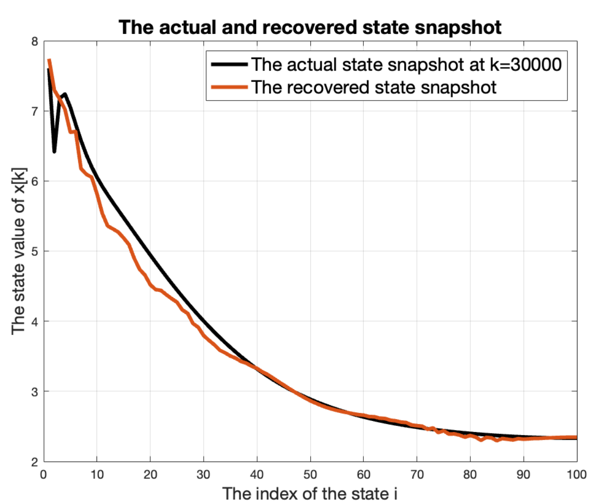

Fig. 7. A recovered state snapshot based on the Gramian spectrum basis from an actual state snapshot at $k = 30000$.

representation of string system state snapshots. It is important to emphasize that the Laplacian basis permits a parameter-free representation, requiring only the system scale. The Gramian basis, in comparison, provides an advantage in recovery when only low-dimensional state data is available. However, capturing the Gramian spectrum basis requires model knowledge, meaning the system matrices are necessary.

VI. Conclusion

The compressibility of state snapshots of linear string systems is analyzed in two spectral bases of the Laplacian and the Gramian. The analyses show that string system states are highly compressible in both bases. This may be valuable for the reconstruction of string states from limited data, effective control, and other goals. This paper focuses only very simple string processes, future work will need to consider more general models.

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