2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Adaptation for Validation of Consolidated Control Barrier Functions

Mitchell Black[1] Dimitra Panagou[2]

***Abstract*— We develop a novel adaptation-based technique for safe control design in the presence of multiple state constraints. Specifically, we introduce an approach for synthesizing any number of candidate control barrier functions (CBFs), each encoding a different state constraint, into one consolidated CBF (C-CBF) candidate. We then propose a parameter adaptation law for the weights of the C-CBF's constituent functions such that its controllable dynamics are non-vanishing. We prove that the adaptation law certifies the consolidated CBF candidate as valid for a class of nonlinear, control-affine, multi-agent systems, which permits its use in a quadratic program based control law. We highlight the success of our approach in simulation on a multi-robot goal-reaching problem in a warehouse environment, and further demonstrate its efficacy via a laboratory study with an AION ground rover operating amongst other vehicles behaving both aggressively and conservatively.**

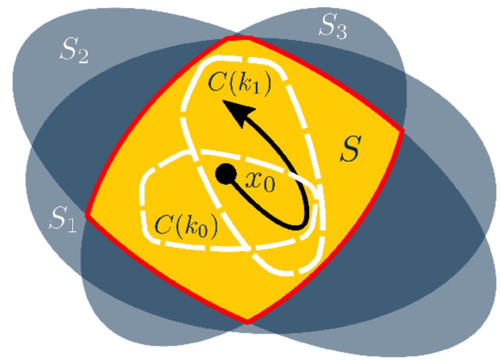

Fig. 1: Parameter adaptation for our C-CBF leads to a gain-dependent (and time-varying) controlled-invariant set $C(k) \subset S = \bigcap_{i=1}^{c} S_i$. $C(k)$ is shown here with a dotted white boundary for gains k_0 at time t_0 and k_1 at t_1.

I. Introduction

In the context of dynamical systems, the notion of safety may be equated to the containment of the system trajectories within a set of safe states. It is no coincidence, then, that as a class of certificate functions for set invariance control barrier functions (CBFs) have been studied extensively in the context of control design for safety-critical systems [1]–[4]. True to their name, CBFs enforce a barrier-like inequality condition on the evolution of the system trajectories with respect to the zero-level of a set of interest, and thereby contain the system within the set. The success of CBFs in rectifying some unsafe legacy controller has been demonstrated in a variety of practical applications, including including mobile robots [5], [6], unmanned aerial vehicles (UAVs) [7], [8], and autonomous driving [9], [10]; however, the verification of *candidate* CBFs as *valid*, i.e., proving that the specified barrier-like inequality condition is satisfiable via available control authority in perpetuity, is a challenging problem.

While for isolated CBF constraints various works have proven the viability of their associated controllers under certain conditions for systems with either unlimited [1] or bounded control authority [11], [12], these methods do not generally extend to control systems seeking to jointly satisfy multiple state constraints. Recent approaches to control design in the presence of multiple CBF constraints have mainly circumvented this challenge by considering only one such constraint at a given time instance, either by assumption [13] or construction in a non-smooth manner [14], [15]. In contrast, the authors of [16] and [17] each propose smoothly synthesizing one candidate CBF for the joint satisfaction of multiple constraints. No attempts are made, however, to validate these candidate functions. Other works propose synthesizing and/or verifying a CBF for a set of states using offline tools like sum-of-squares optimization [18], supervised machine learning [19], [20], and Hamilton-Jacobi-Bellman reachability analysis [21]. The first offline approach for synthesis of multiple compatible CBFs recently appeared in [22]. In contrast to offline methods, however, online approaches to control design are more responsive to environmental changes or unmodeled phenomena.

The problem of online safe control design under a multitude of constraints is especially relevant in practical applications involving autonomous mobile robots, where the main challenge is in the robot completing its nominal objective while satisfying constraints related to collision avoidance with respect to obstacles both static and dynamic. It is with this problem in mind that we propose a consolidated CBF (C-CBF) based approach to control design for multi-agent systems in the presence of both non-communicative and non-responsive (though non-adversarial) agents. Constructed by smoothly synthesizing any arbitrary number of candidate CBFs into one, our C-CBF defines a new super-level set that can under-approximate the intersection of its constituent sets arbitrarily closely (see Figure 1). We further propose a parameter adaptation law for the weighting of the constituent functions, and prove that its use renders our C-CBF valid and the super-level set forward invariant for the class of nonlinear, control-affine, multi-agent systems under consideration. And while various works have utilized parameter adaptation in the context of control for safety-critical systems, usually in an attempt to either learn [23], [24] or compensate for [25] unknown parameters in the

The authors would like to acknowledge the support of the National Science Foundation award number 1931982.

[1]Dept. of Aerospace Engineering, Univ. of Michigan, 1320 Beal Ave, Ann Arbor, MI 48109, USA; mblackjr@umich.edu.

[2]Dept. of Robotics and Dept. of Aerospace Engineering, Univ. of Michigan, Ann Arbor, MI 48109, USA; dpanagou@umich.edu.

979-8-3503-0123-6/23/$31.00 ©2023 IEEE

system dynamics, our proposed adaptation law is the first to our knowledge to be used for the simultaneous satisfaction of multiple CBF constraints. To show its effectiveness, we study a decentralized multi-robot goal-reaching problem in a warehouse environment amongst non-responsive agents. We further tested our controller experimentally on a collection of ground rovers in the laboratory setting and found that it succeeded in safely driving the rovers to their goal locations amongst non-responsive agents behaving both aggressively and conservatively.

The paper is organized as follows. Section II introduces some preliminaries, including set invariance, optimization based control, and our first problem statement. In Section III, we introduce the form of our C-CBF and propose a parameter adaptation law for rendering it valid. Sections IV and V contain the results of our simulated and experimental case studies respectively, and in Section VI we conclude with final remarks and directions for future work.

II. Mathematical Preliminaries

We use the following notation throughout the paper. $\mathbb{R}$ denotes the set of real numbers. The set of integers between i and j (inclusive) is $[i..j]$. $\|\cdot\|$ represents the L^2 norm. A function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to belong to class $\mathcal{K}_\infty$ if $\alpha(0) = 0$ and α is increasing on the interval $(-\infty, \infty)$. A function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to belong to class $\mathcal{LL}$ if for each fixed r (resp. s), the function $\phi(r, s)$ is decreasing with respect to s (resp. r) and is such that $\phi(r, s) \to 0$ for $s \to \infty$ (resp. $r \to \infty$). The Lie derivative of a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ along a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is denoted $L_f V(x) \triangleq \frac{\partial V}{\partial x} f(x)$.

In this paper we consider a multi-agent system, each of whose A constituent agents may be modelled by the following class of nonlinear, control-affine dynamical systems:

$$\dot{\boldsymbol{x}}_i = f_i(\boldsymbol{x}_i) + g_i(\boldsymbol{x}_i)\boldsymbol{u}_i, \tag{1}$$

where $\boldsymbol{x}_i \in \mathbb{R}^n$ and $\boldsymbol{u}_i \in \mathcal{U}_i \subseteq \mathbb{R}^m$ are the state and control input vectors for the ith agent, with $\mathcal{U}_i$ the input constraint set, and where $f_i : \mathbb{R}^n \to \mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are known, locally Lipschitz, and not necessarily homogeneous $\forall i \in \mathcal{A} = [1..A]$. We denote the concatenated state vector as $\boldsymbol{x} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_A]^\top \in \mathbb{R}^N$, the concatenated control input vector as $\boldsymbol{u} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_A]^\top \in \mathcal{U} \subseteq \mathbb{R}^M$, and as such express the full system dynamics as

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}(t)) + G(\boldsymbol{x}(t))\boldsymbol{u}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2}$$

where $F = [f_1, \ldots, f_A]^\top : \mathbb{R}^N \to \mathbb{R}^N$ and $G = \mathrm{diag}([g_1, \ldots, g_A]) : \mathbb{R}^N \to \mathbb{R}^{N\times M}$. We assume that a (possibly empty) subset of the agents are communicative, denoted $j \in \mathcal{A}_{com} = [1..A_{com}]$, in the sense that they share information (e.g., states, control objectives, etc.) with one another and are thus able to use centralized controllers, and that the remaining agents are non-communicative, denoted $k \in \mathcal{A}_{nc} = [(A_{com} + 1)..A]$, in that they do not share information and therefore must resort to decentralized control schemes. $A_{com} \geq 0$ and $A_{nc} = A - A_{com} \geq 0$ are the number of communicative and non-communicative agents respectively. We further assume that all agents are non-adversarial in that they do not seek to damage or otherwise deceive others, though there may be non-communicative agents which are non-responsive ($l \in \mathcal{A}_{ncnr} \subseteq \mathcal{A}_{nc}$) in that they do not actively avoid unsafe situations.

A. Safety and Forward Invariance

Consider a set of states $\mathcal{S}$ defined implicitly by a continuously differentiable constraint function $h : \mathbb{R}^N \to \mathbb{R}$, as follows:

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^N \mid h(\boldsymbol{x}) \geq 0\}, \tag{3}$$

where the boundary and interior of $\mathcal{S}$ are denoted by $\partial\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^N \mid h(\boldsymbol{x}) = 0\}$ and $\mathrm{int}(\mathcal{S}) = \{\boldsymbol{x} \in \mathbb{R}^N \mid h(\boldsymbol{x}) > 0\}$ respectively, and for which it is known that $\frac{\partial h}{\partial \boldsymbol{x}} \neq 0, \forall \boldsymbol{x} \in \partial\mathcal{S}$. The function h may be used to encode safety in a particular sense, e.g., inter-agent collision avoidance, obeying a speed limit, etc. Thus, in many works (e.g., [26], [27]), the set $\mathcal{S}$ is referred to as *safe* if $\mathcal{S}$ is *forward-invariant*, i.e., if $\boldsymbol{x}(0) \in \mathcal{S} \implies \boldsymbol{x}(t) \in \mathcal{S}$, $\forall t \geq 0$. Nagumo's Theorem provides a necessary and sufficient condition for rendering such a set forward-invariant for the system (2).

Lemma 1 (Nagumo's Theorem [28])**.** *Suppose that there exists $\boldsymbol{u}(\boldsymbol{x}) \in \mathcal{U}$ such that (2) admits a globally unique solution for each $\boldsymbol{x}_0 \in \mathcal{S}$. Then, the set $\mathcal{S}$ is forward-invariant for the controlled system (2) if and only if*

$$L_F h(\boldsymbol{x}) + L_G h(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) \geq 0, \ \forall \boldsymbol{x} \in \partial\mathcal{S}. \tag{4}$$

One way to satisfy Nagumo's Theorem is to use CBFs in the control design.

Definition 1. *[1, Def. 5] Given a set $\mathcal{S} \subset \mathbb{R}^N$ defined by (3) for a continuously differentiable function $h : \mathbb{R}^N \to \mathbb{R}$, the function h is a **control barrier function** (CBF) defined on a set $\mathcal{D} \supseteq \mathcal{S}$ if there exists a Lipschitz continuous class $\mathcal{K}_\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, for all $\boldsymbol{x} \in \mathcal{D}$,*

$$\sup_{\boldsymbol{u}\in\mathcal{U}} \left[L_F h(\boldsymbol{x}) + L_G h(\boldsymbol{x})\boldsymbol{u}\right] \geq -\alpha(h(\boldsymbol{x})). \tag{5}$$

In this paper, we assume that $\frac{\partial h}{\partial \boldsymbol{x}}$ is Lipschitz continuous so that $L_F h(\boldsymbol{x})$ and $L_G h(\boldsymbol{x})$ are likewise. In other works (e.g., [29]), the function h defining $\mathcal{S}$ is a CBF if there exists a function $\alpha \in \mathcal{K}_\infty$ satisfying

$$L_G h(\boldsymbol{x}) = \boldsymbol{0}_{1\times M} \implies L_F h(\boldsymbol{x}) + \alpha(h(\boldsymbol{x})) > 0. \tag{6}$$

We observe that with unbounded control authority (i.e., $\mathcal{U} = \mathbb{R}^M$) a sufficient condition for the existence of some $\alpha \in \mathcal{K}_\infty$ satisfying (5), and thus a condition for h to be a CBF, is $L_G h(\boldsymbol{x}) \neq \boldsymbol{0}_{1\times M}$, $\forall \boldsymbol{x} \in \mathcal{S}$. Notably, however, when there are multiple state constraints in need of satisfaction, i.e., for $c > 1$ continuously differentiable functions $h_s : \mathbb{R}^N \to \mathbb{R}$, $s \in [1..c]$ defining sets

$$\mathcal{S}_s = \{\boldsymbol{x} \in \mathbb{R}^N \mid h_s(\boldsymbol{x}) \geq 0\}, \tag{7}$$

which may encode, e.g., a total of $c = \binom{A}{2}$ inter-agent collision avoidance constraints, the condition $L_G h_s(\boldsymbol{x}) \neq \boldsymbol{0}_{1\times M}$, $\forall \boldsymbol{x} \in \mathcal{S}_s$, $\forall s \in [1..c]$ is not sufficient for each h_s to be a CBF.

B. Control Design using CBFs

Decentralized controllers, in which agents compute inputs based on local information, have found empirical success as a control strategy for multi-agent systems of the form (2) [30], [31]. The following is an example of one such controller for agent $i \in \mathcal{A}$ with safety constraints encoded via $c > 1$ CBFs:

$$\boldsymbol{u}_i^* = \underset{\boldsymbol{u}_i \in \mathcal{U}_i}{\arg\min} \frac{1}{2}\|\boldsymbol{u}_i - \boldsymbol{u}_i^0\|^2 \tag{8a}$$
$$\text{s.t.} \quad \forall s \in [1..c]$$
$$a_{s,i} + \boldsymbol{b}_{s,i}\boldsymbol{u}_i \geq 0, \tag{8b}$$

where (8a) seeks to minimize the deviation of the control solution $\boldsymbol{u}_i^*$ from some nominal input $\boldsymbol{u}_i^0$, and (8b) encodes c safety constraints of the form (5), where $a_{s,i} = \frac{\partial h_s}{\partial \boldsymbol{x}_i} f_i(\boldsymbol{x}_i) + \alpha_s(h_s(\boldsymbol{x}))$ and $\boldsymbol{b}_{s,i} = \frac{\partial h_s}{\partial \boldsymbol{x}_i} g_i(\boldsymbol{x}_i)$. As observed in [6], however, for many systems (8) is neither guaranteed to be feasible nor to preserve safety between agents. These issues are mitigated by a centralized controller of the form

$$\boldsymbol{u}_{\mathcal{A}_{com}}^* = \underset{\boldsymbol{u}_{\mathcal{A}_{com}} \in \mathcal{U}_{\mathcal{A}_{com}}}{\arg\min} \frac{1}{2}\|\boldsymbol{u}_{\mathcal{A}_{com}} - \boldsymbol{u}_{\mathcal{A}_{com}}^0\|^2 \tag{9a}$$
$$\text{s.t.} \quad \forall j,k \in \mathcal{A}_{com},\ k \neq j$$
$$a_{s,j} + \boldsymbol{b}_{s,j}\boldsymbol{u}_j \geq 0,\ \forall s \in [1..c_I], \tag{9b}$$
$$a_{s,jk} + \boldsymbol{b}_{s,j}\boldsymbol{u}_j + \boldsymbol{b}_{s,k}\boldsymbol{u}_k \geq 0,\ \forall s \in [c_I..c], \tag{9c}$$

which may be deployed by the subset of communicating agents $i \in \mathcal{A}_{com}$. In this case, $\boldsymbol{u}_{\mathcal{A}_{com}}^0 = [\boldsymbol{u}_1^0, \ldots, \boldsymbol{u}_{A_c}^0]^\top$ is the nominal input vector shared amongst communicative agents, the input constraint set is

$$\mathcal{U}_{\mathcal{A}_{com}} = \{\boldsymbol{u}_{\mathcal{A}_{com}} \in \mathbb{R}^{m \cdot A_{com}} \mid \boldsymbol{u}_1 \in \mathcal{U}_1, \ldots, \boldsymbol{u}_{\mathcal{A}_{com}} \in \mathcal{U}_{\mathcal{A}_{com}}\},$$

(9b) denotes the $c_I \geq 0$ individual CBF constraints for agent j (e.g., speed), and (9c) represents combinations of safety constraints between agents (e.g., collision avoidance), where $a_{s,jk} = \frac{\partial h_s}{\partial \boldsymbol{x}_j} f_j(\boldsymbol{x}_j) + \frac{\partial h_s}{\partial \boldsymbol{x}_k} f_k(\boldsymbol{x}_k) + \alpha_s(h_s(\boldsymbol{x}))$, $\boldsymbol{b}_{s,j} = \frac{\partial h_s}{\partial \boldsymbol{x}_j} g_j(\boldsymbol{x}_j)$, and $\boldsymbol{b}_{s,k} = \frac{\partial h_s}{\partial \boldsymbol{x}_k} g_k(\boldsymbol{x}_k)$. When all agents are communicative, i.e., when $\mathcal{A}_{com} = \mathcal{A}$, the control law (9) is guaranteed to preserve safety provided that it is feasible.

A challenge when it comes to both (8) and (9) is in satisfying every constraint simultaneously, especially when it comes to designing each α_s. In some recent works, authors have proposed setting $\alpha_s(h_s) = p_s h_s$ and including the parameters p_s as decision variables in the QP [32], but performance is still heavily dependent on cost function weights. Others have avoided the issue of multiple candidate CBFs by assuming that only one constraint is in need of satisfaction at once [13] or by synthesizing a single non-smooth candidate CBF [14], [15], both of which may lead to undesirable oscillatory agent motion as the constraints are toggled on and off. We seek to address this open problem, and require the following assumption to do so.

Assumption 1. *The intersection of constraint sets $\mathcal{S}_s$ given by (7) for all $s \in [1..c]$ is non-empty, i.e., $\mathcal{S} = \bigcap_{s=1}^{c} \mathcal{S}_s \neq \emptyset$.*

Without this assumption, it is impossible to satisfy all constraints jointly. We now formally state the first problem.

Problem 1. *Suppose that Assumption 1 holds for a collection of $c > 1$ constraint functions h_s corresponding to constraint sets $\mathcal{S}_s$, $\forall s \in [1..c]$. Design a consolidated control barrier function candidate $H : \mathbb{R}^N \times \mathbb{R}_+^c \to \mathbb{R}$ with constituent gains $\boldsymbol{k} = [k_1, \ldots, k_c]^\top \in \mathbb{R}_+^c$ such that the set $\mathcal{C}(\boldsymbol{k}) = \{\boldsymbol{x} \in \mathbb{R}^N \mid H(\boldsymbol{x}, \boldsymbol{k}) \geq 0\} \subseteq \mathcal{S}$ for any $\boldsymbol{k}$ satisfying $0 < k_s < \infty$, $\forall s \in [1..c]$.*

III. Consolidated CBF based Control

In this section, we introduce our solution to Problem 1, a C-CBF candidate that smoothly synthesizes multiple candidate CBFs into one, and then design an adaptation law to render the candidate C-CBF valid for safe control design.

A. Consolidated CBFs

Let the vector of $c > 1$ candidate CBFs evaluated at a given state $\boldsymbol{x}$ be denoted $\boldsymbol{h}(\boldsymbol{x}) = [h_1(\boldsymbol{x}) \ \ldots \ h_c(\boldsymbol{x})]^\top \in \mathbb{R}^c$, and define a gain vector as $\boldsymbol{k} = [k_1 \ \ldots \ k_c]^\top \in \mathbb{R}_+^c$. Our C-CBF candidate $H : \mathbb{R}^N \times \mathbb{R}_+^c \to \mathbb{R}$ is the following:

$$H(\boldsymbol{x}, \boldsymbol{k}) = 1 - \sum_{s=1}^{c} \phi\Big(h_s(\boldsymbol{x}), k_s\Big), \tag{10}$$

where $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{LL}$, is continuously differentiable, and satisfies $\phi(h_s, 0) = \phi(0, k_s) = \phi(0,0) = 1$. For example, the decaying exponential function, i.e., $\phi(h_s, k_s) = e^{-h_s k_s}$, satisfies these requirements over the domain $\mathbb{R}_+ \times \mathbb{R}_+$. With ϕ possessing these properties, it follows that the set $\mathcal{C}(\boldsymbol{k}) = \{\boldsymbol{x} \in \mathbb{R}^N \mid H(\boldsymbol{x}, \boldsymbol{k}) \geq 0\}$ is a subset of $\mathcal{S}$ (i.e., $\mathcal{C}(\boldsymbol{k}) \subset \mathcal{S}$), where the level of closeness of $\mathcal{C}(\boldsymbol{k})$ to $\mathcal{S}$ depends on the choices of gains $\boldsymbol{k}$. This may be confirmed by observing that if any $h_s(\boldsymbol{x}) = 0$ then $H(\boldsymbol{x}) \leq 1 - 1 - \sum_{j=1, j\neq s}^{c} \phi(h_j(\boldsymbol{x}), k_j) < 0$, and thus for $H(\boldsymbol{x}) \geq 0$ it must hold that $h_s(\boldsymbol{x}) > 0$, for all $s \in [1..c]$.

As such, H defined by (10) is a solution to Problem 1, i.e., H is a C-CBF candidate. This implies via Lemma 1 that if H is a *valid* C-CBF over the set $\mathcal{C}(\boldsymbol{k})$, then $\mathcal{C}(\boldsymbol{k})$ is forward invariant and thus the trajectories of (2) remain safe with respect to each constituent safe set $\mathcal{S}_s$, $\forall s \in [1..c]$. By Definition 1, for a static gain vector (i.e., $\dot{\boldsymbol{k}} = \boldsymbol{0}_{c\times 1}$) the function H is a CBF on the set $\mathcal{S}$ if there exists $\alpha_H \in \mathcal{K}_\infty$ such that the following condition holds for all $\boldsymbol{x} \in \mathcal{S} \supset \mathcal{C}(\boldsymbol{k})$:

$$L_F H(\boldsymbol{x}, \boldsymbol{k}) + L_G H(\boldsymbol{x}, \boldsymbol{k})\boldsymbol{u}(\boldsymbol{x}) \geq -\alpha_H(H(\boldsymbol{x}, \boldsymbol{k})), \tag{11}$$

where from (10) it follows that

$$L_F H(\boldsymbol{x}) = -\sum_{s=1}^{c} \frac{\partial \phi}{\partial h_s} L_F h_s(\boldsymbol{x}), \tag{12}$$
$$L_G H(\boldsymbol{x}) = -\sum_{s=1}^{c} \frac{\partial \phi}{\partial h_s} L_G h_s(\boldsymbol{x}). \tag{13}$$

Again taking $\phi(h_s, k_s) = e^{-h_s k_s}$ as an example, we obtain that $\frac{\partial \phi}{\partial h_s} = -k_s e^{-h_s k_s}$, in which case it is evident that the role of the gain vector $\boldsymbol{k}$ is to weight the constituent candidate CBFs h_s and their derivative terms $L_F h_s$ and $L_G h_s$ in the CBF condition (11). Thus, a higher value k_s indicates a weaker weight in the CBF dynamics, as the exponential decay overpowers the linear growth. Due to the combinatorial nature of these gains, for an arbitrary $\boldsymbol{k}$ there may exist some $\boldsymbol{x} \in \mathcal{C}(\boldsymbol{k})$ such that $L_G H(\boldsymbol{x}) = \mathbf{0}_{1\times M}$, which may violate (6) and lead to the state exiting $\mathcal{C}(\boldsymbol{k})$ (and potentially $\mathcal{S}$ as a result). Using online adaptation of $\boldsymbol{k}$, however, it may be possible to achieve $L_G H(\boldsymbol{x}(t)) \neq \mathbf{0}_{1\times M}$ for all $t \geq 0$, which motivates the following problem.

Problem 2. *Given a C-CBF candidate $H : \mathbb{R}^N \times \mathbb{R}^c_+ \to \mathbb{R}$ defined by (10), design an adaptation law $\dot{\boldsymbol{k}} = \kappa(\boldsymbol{x}, \boldsymbol{k})$ such that $L_G H(\boldsymbol{x}(t)) \neq \mathbf{0}_{1\times M}$, $\forall t \geq 0$.*

B. Adaptation for Control Synthesis

Before proceeding with our main result, we require the following regularity assumption.

Assumption 2. *At all points in the intersection of constraint sets $\mathcal{S}$, the matrix of controlled candidate CBF dynamics $\boldsymbol{L}_G \in \mathbb{R}^{c\times M}$ is not all zero, i.e.,*

$$\boldsymbol{L}_G(\boldsymbol{x}) \triangleq \begin{bmatrix} L_G h_1(\boldsymbol{x}) \\ \vdots \\ L_G h_c(\boldsymbol{x}) \end{bmatrix} \neq \mathbf{0}_{c\times M}, \ \forall \boldsymbol{x} \in \mathcal{S}. \tag{14}$$

The above requires non-zero sensitivity of at least one constraint function h_s to the control input $\boldsymbol{u}$. It is a mild condition, and is easily satisfiable when at least one h_s is of relative-degree one with respect to the system (2).

Now, consider the following QP-based adaptation law:

$$\kappa(\boldsymbol{x}, \boldsymbol{k}) = \underset{\boldsymbol{\mu}\in\mathbb{R}^c}{\arg\min} \ \frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{\mu}_0(\boldsymbol{x}))^\top \boldsymbol{P}(\boldsymbol{\mu} - \boldsymbol{\mu}_0(\boldsymbol{x})) \tag{15a}$$

$$\text{s.t., } \forall s \in [1..c],$$

$$\mu_s + \alpha_k(k_s - k_{s,min}) \geq 0, \tag{15b}$$

$$\boldsymbol{p}^\top(\boldsymbol{x})\boldsymbol{Q}(\boldsymbol{x})\dot{\boldsymbol{p}} + \boldsymbol{p}^\top(\boldsymbol{x})\dot{\boldsymbol{Q}}\boldsymbol{p}(\boldsymbol{x}) + \alpha_p(h_p(\boldsymbol{x})) \geq 0, \tag{15c}$$

where $\boldsymbol{P} \in \mathbb{R}^{c\times c}$ is a positive-definite gain matrix, $\alpha_k, \alpha_p \in \mathcal{K}_\infty$, $\boldsymbol{\mu}_0 \in \mathbb{R}^c$ is the desired solution, $\boldsymbol{k}_{min} = [k_{1,min}, \dots, k_{c,min}]^\top$ is the vector of minimum allowable values $k_{s,min} > 0$, and

$$\boldsymbol{p}(\boldsymbol{x}) \triangleq \begin{bmatrix} \frac{\partial\phi(\boldsymbol{x})}{\partial h_1} & \dots & \frac{\partial\phi(\boldsymbol{x})}{\partial h_c} \end{bmatrix}^\top, \tag{16}$$

$$\boldsymbol{Q}(\boldsymbol{x}) \triangleq \boldsymbol{I} - 2\boldsymbol{N}\boldsymbol{N}^\top - \boldsymbol{N}\boldsymbol{N}^\top\boldsymbol{N}\boldsymbol{N}^\top \tag{17}$$

with $h_p(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{p}^\top(\boldsymbol{x})\boldsymbol{Q}(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}) - \varepsilon$, $\varepsilon > 0$, and

$$\boldsymbol{N} = \boldsymbol{N}(\boldsymbol{x}) \triangleq [\boldsymbol{n}_1(\boldsymbol{x}) \ \dots \ \boldsymbol{n}_r(\boldsymbol{x})], \tag{18}$$

such that $\{\boldsymbol{n}_1(\boldsymbol{x}), \dots, \boldsymbol{n}_r(\boldsymbol{x})\}$ constitutes a basis for the null space of $\boldsymbol{L}_G^\top(\boldsymbol{x})$, i.e., $\mathcal{N}(\boldsymbol{L}_G^\top(\boldsymbol{x})) = \text{span}\{\boldsymbol{n}_1(\boldsymbol{x}), \dots, \boldsymbol{n}_r(\boldsymbol{x})\}$, where $\boldsymbol{L}_G(\boldsymbol{x})$ is given by (14).

The following is one of the main contributions of the paper, and proves how the adaptation law (15) may be used to solve Problem 2 and therefore to render H a valid CBF for the set $\mathcal{C}(\boldsymbol{k}(t))$, for all $t \geq 0$.

Theorem 1. *Suppose that there exist $c > 1$ candidate CBFs $h_s : \mathbb{R}^N \to \mathbb{R}$ defining sets $\mathcal{S}_s = \{\boldsymbol{x} \in \mathbb{R}^N \mid h_s(\boldsymbol{x}) \geq 0\}$, $\forall s \in [1..c]$. Further suppose that Assumptions 1 and 2 hold, and that $\mathcal{U} = \mathbb{R}^M$. If $\boldsymbol{k}(0)$ is such that $L_G H(\boldsymbol{x}(0)) \neq \mathbf{0}_{1\times M}$, then under $\dot{\boldsymbol{k}} = \kappa(\boldsymbol{x}, \boldsymbol{k})$ the controlled CBF dynamics are non-vanishing provided that (15) is feasible, i.e., $L_G H(\boldsymbol{x}(t)) \neq \mathbf{0}_{1\times M}$, $\forall t \geq 0$, and thus the function H defined by (10) is a valid CBF for the set $\mathcal{C}(\boldsymbol{k}(t)) = \{\boldsymbol{x} \in \mathbb{R}^N \mid H(\boldsymbol{x}, \boldsymbol{k}) \geq 0\}$, $\forall t \geq 0$.*

Proof. First, given (10), we have that

$$\begin{aligned} \dot{H} &= -\sum_{s=1}^{c}\left(\frac{\partial\phi}{\partial h_s}\dot{h}_s + \frac{\partial\phi}{\partial k_s}\dot{k}_c\right) \\ &= \boldsymbol{p}^\top\dot{\boldsymbol{h}} + \boldsymbol{q}^\top\dot{\boldsymbol{k}} \\ &= \boldsymbol{p}^\top(\boldsymbol{L}_f + \boldsymbol{L}_G\boldsymbol{u}) + \boldsymbol{q}^\top\dot{\boldsymbol{k}} \end{aligned}$$

where $\boldsymbol{p}$ is given by (16), $\boldsymbol{L}_G$ by (14), $\boldsymbol{L}_f = [L_F h_1 \ \dots \ L_F h_c]^\top$, and $\boldsymbol{q} = [\frac{\partial\phi}{\partial k_1} \ \dots \ \frac{\partial\phi}{\partial k_c}]^\top$. As such, $L_F H = \boldsymbol{p}^\top\boldsymbol{L}_f + \boldsymbol{q}^\top\dot{\boldsymbol{k}}$ and $L_G H = \boldsymbol{p}^\top\boldsymbol{L}_G$. With $\mathcal{U} = \mathbb{R}^M$, it follows that as long as $L_G H \neq \mathbf{0}_{1\times M}$ it is possible to choose $\boldsymbol{u}$ such that $\dot{H}(\boldsymbol{x}, \boldsymbol{u}) \geq -\alpha_H(H)$. We will show that with $\dot{\boldsymbol{k}} = \kappa(\boldsymbol{x}, \boldsymbol{k})$ given by (15) it holds that $L_G H \neq \mathbf{0}_{1\times M}$ and thus H is a CBF for $\mathcal{C}(\boldsymbol{k}(t))$, for all $t \geq 0$.

Since $L_G H = \boldsymbol{p}^\top\boldsymbol{L}_G$, the problem of showing that $L_G H \neq \mathbf{0}_{1\times M}$ is equivalent to proving that $\boldsymbol{p} \notin \mathcal{N}(\boldsymbol{L}_G^\top) = \text{span}\{\boldsymbol{n}_1, \dots, \boldsymbol{n}_r\}$. Since the vector $\boldsymbol{p}$ can be expressed as a sum of vectors perpendicular to and parallel to $\mathcal{N}(\boldsymbol{L}_G^\top)$ (respectively $\boldsymbol{p}^\perp$ and $\boldsymbol{p}^\parallel$), it follows that $\boldsymbol{p} \notin \mathcal{N}(\boldsymbol{L}_G^\top)$ as long as $\|\boldsymbol{p}^\perp\| > 0$, where $\boldsymbol{p}^\perp = \left(\boldsymbol{I} - \boldsymbol{N}\boldsymbol{N}^\top\right)\boldsymbol{p}$ by vector projection, and $\boldsymbol{N}$ is given by (18). Thus, a sufficient condition for $\boldsymbol{p} \notin \mathcal{N}(\boldsymbol{L}_G^\top)$ is that

$$\frac{1}{2}\|(\boldsymbol{I} - \boldsymbol{N}\boldsymbol{N}^\top)\boldsymbol{p}\|^2 = \frac{1}{2}\boldsymbol{p}^\top\boldsymbol{Q}\boldsymbol{p} > \varepsilon \tag{19}$$

for some $\varepsilon > 0$, where $\boldsymbol{Q}$ is given by (17). Then, by defining a function $h_p = \frac{1}{2}\boldsymbol{p}^\top\boldsymbol{Q}\boldsymbol{p} - \varepsilon$, it follows from (5) that when (19) is true at $t = 0$, it is true $\forall t \geq 0$ as long as (15c) holds.

Therefore, gains $\boldsymbol{k}$ adapted according to the law (15) are guaranteed to result in $L_G H \neq \mathbf{0}_{1\times M}$. Thus, H is a CBF for the set $\mathcal{C}(\boldsymbol{k}(t))$, for all $t \geq 0$. This completes the proof. $\square$

Remark 1. *With $\boldsymbol{Q}$ depending on basis vectors spanning $\mathcal{N}(\boldsymbol{L}_G^\top)$, it is not immediately obvious under what conditions $\dot{\boldsymbol{Q}}$ is continuous (or even well-defined). Prior results show that if the rank of $\mathcal{N}(\boldsymbol{L}_G^\top)$ is constant then $\dot{\boldsymbol{Q}}$ varies continuously within an epsilon ball, i.e., $\forall \boldsymbol{x}' \in B_\epsilon(\boldsymbol{x})$ [33], but analytical derivations of $\dot{\boldsymbol{Q}}$ are not available to the best of our knowledge. In practice, we observe that the rank of $\mathcal{N}(\boldsymbol{L}_G^\top)$ is indeed constant, and we approximate $\dot{\boldsymbol{Q}}$ numerically using finite-difference methods.*

Remark 2. *It is worth further noting that the optimization problem (15) is a quadratic program, and thus may be solved very efficiently online using open-source libraries.*

With H consolidating many constraints into one CBF condition, the centralized controller (9) may be replaced by

$$\boldsymbol{u}^*_{\mathcal{A}_{com}} = \underset{\boldsymbol{u}_{\mathcal{A}_{com}}\in\mathcal{U}_{\mathcal{A}_{com}}}{\arg\min} \ \frac{1}{2}\|\boldsymbol{u}_{\mathcal{A}_{com}} - \boldsymbol{u}^0_{\mathcal{A}_{com}}\|^2 \tag{20a}$$

$$\text{s.t.}$$

$$a + \boldsymbol{b}\boldsymbol{u}_{\mathcal{A}_{com}} \geq 0, \tag{20b}$$

where $a = L_F H + \alpha_H(H)$ and $\boldsymbol{b} = L_G H_{[i \in \mathcal{A}_{com}]}$. If all agents are communicative, i.e., $\mathcal{A}_{com} = \mathcal{A}$, then since H is a CBF for the set $\mathcal{C}(\boldsymbol{k}(t)) \subset \mathcal{S}$, $\forall t \geq 0$, the system trajectories are guaranteed to stay within $\mathcal{C}(\boldsymbol{k}(t)) \subset \mathcal{S}$ and thus remain safe. In the presence of non-communicative agents, we replace the decentralized controller (8) with

$$\boldsymbol{u}_i^* = \underset{\boldsymbol{u}_i \in \mathcal{U}_i}{\arg\min} \frac{1}{2}\|\boldsymbol{u}_i - \boldsymbol{u}_i^0\|^2 \tag{21a}$$

s.t.

$$a + \boldsymbol{b}_i \boldsymbol{u}_i \geq d, \tag{21b}$$

where $d = e^{-rH} \max_{\boldsymbol{u} \in \mathcal{U}} \sum_{j=1, j \neq i}^{A} L_G H_{[jm:j(m+1)]} u_j$ with $r > 0$, and $\boldsymbol{b}_i = L_G H_{[mi:m(i+1)]}$. While for unbounded control authority d is unbounded, in practice it is reasonable to assume that agents have actuation limits and thus to use (21) assuming some bounded $\mathcal{U}$. In addition, the conservatism introduced by this robustness term d may be decreased by increasing the gain r appearing in the exponential.

IV. Multi-Robot Numerical Study

In this section, we demonstrate our C-CBF controller on a decentralized multi-robot goal-reaching problem.

Consider a collection of 3 non-communicative, but responsive robots ($i \in \mathcal{A}_{nc} \setminus \mathcal{A}_{ncnr}$) in a warehouse environment seeking to traverse a narrow corridor intersected by a passageway occupied with 6 non-responsive agents ($j \in \mathcal{A}_{ncnr}$). The non-responsive agents may be, e.g., humans walking or some other dynamic obstacles. Let $\mathcal{F}$ be an inertial frame with a point s_0 denoting its origin, and assume that each robot may be modeled according to the following kinematic bicycle model described by [34, Ch. 2]:

$$\dot{x}_i = v_i \left(\cos\psi_i - \sin\psi_i \tan\beta_i\right) \tag{22a}$$

$$\dot{y}_i = v_i \left(\sin\psi_i + \cos\psi_i \tan\beta_i\right) \tag{22b}$$

$$\dot{\psi}_i = \frac{v_i}{l_r}\tan\beta_i \tag{22c}$$

$$\dot{\beta}_i = \omega_i \tag{22d}$$

$$\dot{v}_i = a_i, \tag{22e}$$

where x_i and y_i denote the position (in m) of the center of gravity (c.g.) of the ith robot with respect to s_0, ψ_i is the orientation (in rad) of its body-fixed frame, $\mathcal{B}_i$, with respect to $\mathcal{F}$, β_i is the slip angle[1] (in rad) of the c.g. of the vehicle relative to $\mathcal{B}_i$ (assume $|\beta_i| < \frac{\pi}{2}$), and v_i is the velocity of the rear wheel with respect to $\mathcal{F}$. The state of robot i is denoted $\boldsymbol{z}_i = [x_i \; y_i \; \psi_i \; \beta_i \; v_i]^\top$, and its control input is $\boldsymbol{u}_i = [a_i \; \omega_i]^\top$, where a_i is the acceleration of the rear wheel (in m/s^2), and ω_i is the angular velocity (in rad/s) of β_i.

The challenges of this scenario relate to preserving safety despite multiple non-communicative and non-responsive agents present in a constrained environment. A robot is safe if it 1) obeys the speed restriction, 2) remains inside the corridor area, and 3) avoids collisions with all other robots. Speed is addressed with the following candidate CBF:

$$h_v(\boldsymbol{z}_i) = s_M - v_i, \tag{23}$$

where $s_M > 0$, while for corridor safety and collision avoidance we used forms of the relaxed future-focused CBF introduced in [9] for roadway intersections, namely

$$\begin{aligned} h_c(\boldsymbol{z}_i) = (m_L(x_i + \dot{x}_i) + b_L - (y_i + \dot{y}_i)) \cdot \\ (m_R(x_i + \dot{x}_i) + b_R - (y_i + \dot{y}_i)), \end{aligned} \tag{24}$$

$$\begin{aligned} h_r(\boldsymbol{z}_i, \boldsymbol{z}_j) = D(\boldsymbol{z}_i, \boldsymbol{z}_j, t + \hat{\tau})^2 \\ + \epsilon D(\boldsymbol{z}_i, \boldsymbol{z}_j, t)^2 - (1+\epsilon)(2R)^2, \end{aligned} \tag{25}$$

where (24) prevents collisions with the corridor walls (defined as lines in the xy-plane via $m_L, b_L, m_R, b_R \in \mathbb{R}$), and (25) prevents inter-robot collisions and is defined $\forall i \in \mathcal{A}_{nc} \setminus \mathcal{A}_{ncnr}$, $\forall j \in \mathcal{A}_{nc}$, where $\epsilon > 0$, $D(\boldsymbol{z}_i, \boldsymbol{z}_j, t_a)$ is the Euclidean distance between agents i and j at arbitrary time t_a, and $\hat{\tau}$ denotes the time in the interval $[0, T]$ at which the minimum inter-agent distance will occur under constant velocity future trajectories. For a more detailed discussion on future-focused CBFs, see [9]. As such, (23), (24), and (25) define the sets

$$\mathcal{S}_{v,i} = \{\boldsymbol{z}_i \in \mathbb{R}^n \mid h_v(\boldsymbol{z}_i) \geq 0\},$$
$$\mathcal{S}_{c,i} = \{\boldsymbol{z}_i \in \mathbb{R}^n \mid h_c(\boldsymbol{z}_i) \geq 0\},$$
$$\mathcal{S}_{r,i} = \bigcap_{j=1, j\neq i}^{A} \{\boldsymbol{z} \in \mathbb{R}^N \mid h_r(\boldsymbol{z}_i, \boldsymbol{z}_j) \geq 0\},$$

the intersection of which constitutes the safe set for agents i, i.e., $\mathcal{S}_i(t) = \mathcal{S}_{v,i} \cap \mathcal{S}_{c,i} \cap \mathcal{S}_{r,i}$.

We control robots $i \in \mathcal{A}_{nc} \setminus \mathcal{A}_{ncnr}$ using a C-CBF based decentralized controller of the form (21) with constituent functions h_c, h_s, h_r, an LQR based nominal control input (see [9, Appendix 1]), and initial gains $\boldsymbol{k}(0) = \mathbf{1}_{10\times 1}$. The non-responsive agents used a similar LQR controller to move through the passageway in pairs of two, with the first two pairs passing through the intersection without stopping and the last pair stopping at the intersection before proceeding. This may model robots stopping to complete a task or, for example, pairs of humans walking together and stopping to converse, ignoring the robots all the while.

As shown in Figure 2, the non-communicative robots traverse both the narrow corridor and the busy intersection to reach their goal locations safely. The trajectories of the gains $\boldsymbol{k}$ for each warehouse robot are shown in Figure 3, while their control inputs are depicted in Figure 4. It is worth noting that though there is some chattering in the control input, we hypothesize that this is due more to the decentralized control law than to our adaptation law for the following reason: if, at a given time instance two robots deem it safe to accelerate toward each other, at the next time instance they may each need to decelerate to preserve safety, and repeat. Further, as shown in Figure 3 the weights $\boldsymbol{k}$ are not oscillating in unison with the control inputs. The CBF time histories for the constituent and consolidated functions are highlighted in Figures 5 and 6 respectively, and show that the C-CBF controllers maintained safety at all times.

[1] β_i is related to the steering angle δ_i via $\tan\beta_i = \frac{l_r}{l_r + l_f}\tan\delta_i$, where $l_f + l_r$ is the wheelbase with l_f (resp. l_r) the distance from the c.g. to the center of the front (resp. rear) wheel.

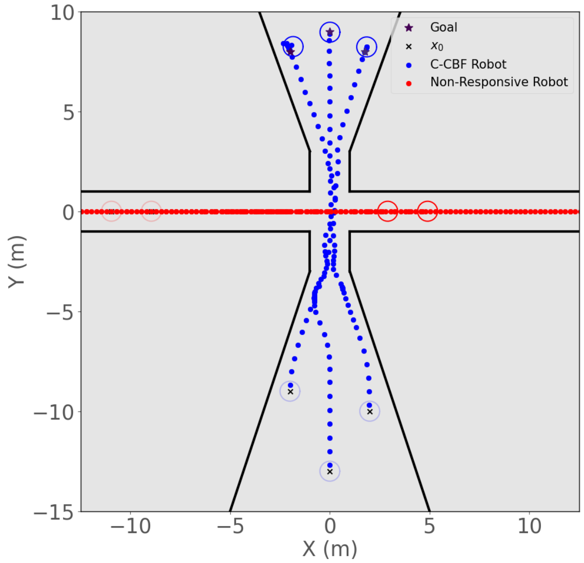

Fig. 2: XY paths for the warehouse robots (blue) and non-responsive agents (red) in the warehouse control problem.

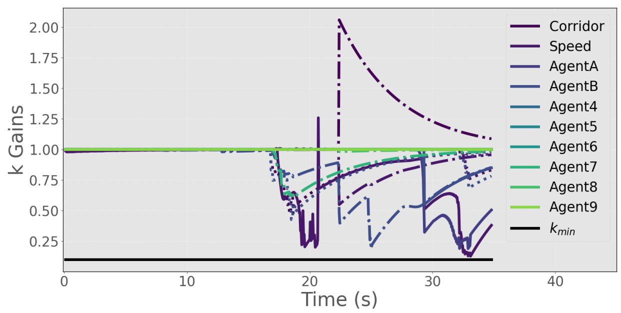

Fig. 3: Gains $\boldsymbol{k}$ for the C-CBF controllers in the warehouse study. Robot 1 denoted with solid lines, dotted for robot 2, dash-dots for robot 3. AgentA and AgentB denote the other two non-communicative robots from the perspective of one (e.g., for robot 2 AgentA=Agent1 and AgentB=Agent3).

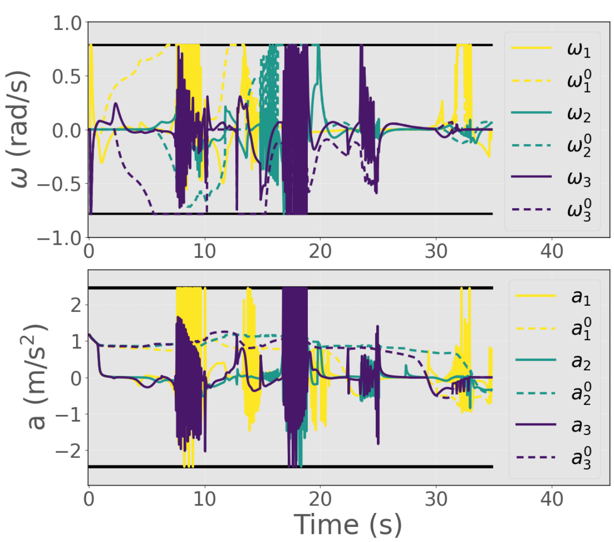

Fig. 4: Warehouse robot controls: accel. (a) and slip angle rate (ω).

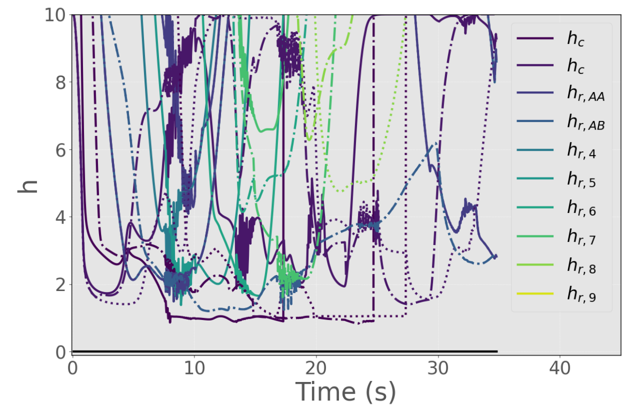

Fig. 5: Evolution of warehouse robot constituent CBF candidates, $h_s \; \forall s \in [1..c]$, synthesized to construct C-CBF.

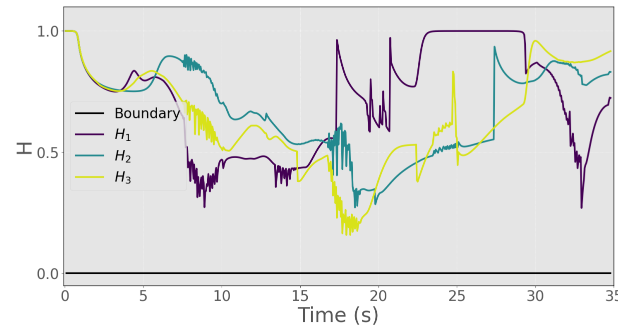

Fig. 6: Evolution of C-CBF H for warehouse robots 1, 2, and 3.

V. Experimental Case Study

For experimental validation of our approach, we used an AION R1 UGV ground rover as an ego vehicle in the laboratory setting and required it to reach a goal location in the presence of two non-responsive rovers: one static and one dynamic. We modeled the rovers as bicycles using (22), and sent angular rate ω_i and velocity v_i (numerically integrated based on the controller's acceleration output) commands to the rovers' on-board PID controllers. The ego rover used our proposed C-CBF (21) with constituent candidate CBFs (23) (with $s_M = 1$ m/s) and the rff-CBF defined in (25) for collision avoidance. The nominal input to the C-CBF controller was the LQR law from the warehouse robot example, as was the controller used by the dynamic non-responsive rover. A Vicon motion capture system was used for position feedback, and the state estimation was performed by extended Kalman filter via the on-board PX4.

For the setup, the static rover was placed directly between the ego rover and its goal, while the dynamic rover was stationary until suddenly moving across the ego's path as it approached its target. As highlighted in Figure 7, the ego rover first headed away from the static rover and then decelerated and swerved to avoid a collision with the second rover before correcting course and reaching its goal. Videos and code for both this experiment and the simulation in Section IV are available on Github[2].

[2]Link to Github repo: github.com/6lackmitchell/CCBF-Control

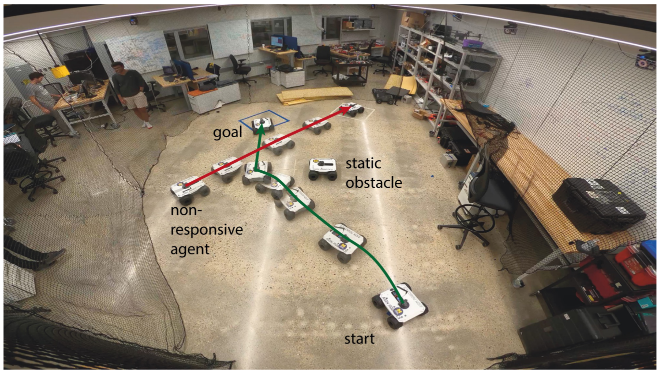

Fig. 7: A rover avoids a static and dynamic rover using our proposed C-CBF controller en route to a target in the laboratory setting.

VI. Conclusion

In this paper, we addressed the problem of safe control under multiple state constraints via a C-CBF based control design. To ensure that the synthesized C-CBF is valid, we introduced a parameter adaptation law on the weights of the C-CBF constituent functions and proved that the resulting controller is safe. We then demonstrated the success of our approach on a multi-robot control problem in a crowded warehouse environment, and further validated our work on a ground rover experiment in the lab.

In the future, we plan to explore conditions under which the C-CBF approach may preserve guarantees in the presence of input constraints, including whether alternative adaptation laws for the weights assist in guarantees of liveness in addition to safety.

References

[1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Trans. on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[2] W. Xiao and C. Belta, "Control barrier functions for systems with high relative degree," in *IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 474–479.

[3] W. S. Cortez, D. Oetomo, C. Manzie, and P. Choong, "Control barrier functions for mechanical systems: Theory and application to robotic grasping," in *IEEE Trans. on Control Systems Tech.*, 2019, pp. 1–16.

[4] P. Wieland and F. Allgöwer, "Constructive safety using control barrier functions," *IFAC Proceedings Vols.*, vol. 40, no. 12, pp. 462–467, 2007.

[5] Y. Chen, A. Singletary, and A. D. Ames, "Guaranteed obstacle avoidance for multi-robot operations with limited actuation: A control barrier function approach," *IEEE Control Systems Letters*, vol. 5, no. 1, pp. 127–132, 2021.

[6] M. Jankovic and M. Santillo, "Collision avoidance and liveness of multi-agent systems with cbf-based controllers," in *2021 60th IEEE Conference on Decision and Control (CDC)*, 2021, pp. 6822–6828.

[7] B. Xu and K. Sreenath, "Safe teleoperation of dynamic uavs through control barrier functions," in *2018 IEEE International Conference on Robotics and Automation (ICRA)*, 2018, pp. 7848–7855.

[8] M. Khan, M. Zafar, and A. Chatterjee, "Barrier functions in cascaded controller: Safe quadrotor control," in *2020 American Control Conference (ACC)*, 2020, pp. 1737–1742.

[9] M. Black, M. Jankovic, A. Sharma, and D. Panagou, "Future-focused control barrier functions for autonomous vehicle control," in *2023 American Control Conference (ACC)*. IEEE, 2023, pp. 3324–3331.

[10] S. Yaghoubi, G. Fainekos, T. Yamaguchi, D. Prokhorov, and B. Hoxha, "Risk-bounded control with kalman filtering and stochastic barrier functions," in *2021 60th IEEE Conference on Decision and Control (CDC)*, 2021, pp. 5213–5219.

[11] J. Breeden and D. Panagou, "High relative degree control barrier functions under input constraints," in *2021 60th IEEE Conference on Decision and Control (CDC)*, 2021, pp. 6119–6124.

[12] W. Xiao, C. Belta, and C. G. Cassandras, "Adaptive control barrier functions," *IEEE Transactions on Automatic Control*, vol. 67, no. 5, pp. 2267–2281, 2022.

[13] W. S. Cortez, X. Tan, and D. V. Dimarogonas, "A robust, multiple control barrier function framework for input constrained systems," *IEEE Control Systems Letters*, vol. 6, pp. 1742–1747, 2022.

[14] P. Glotfelter, J. Cortes, and M. Egerstedt, "Nonsmooth barrier functions with applications to multi-robot systems," *IEEE Control Systems Letters*, vol. 1, pp. 310–315, 10 2017.

[15] Y. Huang and Y. Chen, "Switched Control Barrier Function With Applications to Vehicle Safety Control," ser. Dynamic Systems and Control Conference, vol. 1, 10 2020.

[16] L. Lindemann and D. V. Dimarogonas, "Control barrier functions for signal temporal logic tasks," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 96–101, 2019.

[17] M. Machida and M. Ichien, "Consensus-based control barrier function for swarm," in *2021 IEEE International Conference on Robotics and Automation (ICRA)*, 2021, pp. 8623–8628.

[18] A. Clark, "Verification and synthesis of control barrier functions," in *2021 60th IEEE Conference on Decision and Control (CDC)*. IEEE, 2021, pp. 6105–6112.

[19] A. Robey, H. Hu, L. Lindemann, H. Zhang, D. V. Dimarogonas, S. Tu, and N. Matni, "Learning control barrier functions from expert demonstrations," in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 3717–3724.

[20] M. Srinivasan, A. Dabholkar, S. Coogan, and P. A. Vela, "Synthesis of control barrier functions using a supervised machine learning approach," in *2020 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2020, pp. 7139–7145.

[21] S. Tonkens and S. Herbert, "Refining control barrier functions through hamilton-jacobi reachability," in *2022 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2022, pp. 13 355–13 362.

[22] J. Breeden and D. Panagou, "Compositions of multiple control barrier functions under input constraints," *arXiv preprint arXiv:2210.01354*, 2022.

[23] M. Black, E. Arabi, and D. Panagou, "Fixed-time parameter adaptation for safe control synthesis," *arXiv preprint arXiv:2204.10453*, 2022.

[24] B. T. Lopez, J.-J. E. Slotine, and J. P. How, "Robust adaptive control barrier functions: An adaptive and data-driven approach to safety," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 1031–1036, 2021.

[25] A. J. Taylor and A. D. Ames, "Adaptive safety with control barrier functions," in *2020 American Control Conference (ACC)*, 2020, pp. 1399–1405.

[26] T. Gurriet, A. Singletary, J. Reher, L. Ciarletta, E. Feron, and A. Ames, "Towards a framework for realizable safety critical control through active set invariance," in *9th ACM/IEEE International Conference on Cyber-Physical Systems (ICCPS)*, 2018, pp. 98–106.

[27] S. Kolathaya and A. D. Ames, "Input-to-state safety with control barrier functions," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 108–113, 2019.

[28] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[29] M. Jankovic, M. Santillo, and Y. Wang, "Multi-agent systems with cbf-based controllers – collision avoidance and liveness from instability," *arXiv preprint arXiv:2207.04915*, 2022.

[30] U. Borrmann, L. Wang, A. D. Ames, and M. Egerstedt, "Control barrier certificates for safe swarm behavior," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 68–73, 2015.

[31] L. Wang, A. D. Ames, and M. Egerstedt, "Safety barrier certificates for collisions-free multirobot systems," vol. 33, no. 3. IEEE, 2017, pp. 661–674.

[32] H. Parwana, A. Mustafa, and D. Panagou, "Trust-based rate-tunable control barrier functions for non-cooperative multi-agent systems," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 2222–2229.

[33] T. F. Coleman and D. C. Sorensen, "A note on the computation of an orthonormal basis for the null space of a matrix," *Mathematical Programming*, vol. 29, pp. 234–242, 6 1984.

[34] R. Rajamani, *Vehicle Dynamics and Control*. Springer US, 2012.