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Continuous-Time Adaptive Control with Dynamic Adaptation Gain

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Abstract— **This paper investigates a class of continuous-time parameter identifiers with the so-called "dynamic adaptation gain (DAG)", inspired by its discrete-time counterpart [1]. A modular control law is first presented: this helps re-parametrize the system and highlight the required properties for the identifier. Then, a dynamic adaptation gain is constructed using a strictly passive system, strengthened by a feedthrough path: the resulting DAG is input strictly passive. Analysis shows that the identifier with the proposed DAG satisfies specific signal properties. An integrated analysis of both the control law and the identifier establishes boundedness of all closed-loop signals and state convergence. The convergence of the parameter estimate is also established under a persistence of excitation condition. Finally, an example inspired by a path-following problem demonstrates the convergence improvement achieved by the proposed DAG.**

I. Introduction

Adaptive control describes a class of control algorithms that adapt their parameters or structure to accommodate uncertainties in the system, lack of knowledge about the system, and variations in the operating conditions. In the past decades, adaptive control has undergone extensive research and a comprehensive theory has been built for linear and nonlinear systems (see, *e.g.*, [2]–[6]).

The core of most adaptive control algorithms is the parameter adaptation mechanism. Since adaptive control systems are dynamic systems, unlike what happens for static optimization problems, there is no natural definition of a cost function. A standard practice is to use the current best knowledge of the unknown parameters to predict the evolution of the underlying system. Intuitively, if the current knowledge is close to the "true knowledge", the prediction error should be small. This motivates updating the current parameter estimation in the direction that reduces the prediction error. Viewed from this point, parameter adaptation algorithms are in nature optimization algorithms (see, *e.g.*, [5], [7] for a summary of the underlying optimization algorithms used in adaptive control), and of course, suffer from similar issues such as slow convergence due to "small" gradient and lack of robustness against noise and delays.

To overcome these issues, parameter adaptation algorithms with finer structures have been developed. One of the common remedies is to extend integral-type identifiers to higher-order identifiers (see, *e.g.*, [8]–[10]). Such an idea has been exploited in both discrete-time algorithms (see *e.g.*, [11], [12]) and in continuous-time algorithms (see *e.g.*, [13]–[15]). These methods are proven to help accelerate learning or improve parameter estimation. In [1], a general structure of a discrete-time high-order parameter estimator has been proposed under the name Dynamic Adaptation Gain (DAG)[1]. The static adaptation gain in the classical integral-type adaptation law is replaced by a dynamical operator, which generalizes several known high-order adaptation algorithms. Similar to classical results, the stability and convergence properties are related to passivity of the DAG and of the identifier subsystem.

An alternative method to enhance adaptation performance is to adjust the value of the adaptation gain online based on measurement from the controlled system. The classical least-squares adaptation algorithms (see, *e.g.*, [7] for a discrete-time version and *e.g.*, [5] for a continuous-time version) are their representatives. These algorithms modify the adaptation gain (also known as the *covariance matrix* in the literature) to mitigate the effect of noise on adaptation. Due to the nature of the adaptation gain being adjusted dynamically, sometimes this method is also referred to as dynamic adaptation gain [16][2]. Unlike in [1], the DAGs under such a definition modify only the adaptation gain itself and do not alter the gradient signal fed into the adaptation algorithm directly. A DAG of this type appears in the parameter estimation error dynamics as a time-varying gain instead of a dynamic compensator.

In this paper, we follow a similar idea to show that the continuous-time counterpart[3] of the DAG in [1] exists and has similar properties. In addition, it is shown that by a slight change in the location of the DAG, the passivity-related

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[1]Some special cases of DAG were developed in the 1970s, see *e.g.*, the PID adaptation in [2, Section 3.3]

[2]The motivation for introducing a dynamically updated adaptation gain in [16] and [17] is to use its time derivative to apply *certainty-equivalence principle* for unmatched uncertainties.

[3]The main reason for extending the notion of DAG design to continuous-time systems is because, in modern control systems, the control algorithm is typically running on a real-time operating system, which is mostly event-driven instead of time-driven and does not admit a fixed sampling period (which is the case for classical dedicated digital control systems). In this case, a standard methodology for control design is to first design a continuous-time algorithm with a guaranteed robustness margin, and then fit the effect of variable-rate sampling into the robustness margin by designing the triggering mechanism [18].

requirement on the identifier is easier to fulfil. We also show that the proposed identifier is compatible with a modular design as it has the same signal properties as the classical integral-type identifiers.

Notation: This paper uses standard notation unless stated otherwise. For an n-dimensional vector $v \in \mathbb{R}^n$, $|v|$ denotes the Euclidean 2-norm; $|v|_M = \sqrt{v^\top M v}$, $M = M^\top \succ 0$, denotes the weighted 2-norm with weight M. For $M = M^\top \succeq 0$, $\underline{\lambda}(M)$ denotes the smallest eigenvalue of M (and similarly for $\overline{\lambda}$ denoting the largest eigenvalue); $M^{\frac{1}{2}}$ denotes the unique non-negative square root of M. I denotes the identity matrix. Let $T(s)$ be a transfer function (matrix), $T(s)[u] \triangleq y$, where y is the output signal of the underlying dynamical system driven by u from some initial state. ⋄

II. System Re-parametrization for a Modular Design

Consider a system with *matched*[4] parametric uncertainty, namely the system

$$\dot{x} = Ax + B\left(u + \phi^\top(x)\theta\right), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}$ is the input; $\phi : \mathbb{R}^n \to \mathbb{R}^q$ is a known continuous nonlinear mapping; $\theta \in \mathbb{R}^q$ is a vector of constant unknown parameters; A, B are constant and known; and the pair (A, B) is stabilizable. In what follows we design a feedback controller independent of θ that generates u such that 1) all closed-loop signals are bounded and 2) $\lim_{t\to+\infty} x(t) = 0$.

To begin with, consider the simple scenario in which information on θ is available for control design. It is well known that (1) can be stabilized by a "nominal" control law, that is

$$u = -k^\top x - \phi^\top(x)\theta, \tag{2}$$

where k is a vector of feedback gains such that $P = P^\top \succ 0$ solves the Lyapunov equation

$$\left(A - Bk^\top\right)^\top P + P\left(A - Bk^\top\right) + Q = 0, \tag{3}$$

for some given $Q = Q^\top \succ 0$.

When information on θ is unavailable, one has to design an estimate $\hat{\theta}$ to compensate for the unknown θ and the feedback design is more complicated due to the nonlinear state-dependent uncertainty $\phi^\top(x)\theta$. Without a growth condition imposed on ϕ, even an exponentially converging parameter estimation error $\tilde{\theta} \triangleq \hat{\theta} - \theta$ can lead to finite escape time. A standard practice (see *e.g.*, [4]) is to add nonlinear damping terms to the feedback law to prevent such a phenomenon, namely, to modify (2) as

$$u = -k^\top(x)x - \phi^\top(x)\hat{\theta}, \tag{4}$$

where $k(x) = \left(k_L + k_\phi|\phi(x)|^2\right)PB$, with $k_L > 0$, $k_\phi > 0$, and $P = P^\top \succ 0$ the solution of the algebraic Riccati equation

$$A^\top P + PA - 2k_L PBB^\top P + Q = 0, \tag{5}$$

[4]Only matched uncertainties are considered to avoid unnecessary complexity. The modular design in this section is inspired by and can be extended to the unmatched case discussed in [4], with some tedious but straightforward computation.

for some given $Q = Q^\top \succ 0$. The resulting closed-loop system has the useful property stated hereafter.

Lemma 1: Consider the system (1) controlled by (4). The resulting closed-loop system is *input-to-state stable* (ISS) with respect to the input $\tilde{\theta}$. ⋄

From now on we consider the controlled plant dynamics (which do not explicitly contain u) described by

$$\dot{x} = A_{cl}(x)x - B\phi^\top(x)\tilde{\theta}, \tag{6}$$

where $A_{cl} \triangleq A - Bk^\top(x)$. From (5) we know that, with the same P solving (5), the state-dependent Lyapunov inequality (a nonlinear counterpart of (3))

$$A_{cl}^\top(x)P + PA_{cl}(x) + Q \preceq 0, \tag{7}$$

holds for all $x \in \mathbb{R}^n$. Since θ is unknown, in an adaptive control architecture, one can design an identifier that updates the estimate $\hat{\theta}$ such that the control law (4) achieves the control objectives. To do this, we first need a parametric model of the system. Typically, it is more convenient to study the convergence of parameter estimates if an algebraic relation between the parameter and the output of the parametric model can be established (in contrast, the relation between x and θ in (1) is differential). To this end, consider the filters

$$\dot{\omega}_0 = A_{cl}(x)\omega_0 - B\phi^\top(x)\hat{\theta}, \tag{8a}$$

$$\dot{\Omega}^\top = A_{cl}(x)\Omega^\top + B\phi^\top(x). \tag{8b}$$

One can see that the parametric model $x = \omega_0 + \Omega^\top\theta - \varepsilon$ holds because the parametrization error ε is governed by the equation $\dot{\varepsilon} = \dot{\omega}_0 + \dot{\Omega}^\top\theta - \dot{x} = A_{cl}(x)\left(\omega_0 + \Omega^\top\theta\right) - B\phi^\top(x)(\hat{\theta} - \theta) - A_{cl}(x)x + B\phi^\top(x)(\hat{\theta} - \theta)) = A_{cl}(x)\varepsilon$, hence ε exponentially decays to 0 by (7). In other words, as $t \to +\infty$, one has that $x \to \omega_0 + \Omega^\top\theta$, which establishes an algebraic relation between x and θ. This is also the reason why such a parametrization is called a *static parametric model* (see *e.g.*, [19]). Furthermore, we can use this algebraic relation to create a "prediction" $\hat{x} \triangleq \omega_0 + \Omega^\top\hat{\theta}$ for x. This leads to the alternative algebraic parametric model

$$\tilde{x} = \Omega^\top\tilde{\theta} + \varepsilon, \tag{9}$$

where $\tilde{x} \triangleq \hat{x} - x$ is the prediction error. We will show that while $\tilde{x}$ is crucial for the realization of the identifier, its properties are also determined by the identifier.

Before designing the identifier, we discuss what properties an "ideal" identifier should possess. Since the control design aims to make x converge to 0, a sufficient convergence condition is that both $\hat{x}$ and $\tilde{x}$ converge to 0 (due to (9)), while keeping all other closed-loop variables bounded.

Proposition 1: Consider the partially closed-loop system[5] consisting of (6), (8a), (8b), and the identifier to be designed. Such a system satisfies the following properties.

1) Ω and ε are bounded.
2) All signals are bounded, provided that the signals of the identifier subsystem are bounded, $\tilde{\theta} \in \mathcal{L}_\infty$, and $\dot{\hat{\theta}} \in \mathcal{L}_\infty$.

[5]The system is partially closed-loop because the identifier is still to be determined and its dynamical properties have not yet been analyzed.

3) $\lim_{t\to+\infty} x(t) = 0$, provided that in addition to the conditions in 2), $\tilde{x} \in \mathcal{L}_2$, and $\dot{\hat{\theta}} \in \mathcal{L}_2$.

◇

Proposition 1 indicates that the control law design can provide bounded Ω and ε, independent of the properties of the identifier. As long as the identifier can exploit such boundedness to achieve $\tilde{\theta} \in \mathcal{L}_\infty$, $\tilde{x} \in \mathcal{L}_2$ and $\dot{\hat{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, the control objectives can be achieved. This is the reason why the control law (4), together with the auxiliary filters (8a) and (8b) yields a modular design: any identifier satisfying the requirements in Proposition 1 can be integrated into the controller in a plug-and-play manner.

III. Identifier with Dynamic Adaptation Gain

In the classical scenario, the negative gradient of the quadratic cost of the instantaneous prediction error $J_1 = \frac{1}{2}|\tilde{x}|^2 = \frac{1}{2}\left|\Omega^\top(\hat{\theta}-\theta)+\varepsilon\right|^2$ is integrated to obtain $\hat{\theta}$, namely:

$$\dot{\hat{\theta}} = -\Gamma_1 \left(\frac{\partial J_1}{\partial \hat{\theta}}\right)^\top = -\Gamma_1 \Omega \tilde{x}, \tag{10}$$

where $\Gamma_1 = \Gamma_1^\top \succ 0$ is the adaptation gain to adjust the rate of the "learning" process. An alternative way to adjust the learning rate is to consider a prediction error "weighted" by $\Gamma_2 = \Gamma_2^\top \succ 0$, that is, to minimize $J_2 = \frac{1}{2}|\tilde{x}|^2_{\Gamma_2}$. The resulting gradient descent law is

$$\dot{\hat{\theta}} = -\left(\frac{\partial J_2}{\partial \hat{\theta}}\right)^\top = -\Omega \Gamma_2 \tilde{x}. \tag{11}$$

In both cases, Γ_1 and Γ_2 are constant positive definite matrices, or in other words, static adaptation gains. These gains are special cases of a more general class of dynamic adaptation gains (DAG) [20], which are dynamic operators.

In the spirit of the discrete-time designs in [1], in what follows we develop an identifier similar to (11) but with a DAG. To this end, consider an identifier described by the equations

$$\dot{\xi} = F\xi + Gv, \tag{12a}$$

$$\eta = H\xi + \Gamma v, \tag{12b}$$

$$\dot{\hat{\theta}} = \Omega\eta, \tag{12c}$$

where $\xi(t) \in \mathbb{R}^{n_\xi}$ is the state of the DAG; $v(t) \in \mathbb{R}^n$ is the input to the identifier; $\eta(t) \in \mathbb{R}^n$ is the output of the DAG; $\hat{\theta}(t) \in \mathbb{R}^q$ is the vector of parameter estimates for θ, and $\Omega(t) \in \mathbb{R}^{q\times n}$ is the filtered regressor matrix generated by (8b). The matrices in (12a)–(12c) are selected such that (F, G, H) is minimal[6], and the transfer function $T_{(H\xi)v}(s) = H(sI-F)^{-1}G$ is strictly positive real (SPR). In addition, $\Gamma = \Gamma^\top \succ 0$.

It is not difficult to see that, if one lets $v = -\tilde{x}$, the DAG described by (12a) and (12b) yields a dynamic counterpart of the gain Γ_2 in (11), since $\dot{\hat{\theta}} = \Omega T_{\eta v}(s)[\tilde{x}]$. In fact, the proposed identifier reduces to (11) if one removes the state variables in (12a) and sets $H = 0$, $\Gamma = \Gamma_2$.

[6]This means that (F, G) is controllable and (H, F) is observable.

Note now that since $\dot{\tilde{\theta}} = \dot{\hat{\theta}} - 0 = \dot{\hat{\theta}}$ and $\tilde{x} = \Omega^\top\tilde{\theta} + \varepsilon$, the identifier dynamics can be described by the negative feedback interconnection in Fig 1.

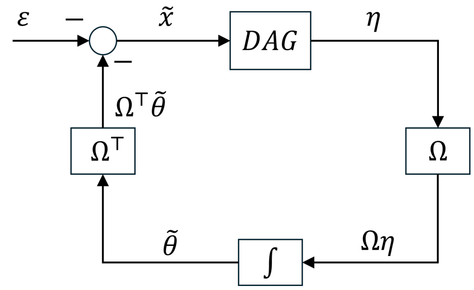

Fig. 1. Schematic interpretation of the identifier subsystem.

From the classical Lyapunov/SPR framework for adaptive control design, we know that the interconnected system is stable and the state of the forward-path system converges to zero, provided the forward-path system is strictly passive and the feedback-path system is passive. The difference here is that the forward path is now the DAG instead of a plant to be controlled. The convergence of ξ is not sufficient for the identifier design because we still need to fulfil the other requirements in Proposition 1, especially the $\mathcal{L}_2$ property of $\tilde{x}$ and $\dot{\hat{\theta}}$. Fortunately, one can show that the DAG described by (12a) and (12b) has a useful property as stated in the following result.

Lemma 2: The DAG is both strictly passive and input strictly passive[7] with respect to the input v, the output η, with storage function $V_\xi(\xi) = \frac{1}{2}|\xi|^2_X$, where $X = X^\top \succ 0$ solves the equations

$$F^\top X + XF = Y, \tag{13a}$$

$$XG = H^\top, \tag{13b}$$

for some given $Y = Y^\top \succ 0$. ◇

Proof: Since $T_{(H\xi)v}(s) = H(sI-F)^{-1}G$ is SPR, one can compute X by solving (13a) and (13b) (this is due to the Lefschetz-Kalman-Yakubovich Lemma, see, *e.g.*, Lemma 3.5.3 in [5] and the references therein). Differentiating V_ξ with respect to time along the trajectories of the system yields

$$\begin{aligned}\dot{V}_\xi &= \frac{1}{2}\xi^\top(F^\top X + XF)\xi + \xi^\top XGv \\ &\le -|\xi|^2_Y + (H\xi+\Gamma v)^\top v - |v|^2_\Gamma \\ &\le -\bar{a}_\xi|\xi|^2 - \bar{\sigma}_{\xi v}|v|^2 + \eta^\top v, \end{aligned} \tag{14}$$

where $\bar{a}_\xi \triangleq \frac{1}{2}\underline{\lambda}(Y) > 0$, $\bar{\sigma}_{\xi v} \triangleq \underline{\lambda}(\Gamma) > 0$. In particular, the fact that $\bar{a}_\xi > 0$ proves strict passivity and $\bar{\sigma}_{\xi v} > 0$ proves input strict passivity. ■

We are now ready to present the properties of the identifier subsystem.

Proposition 2: Consider the identifier described by (12a)–(12c), with input $v = -\tilde{x}$. The following properties hold.

1) All signals within the identifier are bounded.

[7]See, *e.g.*, Definition 6.3 in [21] for the definitions of passivity properties.

2) $\tilde{\theta} \in \mathcal{L}_\infty$, $\dot{\hat{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and $\tilde{x} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.
3) $\lim_{t\to+\infty} \xi(t) = 0$.

◇

Proof: Recall the property 1) of Proposition 1. Let $V_\varepsilon(\varepsilon) = |\varepsilon|_P^2$ and note that

$$\dot{V}_\varepsilon \leq -a_\varepsilon |\varepsilon|^2, \quad (15)$$

with $a_\varepsilon \triangleq \underline{\lambda}(Q)$. Consider the storage function $V_{\tilde{\theta}}(\tilde{\theta}) = \frac{1}{2}|\tilde{\theta}|^2$ and differentiate it with respect to time along the trajectories of (12c) (noting that $\dot{\tilde{\theta}} = \dot{\hat{\theta}}$). This yields $\dot{V}_{\tilde{\theta}} = \tilde{\theta}^\top(\Omega\eta) = (\Omega^\top\tilde{\theta})^\top\eta$. Recall Lemma 2 and the dissipation inequality (14). We can then write $\dot{V}_\xi \leq -\bar{a}_\xi|\xi|^2 - \bar{\sigma}_{\xi v}|\tilde{x}|^2 + \eta^\top(-\Omega^\top\tilde{\theta} - \varepsilon) = -\bar{a}_\xi|\xi|^2 - \bar{\sigma}_{\xi v}|\tilde{x}|^2 + \eta^\top(-\Omega^\top\tilde{\theta}) + (H\xi - \Gamma\tilde{x})^\top(-\varepsilon) \leq -a_\xi|\xi|^2 - \sigma_{\xi v}|\tilde{x}|^2 + \eta^\top(-\Omega^\top\tilde{\theta}) + b_{\xi\varepsilon}|\varepsilon|^2$, where $a_\xi \triangleq \bar{a}_\xi - \frac{1}{2\epsilon_{\xi\varepsilon}}$, $\sigma_{\xi v} \triangleq \bar{\sigma}_{\xi v} - \frac{1}{2\epsilon_{\tilde{x}\varepsilon}}$, $b_{\xi\varepsilon} \triangleq \frac{1}{2}\left(\epsilon_{\xi\varepsilon}|H|_2^2 + \epsilon_{\tilde{x}\varepsilon}|\Gamma|_2^2\right)$, and the $\epsilon_{(\cdot)} > 0$ are balancing coefficients to guarantee that $a_\xi > 0$ and $\sigma_{\xi v} > 0$. Let $V_{\xi\tilde{\theta}} = V_\xi + V_{\tilde{\theta}}$ and note that

$$\dot{V}_{\xi\tilde{\theta}} \leq -a_\xi|\xi|^2 - \sigma_{\xi v}|\tilde{x}|^2 + b_{\xi\varepsilon}|\varepsilon|^2. \quad (16)$$

This, together with (15), proves that ξ, $\tilde{\theta}$ and all other signals generated by the identifier subsystem are bounded. In addition, note that $\tilde{x} = \Omega^\top\tilde{\theta} + \varepsilon$, which is also bounded due to boundedness of Ω and ε. Therefore, the property 1) has been proven.

To prove 2), we first note that $\tilde{x} \in \mathcal{L}_2$, by (16) and (15). $\dot{\hat{\theta}} \in \mathcal{L}_2$ follows due to boundedness of Ω. Finally, using the boundedness properties established in 1) proves the claim.

To prove 3), we note that $\xi \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ by (16) and (15). In addition, $\dot{\xi}$ is bounded by (12a), with $v = -\tilde{x}$. Hence by the standard convergence argument in Barbalat's lemma, we conclude that $\lim_{t\to+\infty} \xi(t) = 0$, which completes the proof. ■

An intuitive interpretation of the reason why a DAG could be a better choice is to consider a scalar time-invariant version of Fig. 1, that is, let $\Omega = \Omega^\top = 1$. One can view the DAG as a controller/compensator for the integrator. In the classical case, the DAG reduces to a proportional controller, which can only shift the Bode magnitude plot of the loop transfer function to adjust the cut-off frequency. The proposed DAG acts as a passive compensator. Consider a first-order compensator as an example. One can let the DAG be

$$T_{\eta v}(s) = \gamma + \frac{\beta}{\alpha s + 1} = \frac{\gamma\alpha s + \beta + \gamma}{\alpha s + 1}, \quad (17)$$

with $\alpha > 0$, $\beta > 0$, $\gamma > 0$. In this case, the high-frequency gain γ and the DC gain $\beta + \gamma$ can be adjusted separately and the cut-off frequency can be adjusted by selecting α. Since Ω is in general time-varying and not a scalar, the tuning procedure, in that case, can be more complicated than the one discussed in the linear case.

Remark 1: Compared to the discrete-time DAG in [1] and the continuous-time DAG in [22], which are associated with (10), the passivity condition for the proposed DAG associated with (11) is easier to be satisfied, from a design perspective. For the DAG associated with (10), not only the DAG should be SPR, but the cascade of the DAG and the integrator should be positive real. This makes the design procedure of the DAG implicit as it has to satisfy two conditions simultaneously. In comparison, the proposed DAG only requires finding an SPR component and combining it with a feedthrough path, which is an explicit procedure. ◇

IV. Closed-Loop Analysis and Parameter Convergence

With the properties of the partially closed-loop system established in Section II and the properties of the identifier established in Section III, we are ready to analyze the overall closed-loop system.

Theorem 1: Consider the closed-loop system consisting of (6), (8a), (8b), and the identifier described by (12a)–(12c) (with $v = -\tilde{x}$). Then, all trajectories of the closed-loop system are bounded, $\lim_{t\to+\infty} \tilde{x}(t) = 0$, and $\lim_{t\to+\infty} x(t) = 0$. ◇

Proof: The proof is a straightforward consequence of Proposition 1 and Proposition 2. ■

Remark 2: One may have noted that the formulations of Proposition 1 and 2 are to achieve modularity, which allows the user to use an alternative identifier. In the proposed design, $\dot{\hat{\theta}} \in \mathcal{L}_2$ is already guaranteed by $\tilde{x} \in \mathcal{L}_2$. In addition, such a proof based on a cascade of implications for signal properties can be fragile if the system contains any unmodelled uncertainties that may result in circular signal dependencies (see, *e.g.*, [23]). In this case, one may perform a network small-gain-like synthesis by computing the dissipation inequalities of all subsystems and invoking Theorems 2 and 3 in [24] to derive an alternative proof for Theorem 1. ◇

Aside from the control objectives mentioned at the beginning of Section II, one may also want the controller to estimate the true value of θ, for long-term control performance [25]. To establish the convergence of $\hat{\theta}$ to the true parameter θ, we impose a classical condition.

Assumption 1: (Persistence of Excitation) The filtered regressor $\Omega : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q\times n}$ is persistently exciting (PE), that is, there exist positive constants $\underline{\alpha}$, $\overline{\alpha}$ and T such that

$$\underline{\alpha}\ TI \preceq \int_t^{t+T} \Omega(\tau)\Omega^\top(\tau)d\tau \preceq \overline{\alpha}\ TI, \quad (18)$$

for all $t \geq 0$. ◇

Remark 3: Assumption 1 essentially says that even if $\Omega(\tau)\Omega^T(\tau)$ can be singular at each τ, it is cumulatively positive-definite over a finite and receding time window. An obvious concern is that if x converges to 0, $\phi(x)$ converges to the constant vector $\phi(0)$, which may render Assumption 1 unachievable. There are two ways to deal with this issue. One is to add a probing signal, which leads to the so-called dual control [25]. The other is to design a reference signal for x to track such that Ω is PE (see an example in Section V). ◇

The PE condition is closely related to uniformly complete observability (UCO) of linear time-varying systems (LTV) (see, *e.g.*, Definition 3.3.3 in [5] or the definition in [26]). In what follows we exploit such a property to conclude convergence of $\tilde{\theta}$ to 0 in the presence of the DAG.

Theorem 2: Consider the same closed-loop system as in Theorem 1 under Assumption 1. Then $\tilde{\theta}$ converges to 0 exponentially. ◇

V. An Illustrative Example

Consider the path-following kinematic model (see *e.g.*, Section 8.8 in [19]), described by the equations

$$\dot{y} = V_c \sin\psi, \\ \dot{\psi} = -\frac{\kappa}{1-\kappa y} V_c \cos\psi + \omega, \tag{19}$$

where y is the lateral deviation from the desired path[8]; ψ is the deviation between the yaw angle of the agent and the direction of the desired path; ω is the angular speed of the yaw angle, which is used as a control input; V_c is the speed at the center of rotation; and κ is the curvature of the path at the location of the agent.

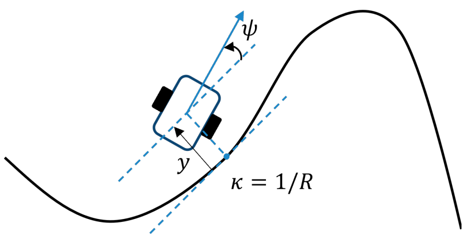

Fig. 2. Graphical illustration of y, ψ, and κ.

The model is valid for path-following problems (such as dual-wheel robots, quadrotors operated in planar space) in which the rotation of the agent can be independently controlled and the speed of advance is approximately constant. The task is twofold: to follow the path by letting y track a C^2 reference signal r (with r, $\dot{r}$, and $\ddot{r}$ known), and to identify the unknown curvature κ.

In what follows, we assume that the curvature of the desired path is constant (see [27] for a solution for the time-varying curvature scenario), the curvature is relatively small compared to y (*i.e.* $y \ll \frac{1}{\kappa}$) (in which case $\frac{1}{1-\kappa y} \approx 1+\kappa y$), and the angular deviation $\psi \ll 1$ (therefore $\sin\psi \approx \psi$). This allows simplifying the model into the form of equation (1), yielding

$$\dot{x}_1 = x_2, \\ \dot{x}_2 = \phi^\top(y,\psi)\theta + u, \tag{20}$$

where $x_1 \triangleq y - r$, $x_2 \triangleq V_c\psi - \dot{r}$, $u \triangleq V_c\omega - \ddot{r}$, $\phi^\top(y,\psi) \triangleq V_c^2[\cos(\psi), y\cos(\psi)]$, and $\theta^\top \triangleq [-\kappa, -\kappa^2]$. The tracking problem is therefore converted to the stabilization problem discussed in the previous sections. Furthermore, let $\kappa = -0.3$ (unknown to the controller), $V_c = 1$ and $r(t) = 0.14\sin(2\pi \times 0.2t)$.

To demonstrate the advantages of the DAG in comparison with a classical static adaptation gain, consider two controllers with the same control law ($k_L = 1, k_\phi = 1.6$, $Q = \mathrm{diag}(2,1)$), the same feedthrough matrix $\Gamma = I$, except that the classical controller does not have the strictly passive subsystem (12a), whereas for the DAG controller, the triple (H,F,G) is a realization of the SPR transfer function $T_{(H\xi)v}(s) \triangleq \mathrm{diag}\left(\frac{1500}{0.05s+1}, \frac{500}{0.05s+1}\right)$. Time histories of the states of system (20) controlled by the two controllers are plotted in Fig. 3. One can see from the plots that only the controller with DAG can identify the parameters $\theta_1 = -\kappa = 0.3$ and $\theta_2 = -\kappa^2 = -0.09$ within the 12-sec interval and the classical controller shows a large tracking error (x_1) due to slow parameter adaptation (verified by the slow convergence of $\hat{\theta}$). One can, of course, use a larger constant Γ for faster adaptation, whereas this may impact robustness [28]. To see this, we set $\Gamma = 10^3 I$ for the classical controller so that its identification capability is comparable to the DAG controller with $\Gamma = I$. To investigate their robustness against noise, we consider a high-frequency noise (represented by $0.01\sin 200t$) element-wise added to the measurement of x. It can be observed from Fig. 4 that with a much larger Γ the classical controller can achieve similar tracking performance as the DAG controller, whereas the parameter estimation is much more sensitive to high-frequency noise due to the large Γ.

VI. Conclusions and Future Work

This paper has proposed a continuous-time adaptive controller design exploiting a class of identifiers with dynamic adaptation gains. Such a class of identifiers have the same signal properties as the classical pure integral-type identifiers, which guarantees boundedness of the closed-loop signals and convergence of the state to zero. In addition, under the classical PE condition, the parameter estimation error converges to zero exponentially. The paper also presents an intuitive interpretation of the role of the DAG in the identifier loop. A path-following example has been presented to demonstrate the effectiveness of the DAG in improving parameter convergence.

Systematic quantitative analysis to rigorously explain such performance improvement will be the focus of the next step of this work.

[8] See the illustration in Fig. 2.

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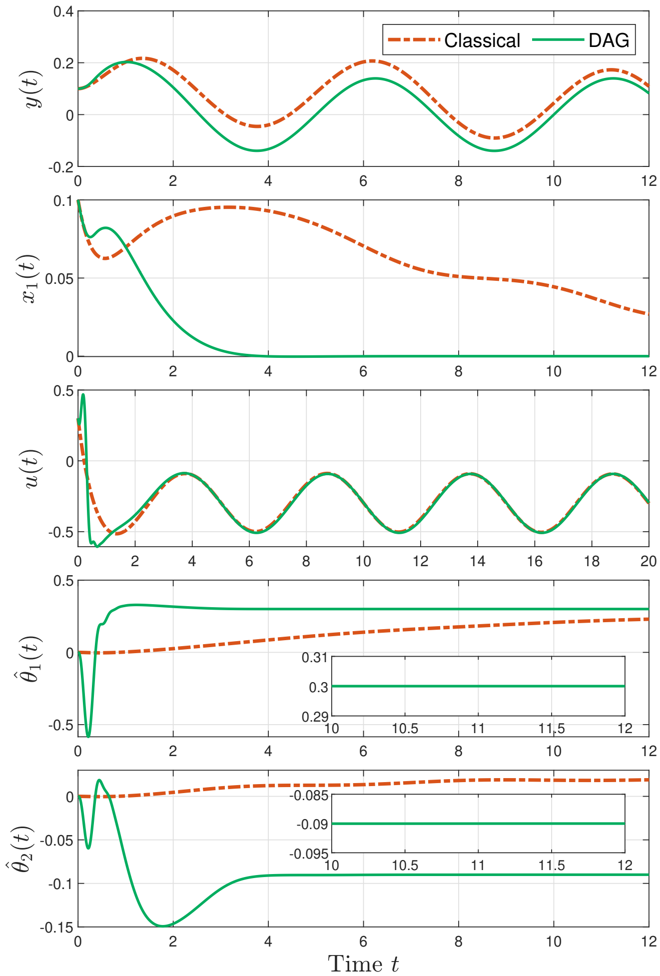

Fig. 3. Time histories of the states of the system (20) controlled by the adaptive controllers with and without DAG ($\Gamma = I$ for both).

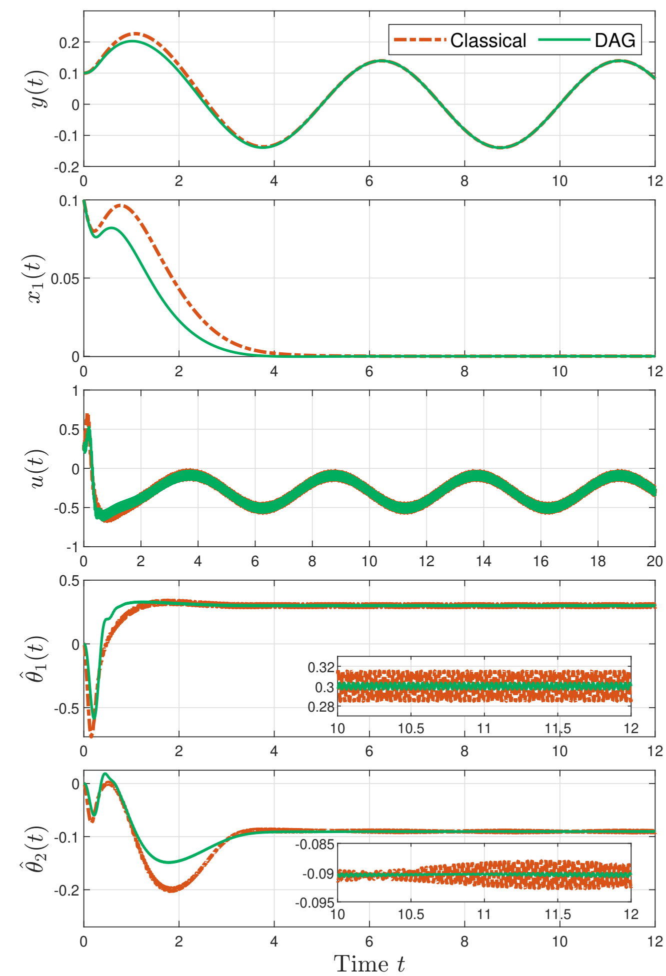

Fig. 4. Time histories of the states of the system (20) controlled by the adaptive controllers with DAG ($\Gamma = I$) and without DAG ($\Gamma = 10^3 I$).

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