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The Optimal Performance Loss of the Finite-Horizon Discrete-Time Linear-Quadratic Controller Driven by a Reduced-Order Observer

Verica Radisavljevic-Gajic *Member, IEEE*

***Abstract*— In this paper we formulate and solve a problem encountered in engineering practice when a discrete-time linear-quadratic optimal feedback controller uses the state estimates obtained via a discrete-time reduced-order observer. Due to the use of state estimates instead of the actual state variables, the optimal quadratic performance is degraded in a pretty complex manner. In the paper, we derive the exact formula for the optimal performance degradation for the finite time horizon optimization problem in terms of solution of a reduced-order difference Lyapunov equation. An aircraft example demonstrates that the optimal performance loss can be significant when a reduced-order observer is used. The optimal performance degradation can be considerably reduced by using the least square method to set-up the reduced-order observer initial condition.**

Index Terms— Reduced-order discrete-time observer, discrete-time finite horizon optimal observer-based controller, reduced-order observer initial condition

I. INTRODUCTION

THE full- and reduced-order linear observers were introduced in the work of Luenberger, [1]. The design of observers is very well documented in engineering. There are numerous applications of full- and reduced-order observers, especially to solve practical problems in industry. For example, paper [2] developed a discrete-time extended observer for an unmanned helicopter with the purpose of constructing a PID-like adaptive controller. Paper [3] applied a reduced-order observer for a vehicle active suspension system. The paper uses also the linear-quadratic (LQ) optimization for the controller design. The use of observers in Internet of Things was considered in [4], and an observer for distributed parameter systems in [5]. An overview on the use of observers for linear, nonlinear, and distributed parameter energy systems has been presented in [6]. Paper [7] presented a reduced-order observer for detection of false data injection attacks in cyber physical systems for smart power grids.

Consider discrete-time invariant linear system defined by

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \tag{1}$$

where $x(k) \in \Re^n$ are the state-space variables and $u(k) \in \Re^m$ is the control input. Matrices $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ are constant (time invariant system). When all state variables are available for feedback, a feedback control input is given by

$$u(x(k)) = -Fx(k) \tag{2}$$

where F is a feedback gain matrix. The system (1) with perfect full-state feedback control (2) has the closed-loop eigenvalues $\lambda(A-BF)$. The output signal $y(k) \in \Re^l$ is defined by

$$y(k) = Cx(k) \tag{3}$$

In practice, $\dim\{y(k)\} = l < n = \dim\{x(k)\}$. In the case of non-redundant measurements, we have that $l = c = \text{rank}\{C\}$. If the pair (A, C) is observable [1] then an observer can be designed to estimate all state variables at all times and provide $x(t) \approx \hat{x}(t)$, where $\hat{x}(t)$ is the signal produced by the observer. In such a case, the actual control signal applied to the system is

$$u(\hat{x}(k)) = -F\hat{x}(k) = -Fx(k) + Fe(k) \tag{4}$$

The full-order discrete-time observer has the form [8]

$$\hat{x}(k+1) = (A-KC)\hat{x}(k) + Bu(k) + Ky(k) \tag{5}$$

The signal $e(k)$ is observation error defined by

$$e(k) = x(k) - \hat{x}(k) \tag{6}$$

It is important to indicate that the system-observer configuration has the closed-loop eigenvalues separated into the *system closed-loop eigenvalues* under perfect full-state feedback defined by $\lambda(A-BF)$, and the *observer closed-loop eigenvalues* defined by $\lambda(A-KC)$. This property is known as the *separation principle*.

From $y(k) = Cx(k)$, there are c equations for n unknows of $x(k)$. An observer of order $r = n - c$ can be used to estimate the remaining r states. Our goal in this paper is to consider the impact of feedback control defined in (4) with the estimate $\hat{x}(k)$ obtained from a reduced-order observer to be presented in the next section. The corresponding optimal control problem in the continuous-time domain using a reduced-order observer was considered in [9]-[10].

Verica Radisavljevic-Gajic is with Department of Mechanical Engineering, Ajman University UAE, (e-mail: v.gajic@ajman.ac.ae)

II. PRELIMINARY RESULTS

In this section we first summarize the procedure for design of reduced-order observers, and then derive some preliminary results needed for the rest of the paper.

In the first step design, a constant matrix $C_1 \in \mathcal{R}^{r\times n}$, whose rank is equal to $r = n - l$, is selected such that

$$\operatorname{rank}\begin{bmatrix} C \\ C_1 \end{bmatrix} = n \tag{7}$$

Let $p(k) \in \mathcal{R}^r$ be defined by

$$p(k) = C_1 x(k) \tag{8}$$

From (3) and (8), it follows

$$\begin{bmatrix} y(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} C \\ C_1 \end{bmatrix} x(k) \tag{9}$$

$$x(k) = \begin{bmatrix} C \\ C_1 \end{bmatrix}^{-1} \begin{bmatrix} y(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} L & L_1 \end{bmatrix} \begin{bmatrix} y(k) \\ p(k) \end{bmatrix} = Ly(k) + L_1 p(k) \tag{10}$$

with $L \in \mathcal{R}^{n\times l}$, $L_1 \in \mathcal{R}^{n\times r}$. From (10), an estimate for $x(k)$ is

$$\hat{x}(k) = Ly(k) + L_1 \hat{p}(k) \tag{11}$$

where the unknown vector $\hat{p}(k) \in \mathcal{R}^r$ has to be estimated. The useful relations can be derived from formulas in (10)

$$\begin{gathered}\begin{bmatrix} C \\ C_1 \end{bmatrix}\begin{bmatrix} C \\ C_1 \end{bmatrix}^{-1} = I_n = \begin{bmatrix} C \\ C_1 \end{bmatrix}\begin{bmatrix} L & L_1 \end{bmatrix} = \begin{bmatrix} CL & CL_1 \\ C_1 L & C_1 L_1 \end{bmatrix} = \begin{bmatrix} I_c & 0 \\ 0 & I_r \end{bmatrix} \\ \Rightarrow \quad CL = I_c, \qquad C_1 L_1 = I_r, \qquad CL_1 = 0, \qquad C_1 L = 0 \end{gathered}\tag{12}$$

Using (3), (10), and (12), it follows that the signal $y(k) = Cx(k) = CLy(k) + CL_1 p(k) = y(k) + 0p(k)$ does not contain information about $p(k)$. It can be shown that the forwarded value of the output signal $y(k+1)$ contains information about $p(k)$. With the change of variables as

$$\hat{q}(k) = \hat{p}(k) - K_1 y(k) \tag{13}$$

the future value $y(k+1)$ of the system measurements will not be needed. The gain K_1 will be determined to make reduced-order discrete-time observer asymptotically stable. It can be shown, after some algebra, that the reduced-order discrete-time observer for $\hat{q}(k)$ is given by

$$\hat{q}(k+1) = A_q \hat{q}(k) + B_q u(k) + K_q y(k) \tag{14}$$

$$\begin{aligned} A_q &= \left(C_1 - K_1 C\right) A L_1 = \left(C_1 A L_1\right) - K_1 (CAL_1) \\ B_q &= (C_1 - K_1 C)B, \quad K_q = \left(C_1 - K_1 C\right) A (L + L_1 K_1) \end{aligned}\tag{15}$$

Having generated the values for $\hat{q}(k)$ from (14)-(15) for all discrete-time instants, the required discrete-time estimate for $p(k)$ is obtained from (13) as $\hat{p}(k) = \hat{q}(k) + K_1 y(k)$, and then from (11), the estimate for $x(k)$ follows as

$$\begin{aligned} \hat{x}(k) &= Ly(k) + L_1 \hat{p}(k) = Ly(k) + L_1(\hat{q}(k) + K_1 y(k)) \\ &= \left(L + L_1 K_1\right) y(k) + L_1 \hat{q}(k) \end{aligned}\tag{16}$$

The selected reduced-order observer gain K_1 must be stabilizing and set the closed-loop eigenvalues in the desired locations such that the reduced-order observer is considerably faster than the system. This can be achieved using the eigenvalue assignment technique [10]. To achieve this goal the pair $\left(C_1 A L_1, CAL_1\right)$ in matrix A_q in formula (15) must be observable. This result was shown in [9]-[10].

To simplify the derivations and provide a better understanding of what follows, the problem under is mapped into new coordinates via the following change of variables as

$$Mx(k) = z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \quad M = \begin{bmatrix} C \\ C_1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} L & L_1 \end{bmatrix} \tag{17}$$

Using the results from (12), it follows

$$\begin{aligned} &y(k) = Cx(k) = CM^{-1}Mx(k) = CM^{-1}z(k) \\ &= [CL \quad CL_1]z(k) = [I_c \quad 0]\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = \bar{C}z(k) = z_1(k) \end{aligned}\tag{18}$$

Since in the new coordinates the vector $z_1(k) \in \Re^c$ is directly measured, its observer is simply $\hat{z}_1(k) = y(k)$. An observer is now needed only for the vector $z_2(k) \in \Re^r$. Note that $c + r = n$. The difference equation for the vector $z(k)$ is

$$\begin{aligned} z(k+1) &= \bar{A}\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \bar{B}u(k) = MAM^{-1}\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + MBu(k) \\ &= \begin{bmatrix} CAL & CAL_1 \\ C_1 AL & C_1 AL_1 \end{bmatrix}\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} CB \\ C_1 B \end{bmatrix} u(k) \\ &= \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \\ z(0) &= Mx(0) = z_0 \\ y(k) &= z_1(k), \qquad \bar{C} = \begin{bmatrix} I & 0 \end{bmatrix} \end{aligned}\tag{19}$$

It can be seen from (19) that $y(k)$ carries no information about $z_2(k)$. An observer for the vector $z_2(k)$ can be designed as

$$\begin{aligned} &\hat{z}_2(k+1) = A_{21}y(k) + A_{22}\hat{z}_2(k) + B_2 u(k) + K_1\left(y(k+1) - \hat{y}(k+1)\right) \\ &\hat{y}(k+1) = z_1(k+1) = A_{12}\hat{z}_2(k) + A_{11}y(k) + B_1 u(k) \end{aligned}\tag{20}$$

The need to $y(k+1)$ in (20) can be eliminated by introducing a change of variables similar to in (13), that is

$$\hat{z}_2(k) = \hat{q}_2(k) - K_1 y(k) \tag{21}$$

which leads to the reduced-order observer

$$\hat{q}_2(k+1) = A_z \hat{q}_2(k) + B_z u(k) + K_z y(k) \tag{22}$$

$$\begin{aligned} A_z &= C_1 AL_1 - K_1 CAL_1 = A_{22} - K_1 A_{12} \\ B_z &= C_1 B - K_1 CB = B_2 - K_1 B_1 \\ K_z &= C_1 AL + C_1 AL_1 K_1 - K_1 CAL - K_1 CAL_1 K_1 \\ &= A_{21} + A_{22}K_1 - K_1 A_{11} - K_1 A_{12} K_1 \end{aligned}\tag{23}$$

Having obtained $\hat{q}_2(k)$, the estimate for $z(k)$ is

$$\hat{z}_2(k) = \hat{q}_2(k) + K_1 y(k) \quad \Rightarrow$$
$$\hat{z}(k) = \begin{bmatrix} \hat{z}_1(k) \\ \hat{z}_2(k) \end{bmatrix} = \begin{bmatrix} y(k) \\ \hat{q}_2(k) + K_1 y(k) \end{bmatrix} \tag{24}$$

Let us define the observation errors by

$$e_2(k) = z_2(k) - \hat{z}_2(k),\ e(k) = z(k) - \hat{z}(k) = \begin{bmatrix} 0 \\ e_2(k) \end{bmatrix} \tag{25}$$

The reduced-order observer estimation error can be obtained from $e_2(k+1) = x_2(k+1) - \hat{x}_2(k+1)$. Using (19)-(22)

$$e_2(k+1) = (A_{22} - K_1 A_{12}) e_2(k) \tag{26}$$

Difference equation (26) indicates that $e_2(k)$ goes to zero if $A_{22} - K_1 A_{12}$ is asymptotically stable. This can be achieved if eigenvalues of $A_{22} - K_1 A_{12}$ are placed in the desired locations inside of the unit circle (asymptotic stability region) using the eigenvalue assignment technique [11], which requires that the pair (A_{22}^T, A_{12}^T) is controllable. Since the state transformation (17) preserves system observability, [11], that is, observability of (A, C) is equivalent to observability of $(\bar{A}, \bar{C})$. It was shown that observability of (A, C) implies observability of (A_{22}, A_{12}), and controllability of the pair (A_{22}^T, A_{12}^T).

III. REDUCED-ORDER OBSERVER DRIVEN FINITE-TIME LQ OPTIMAL CONTROLLER PERFORMANCE

In this section, we use a discrete-time reduced-order observer based optimal controller to minimize a quadratic performance criterion over a finite time horizon, and study its performance. The quadratic performance criterion is defined by

$$J = \min_u \left\{ \frac{1}{2} \sum_{k=k_0}^{k_f - 1} \left[x^T(k) Q x(k) + u^T(k) R u(k) \right] \right\} + \frac{1}{2} x^T(k_f) P_{k_f} x(k_f) \tag{27}$$
$$Q = Q^T \geq 0, \quad P_{k_f} \geq 0, \quad R = R^T > 0$$

k_0 and k_f represent initial and final discrete-time instants, and Q, P_{k_f}, and R are weighted matrices. When no observer is used, that is, $u(x(k)) = -Fx(k)$, the optimal controller formulas are well-known and given by [8]

$$\begin{aligned} u^{opt}(x(k)) &= -\left(R + B^T P(k+1) B\right)^{-1} B^T P(k+1) A x^{opt}(k) \\ &= -F^{opt}(k+1) x^{opt}(k) \\ P(k) &= A^T P(k+1) A + Q \\ &-A^T P(k+1) B \left(R + B^T P(k+1) B\right)^{-1} B^T P(k+1) A,\ P(k_f) = P_{k_f} \\ x^{opt}(k+1) &= \left(A - BF^{opt}(k+1)\right) x^{opt}(k),\ \ x^{opt}(k_0) = x(k_0) = x_0 \\ J^{opt} &= \frac{1}{2} x^T(k_0) P(k_0) x(k_0) \end{aligned} \tag{28}$$

where the difference equation for $P(k)$ is the Riccati difference equation [8]. Since the *separation principle holds for the system-observer configuration*, formula (28) for the optimal feedback gain holds in the case when an observer-based feedback controller is used. Mapping (28) into the new coordinates does not change the value for the optimal performance criterion (28). The transformation (17), that is, $\bar{A} = MAM - 1$, $\bar{B} = MB$, changes the differential Riccati equation (28) into

$$\begin{aligned} P(k) &= M^T \bar{A}^T M^{-T} P(k+1) M^{-1} \bar{A} M + Q \\ &- M^T \bar{A}^T M^{-T} P(k+1) M^{-1} \bar{B} \left(R + \bar{B}^T M^{-T} P(k+1) M^{-1} \bar{B}\right)^{-1} \\ &\times \bar{B}^T M^{-T} P(k+1) M^{-1} \bar{A} M, \quad P(k_f) = P_{k_f} \end{aligned} \tag{29}$$

Multiplying the last equation from the left by M^{-T} and from the right by M^{-1} produces

$$\begin{aligned} M^{-T} P(k) M^{-1} &= \bar{A}^T M^{-T} P(k+1) M^{-1} \bar{A} + M^{-T} Q M^{-1} \\ &- \bar{A}^T M^{-T} P(k+1) M^{-1} \bar{B} \left(R + \bar{B}^T M^{-T} P(k+1) M^{-1} \bar{B}\right)^{-1} \\ &\times \bar{B}^T M^{-T} P(k+1) M^{-1} \bar{A}, \quad P(k_f) = P_{k_f} \end{aligned}$$

or

$$\begin{aligned} \bar{P}(k) &= \bar{A}^T \bar{P}(k+1) \bar{A} + \bar{Q} - \bar{A}^T \bar{P}(k+1) \bar{B} \left(R + \bar{B}^T \bar{P}(k+1) \bar{B}\right)^{-1} \\ &\times \bar{B}^T \bar{P}(k+1) \bar{A}, \quad \bar{P}(k_f) = M^{-T} P_{k_f} M^{-1} \\ \bar{Q} &= M^{-T} Q M^{-1} \end{aligned} \tag{30}$$

with

$$\bar{P}(k) = M^{-T} P(k) M^{-1} \quad \Rightarrow \quad P(k) = M^T \bar{P}(k) M \tag{31}$$

The discrete-time optimal control in the new coordinates under the full-state feedback is given by

$$\begin{aligned} u^{opt}(z(k)) &= -\left(R + \bar{B}^T \bar{P}(k+1) \bar{B}\right)^{-1} \bar{B}^T \bar{P}(k+1) \bar{A} z^{opt}(k) \\ &= -\bar{F}^{opt}(k+1) z^{opt}(k) \\ z^{opt}(k+1) &= \left(\bar{A} - \bar{B}\bar{F}^{opt}(k+1)\right) z^{opt}(k) \\ z^{opt}(k_0) &= M x^{opt}(k_0) = M x_0 = z_0 \end{aligned} \tag{32}$$

The optimal performance criterion in the new coordinates is

$$\begin{aligned} \bar{J}^{opt} &= \frac{1}{2} \sum_{k=k_0}^{k_f - 1} z^{optT}(k) \left(\bar{Q} + \bar{F}(k+1)^{opt^T} R \bar{F}^{opt}(k+1) \right) z^{opt}(k) \\ &+ \frac{1}{2} z^T(k_f) M^{-T} P_{k_f} M^{-1} z(k_f) \\ &= \frac{1}{2} z^T(k_0) \bar{P}(k_0) z(k_0) = \frac{1}{2} x^T(k_0) M^T M^{-T} P(k_0) M^{-1} M x(k_0) \\ &= \frac{1}{2} x^T(k_0) P(k_0) x(k_0) = J^{opt} \end{aligned} \tag{33}$$

The observer driven controller formula (4), using the reduced-order observer is modified into

$$
\begin{aligned}
u(z(k)) &= -\bar{F}(k+1)\hat{z}(k) \\
&= -\begin{bmatrix} \bar{F}_1(k+1) & \bar{F}_2(k+1) \end{bmatrix}\begin{bmatrix} z_1(k) \\ z_2(k)-e_2(k) \end{bmatrix} \\
&= -\begin{bmatrix} \bar{F}_1(k+1) & \bar{F}_2(k+1) \end{bmatrix}\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \bar{F}_2(k+1)e_2(k) \\
&= -\bar{F}(k+1)z(k) + \bar{F}_2(k+1)e_2(k)
\end{aligned} \tag{34}
$$

When an observer driven controller is used, the performance is

$$
\hat{J} = \frac{1}{2}\sum_{k=k_0}^{k_f-1}\left[z^T(k)\bar{Q}z(k) + \hat{z}^T(k)\bar{F}(k+1)^T R\bar{F}(k+1)\hat{z}(k)\right] + \frac{1}{2}z^T(k_f)M^{-T}P_{k_f}M^{-1}z(k_f) \tag{35}
$$

Using (34) in (35), the performance criterion becomes

$$
\begin{aligned}
\hat{J} = \frac{1}{2}\sum_{k=k_0}^{k_f-1} & [z^T(k)(\bar{Q}+\bar{F}(k+1)^T R\bar{F}(k+1))z(k) \\
& -2z^T(k)\bar{F}(k+1)^T R\bar{F}_2(k+1)e_2(k) \\
& +e_2^T(k)\bar{F}_2(k+1)^T R\bar{F}_2(k+1)e_2(k)] \\
& +\frac{1}{2}z^T(k_f)M^{-T}P_{k_f}M^{-1}z(k_f)
\end{aligned} \tag{36}
$$

Using notation

$$
\mathbf{R}(k+1) = \begin{bmatrix} \bar{Q}+\bar{F}(k+1)^T R\bar{F}(k+1) & -\bar{F}(k+1)^T R\bar{F}_2(k+1) \\ -\bar{F}_2(k+1)^T R\bar{F}(k+1) & \bar{F}_2(k+1)^T R\bar{F}_2(k+1) \end{bmatrix}
$$

we have

$$
\begin{aligned}
\hat{J}_{red} &= \frac{1}{2}\sum_{k=k_0}^{k_f-1}\begin{bmatrix} z(k) \\ e_2(k) \end{bmatrix}^T \mathbf{R}(k+1)\begin{bmatrix} z(k) \\ e_2(k) \end{bmatrix} \\
&+\frac{1}{2}\begin{bmatrix} z(k_f) \\ e_2(k_f) \end{bmatrix}^T\begin{bmatrix} M^{-T}P_{k_f}M^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z(k_f) \\ e_2(k_f) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} z(k_0) \\ e_2(k_0) \end{bmatrix}^T \sum_{k=k_0}^{k_f-1}\Phi^T(k,k_0)\mathbf{R}(k+1)\Phi(k,k_0)\begin{bmatrix} z(k_0) \\ e_2(k_0) \end{bmatrix} \\
&\quad +\frac{1}{2}\begin{bmatrix} z(k_f) \\ e_2(k_f) \end{bmatrix}^T\begin{bmatrix} M^{-T}P_{k_f}M^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z(k_f) \\ e_2(k_f) \end{bmatrix}
\end{aligned} \tag{37}
$$

$\Phi(k,k_0)$ is the transition matrix of the augmented discrete-time system. The solution for the augmented vector $\begin{bmatrix} z(k) \\ e_2(k) \end{bmatrix}$ is

$$
\begin{aligned}
\begin{bmatrix} z(k+1) \\ e_2(k+1) \end{bmatrix} &= \begin{bmatrix} \bar{A}-\bar{B}\bar{F}(k+1) & \bar{B}\bar{F}_2(k+1) \\ 0 & A_{22}-K_1A_{12} \end{bmatrix}\begin{bmatrix} z(k) \\ e_2(k) \end{bmatrix} \\
&= \mathbf{A}(k+1)\begin{bmatrix} z(k) \\ e_2(k) \end{bmatrix}
\end{aligned} \tag{38}
$$

$$
\bar{A} = MAM^{-1}, \quad \bar{B} = MB
$$

$$
\Phi(k+1,k_0) = \mathbf{A}(k+1)\mathbf{A}(k)\cdots\mathbf{A}(1)\mathbf{A}(k_0)
$$

After some algebra, it can be shown that the infinite sum (38) can be evaluated along trajectories of the augmented discrete-time linear system (34) via the solution of the following difference Lyapunov equation [10]

$$
\begin{aligned}
\mathbf{P}(k) &= \mathbf{A}(k+1)^T\mathbf{P}(k+1)\mathbf{A}(k+1) + \mathbf{R}(k+1) \\
\mathbf{P}(k_f) &= \begin{bmatrix} M^{-T}P_{k_f}M^{-1} & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{39}
$$

The solution matrix $\mathbf{P}(k)$ is appropriately partitioned

$$
\mathbf{P}(k) = \begin{bmatrix} P_{11}(k) & P_{12}(k) \\ P_{12}^T(k) & P_{22}(k) \end{bmatrix} \tag{40}
$$

Performing the corresponding block matrix multiplications in (39), using the expressions for $\mathbf{A}(k)$ and $\mathbf{P}(k)$, a system of three algebraic equations is obtained

$$
\begin{aligned}
P_{11}(k) &= (\bar{A}-\bar{B}\bar{F}(k+1))^T P_{11}(k+1)(\bar{A}-\bar{B}\bar{F}(k+1)) \\
&\quad -P_{11}+\bar{Q}+\bar{F}^T(k+1)R\bar{F}(k+1) \\
P_{11}(k_f) &= M^{-T}P_{k_f}M^{-1}
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
P_{12}(k) &= (\bar{A}-\bar{B}\bar{F}(k+1))^T \\
&\quad \times\left[P_{11}(k+1)\bar{B}\bar{F}_2(k+1) + P_{12}(k+1)(A_{22}-K_1A_{12})\right] \\
&\quad -\bar{F}^T(k+1)R\bar{F}_2(k+1), \qquad P_{12}(k_f) = 0
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
P_{22}(k) &= (A_{22}-K_1A_{12})^T P_{22}(k+1)(A_{22}-K_1A_{12}) \\
&+\bar{F}_2^T(k+1)\bar{B}^T P_{11}(k+1)\bar{B}\bar{F}_2(k+1) + \bar{F}_2^T(k+1)R\bar{F}_2(k+1) \\
&+(A_{22}-K_1A_{12})^T P_{12}^T(k+1)\bar{B}\bar{F}_2(k+1) \\
&+\bar{F}_2^T(k+1)\bar{B}^T P_{12}(k+1)(A_{22}-K_1A_{12}), \qquad P_{22}(k_f) = 0
\end{aligned} \tag{43}
$$

Equation (41) is the difference Riccati equation equal to the original difference Riccati equation (28), which can be established when the optimal feedback gain given by $\bar{F}(k+1) = (R+\bar{B}^T P_{11}(k+1)\bar{B})^{-1}\bar{B}^T P_{11}(k+1)\bar{A}$ is plugged into (41) producing

$$
\begin{aligned}
P_{11}(k) &= \bar{A}^T P_{11}(k+1)\bar{A} + \bar{Q} \\
&-\bar{A}^T P_{11}(k+1)\bar{B}\left(R+\bar{B}^T P_{11}(k+1)\bar{B}\right)^{-1}\bar{B}^T P_{11}(k+1)\bar{A} \\
P_{11}(k_f) &= M^{-T}P_{k_f}M^{-1}
\end{aligned} \tag{44}
$$

With the optimal gain equation (42) becomes

$$
P_{12}(k) = (\bar{A}-\bar{B}\bar{F}^{opt}(k+1))^T P_{12}(k+1)(A_{22}-K_1A_{12}), P_{12}(k_f) = 0 \tag{45}
$$

Note that the remaining terms in (42) cancel out. Since matrices $A_{22}-K_1A_{12}$ and $\bar{A}-\bar{B}\bar{F}^{opt}(k+1)$ are asymptotically stable feedback matrices, the homogeneous difference equation (45), with the terminal condition $P_{12}(k_f) = 0$, has the unique solution $P_{12}(k) = 0$ for all k. Using $P_{12}(k) = 0$ in equation (43) produces the difference Lyapunov equation

$$
\begin{aligned}
P_{22}(k) &= (A_{22}-K_1A_{12})^T P_{22}(k+1)(A_{22}-K_1A_{12}) \\
&+\bar{F}_2^T(k)\left(R+\bar{B}^T P_{11}(k+1)\bar{B}\right)\bar{F}_2(k), \qquad P_{22}(k_f) = 0
\end{aligned} \tag{46}
$$

With the knowledge of $P_{11}(k)$ and $P_{22}(k)$ obtained from (41) and (46), the expression for the performance criterion under the discrete-time reduced-order observer-driven optimal controller is obtained as follows

$$\begin{aligned}\hat{J}^{opt} &= \frac{1}{2}\begin{bmatrix} z(k_0) \\ e_2(k_0)\end{bmatrix}^T \mathbf{P}(k_0)\begin{bmatrix} z(k_0) \\ e_2(k_0)\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} z(k_0) \\ e_2(k_0)\end{bmatrix}^T \begin{bmatrix} P_{11}(k_0) & 0 \\ 0 & P_{22}(k_0)\end{bmatrix}\begin{bmatrix} z(k_0) \\ e_2(k_0)\end{bmatrix} \\ &= \frac{1}{2}z^T(k_0)P_{11}(k_0)z(k_0)+\frac{1}{2}e_2^T(k_0)P_{22}(k_0)e_2(k_0) \\ &= \frac{1}{2}x^T(k_0)M^T\bar{P}(k_0)Mx(k_0)+\frac{1}{2}e_2^T(k_0)P_{22}(k_0)e_2(k_0) \\ &= \frac{1}{2}x^T(k_0)P(k_0)x(k_0)+\frac{1}{2}e_2^T(k_0)P_{22}(k_0)e_2(k_0) \\ &= J^{opt}+\frac{1}{2}e_2^T(k_0)P_{22}(k_0)e_2(k_0) = J^{opt}+\hat{J}_{red}^{loss}\end{aligned} \quad (47)$$

From (47), the optimal performance degradation when the reduced-order observer is used (compared to the case when a full-state optimal feedback controller is implemented) is given by

$$\hat{J}_{red}^{loss} = \frac{1}{2}e_2^T(k_0)P_{22}(k_0)e_2(k_0) \quad (48)$$

with $P_{22}(k_0)$ obtained iteratively from (46).

The performance degradation can be expressed in the original coordinates using formulas (19)-(23), which leads to

$$\begin{aligned} z(k_0) &= \begin{bmatrix} z_1(k_0) \\ z_2(k_0)\end{bmatrix} = Mx(k_0) = \begin{bmatrix} C \\ C_1\end{bmatrix}x(k_0) \\ &= \begin{bmatrix} Cx(k_0) \\ C_1x(k_0)\end{bmatrix} \;\Rightarrow\; z_2(k_0) = C_1x(k_0) \end{aligned} \quad (49)$$

$$\begin{aligned} e_2(k_0) &= z_2(k_0)-\hat{z}_2(k_0) = z_2(k_0)-\left(\hat{q}_2(k_0)+K_1y(k_0)\right) \\ &= \left(C_1-K_1C\right)x(k_0)-\hat{q}_2(k_0)\end{aligned}$$

and

$$\begin{aligned}\hat{J}_{red}^{loss} &= \frac{1}{2}\left((C_1-K_1C)x(k_0)-\hat{q}_2(k_0)\right)^T P_{22}(k_0) \\ &\quad\times\left((C_1-K_1C)x(k_0)-\hat{q}_2(k_0)\right)\end{aligned} \quad (50)$$

The system initial condition $x(0)$ and the reduced-order observer initial condition $\hat{q}_2(0)$ determine the reduced-order observer error at the initial time, that is, $e_2(0) = (C_1-K_1C)x(0)-\hat{q}_2(0)$, where $x(0)$ is in general unknown, and $\hat{q}_2(0)$ is chosen by the control designer.

When the state initial condition $x(0)$ is exactly known, the reduced-order observer initial condition, should be selected using formula (49) as

$$\hat{q}_2(0) = \left(C_1-K_1C\right)x(0) \;\Rightarrow\; e_2(0)=0,\; e_2(k)=0, \forall k\geq 0 \quad (51)$$

This selection makes the performance loss identical to zero, so that $\hat{J}_{new}^{opt} = J^{opt}$. The system initial condition $x(0)$ is in general unknown or even not measurable at all, and only the output signal at the initial time is available. Hence, the following relationship exists $y(0) = Cx(0)$. This equality gives $c < n$ equations for n unknowns of the vector $x(0)$, so that the least-square method [12] can be used to determine a rational choice for the reduced-order observer initial condition, as suggested in [13], see also [14]. The least square method starts with

$$\begin{aligned} Cx(k) = y(k) \quad &\Rightarrow \quad x(t) = (C^TC)^{\#}C^Ty(k) \\ &\Rightarrow \quad x(0) = (C^TC)^{\#}C^Ty(0) \end{aligned} \quad (52)$$

where # stands for the generalized inverse [12]. Equation (52) produces the following expression for the initial condition of the discrete-time reduced-order observer

$$\hat{q}_{2LS}(0) = (C_1-K_1C)(C^TC)^{\#}C^Ty(0) \quad (53)$$

In Section IV, we will show via simulation that formula (53) produces much more superior results for the real physical system, an aircraft.

IV. AN AIRCRAFT OPTIMAL LQ CONTROLLER DRIVEN BY A REDUCED-ORDER OBSERVER

A state space model of the linearized lateral dynamics of a F-16 aircraft pitch rate control system can be found in [15]. The model is discretized using MATLAB and its zero-order hold with the sampling period equal to $T_s = 0.05$. The weighted matrices are selected as

$$Q = I_5 \qquad R = 1, \qquad P_{k_f} = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$$

Matrix C_1 is chosen as

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0\end{bmatrix}$$

such as the augmented matrix $\begin{bmatrix} C \\ C_1\end{bmatrix}$ is nonsingular. The system initial condition is set to $x(0) = [1 \;\; 0 \;\; -1 \;\; 1 \;\; -1]$. The reduced-order observer eigenvalues were set at $\lambda_{obs}^{red} = [-0.1 \;\; 0.1]$. We have experimented with several choices of the reduced-order observer initial conditions since they can be chosen arbitrary by the control engineer. Using a simple choice $\hat{q}_2(0) = [1 \;\; 1]^T$, we have obtained large optimal performance losses, see Table I. The steady state value was at $k_f = 154$ is $J^{opt} = \frac{1}{2}x^T(154)P(154)x(154) = 270.6073$. The initial condition for $\hat{q}_2(0)$ obtained using the least square formula (53) is $\hat{q}_2(0) = [-2.7432 \;\; -5.0025]^T$. This initial condition resulted in much better values for the optimal performance loss, see Table II.

It can be seen from Tables I and II that the use of the least square formula (53) for the reduced-order observer initial conditions considerably reduces the optimal performance degradation.

TABLE I

THE OPTIMAL PERFORMANCE AND ITS LOSS WHEN ALL INITIAL CONDITION ARE $\hat{q}_2(0) = [1 \quad 1]^T$. THE SECOND COLUMN IS $J^{opt} = \frac{1}{2}x^T(k_f)P(k_f)x(k_f)$, THE THIRD COLUMN REPRESENTS THE VALUES OF $\hat{J}_{red}^{loss} = \frac{1}{2}e_2^T(k_f)P_{22}(k_f)e_2(k_f)$, AND THE FOURTH COLUMN IS $\hat{J}_{los}^{loss}/J^{opt}$[%].

Discrete time	2	3	4
$k_f = 5$	13.0236	0.4036	3.1
$k_f = 10$	51.9208	5.9967	11.6
$k_f = 12$	79.2234	19.4848	24.6
$k_f = 14$	109.7235	40.7463	37.1
$k_f = 17$	153.2849	76.9315	50.2
$k_f = 20$	187.9256	105.1011	55.9
$k_f = 30$	245.0481	128.8692	52.6
$k_f = 50$	267.6078	119.6078	44.7
$k_f = 100$	270.5915	119.0581	44.0
$k_f = 150$	270.6072	119.0560	44.0

TABLE II

THE OPTIMAL PERFORMANCE AND ITS LOSS WHEN $\hat{q}_2(0)$ IS OBTAINED FROM THE LEAST SQUARE FORMULA (57). THE SECOND COLUMN IS $J^{opt} = \frac{1}{2}x^T(k_f)P(k_f)x(k_f)$, THE THIRD COLUMN REPRESENTS THE VALUES OF $\hat{J}_{red}^{loss} = \frac{1}{2}e_2^T(k_f)P_{22}(k_f)e_2(k_f)$, AND THE FORTH COLUMN IS $\hat{J}_{los}^{loss}/J^{opt}$[%]

Discrete time	2	3	4
$k_f = 5$	13.0236	0.1209	0.9
$k_f = 10$	51.9208	2.6277	5.1
$k_f = 12$	79.2234	5.8355	7.4
$k_f = 14$	109.7235	10.2306	9.3
$k_f = 17$	153.2849	17.0681	11.1
$k_f = 20$	187.9256	22.0802	11.7
$k_f = 30$	245.0481	26.1124	10.7
$k_f = 50$	267.6078	24.6883	9.2
$k_f = 100$	270.5915	24.5398	9.1
$k_f = 150$	270.6072	24.5395	9.1

V. CONCLUSIONS

The performance loss formula was derived for the case when the discrete-time finite-horizon linear-quadratic optimal controller uses a discrete-time reduced-order observer. Different choices of the reduced-order observer initial conditions were considered. The presented least square formula for the choice of the observer initial conditions produced the best results by providing the smallest optimal performance loss. A challenging future research problem will be to design reduced-order observers and corresponding LQ optimal controllers using the two-stage design technique developed in [16]-[19] that can lead to simplification of the presented methodology and a better understanding of the results obtained.

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