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From Discrete to Continuous Imitation Dynamics

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Abstract— **It has been previously shown that a finite well-mixed population of individuals imitating the highest earners in a binary game can undergo perpetual fluctuations. However, it remains unknown whether the fluctuations in the population proportions of the two strategies persist as population size grows. In this paper, we answer this question for an imitative population with diagonal anticoordination matrices. We show that the collection of Markov chains corresponding to the population dynamics is a family of generalized stochastic approximation process for a good upper semicontinuous differential inclusion. We additionally show that the differential inclusion always converges to an equilibrium. This convergence, based on the available results in the stochastic approximation theory, implies that the lengths of the fluctuations in the population proportions of the two strategies in a finite population of imitators with diagonal anticoordination payoff matrices vanish with probability one as population size grows. Furthermore, taking the same steps for a population of imitators with diagonal coordination payoff matrices results in a similar conclusion, which is consistent with the previously reported results for finite populations of imitators with coordination payoff matrices.**

I. Introduction

On a daily basis, individuals are involved in different decision-making problems, such as whether to share news, sign a petition, or buy a new product. In a variety of contexts, it has been either assumed or reported that individuals are mainly either *best-responders* or *imitators* [1]–[5]. Best-responders go for a decision maximizing their instant benefit while imitators follow the decision made by the highest earners. Whether individuals of either type reach a satisfactory decision or undergo repetitive switching between available options is important [6], [7].

As for finite populations of decision-makers, a finite population of coordinators, those who benefit more from coordinating with majority of others, reaches equilibrium [8]. A finite population of anticoordinators, those who profit by anticoordinating with majority of others will either equilibriate or fluctuate between to adjacent states [9], [10]. A heterogeneous population of coordinating and anticoordinating, however, may equilibrate or undergo perpetual fluctuations [11]. The long-term behaviour of a population of individuals imitating the highest earners with arbitrary payoff matrices was investigated in [12], [13]. It was shown that the population may reach a satisfactory state, that is, equilibrates, or may fluctuate.

Although studying the long-term behaviour of finite populations of decision-makers gives detailed information on the population state, it has been found to be challenging. Approximating the discrete population dynamics with their associated *mean dynamics* is a way to simplify their analysis [14]. The mean dynamics of the best-response update rule are of differential inclusion form [15]–[19]. The associated mean dynamics of the imitation update rules with continuous switching probability are Lipschitz continuous and in some cases reduce to *replicator dynamics* [14], [20]–[23].

How well does the analysis of the steady-state behaviour of the mean dynamics reveal that of the discrete population dynamics? In this regard, a large body of studies have been dedicated to connecting the asymptotic behaviour of finite populations with that of the associated mean dynamics when the size of the population grows [24]–[28].

Recently we used these existing results to link the asymptotic behaviour of a finite heterogeneous population of coordinators and anticoordinator with the associated mean dynamics, a good upper semicontinuous (GUS) differential inclusion [29]. Our analysis suggested that the reported perpetual fluctuations in the population proportions of the two strategies almost surely vanish with population size. However, it remains an open problem whether the reported fluctuations in a population of individuals imitating the highest earners persist as population size grows. Particularly, imitators may be more likely to fluctuate because it was shown that populations of imitating individuals are less likely to converge to an equilibrium [30].

To partly address this, we consider a finite well-mixed population of individuals playing a repetitive binary game asynchronously. All individuals have either diagonal coordination payoff matrices or anti-diagonal anticoordination payoff matrices, but not both within the same population. The individuals imitate the highest earners in the population. First we analyze an exclusive population of individuals with anti-diagonal anticoordination matrices and show that the series of the Markov chains corresponding to the discrete dynamics are *generalized stochastic approximation processes* (GSAPs) for a good upper semicontinuous differential inclusion. We then obtain the Birkhoff center of the dynamical systems induced by the differential inclusion which, based on the work of Roth and Sandholm [28], contains the support of the limit points of the invariant probability measures of the GSAPs. Our analysis show that the lengths of the reported fluctuations in the population proportions of the two strategies [13] converges to zero with probability one. We then analyze an exclusive population of individuals with diagonal coordination matrices. Following similar steps, we come to a similar conclusion, which is consistent with [13].

Our contribution is twofold: First, we obtain the associ-

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ated mean dynamics of discrete population dynamics where individuals imitate the highest-earners and have either anti-diagonal anticoordination or diagonal coordination payoff matrices, but not both in the same population. The mean dynamics are in the form of differential inclusions, unlike many imitation games where the associated mean dynamics are described by differential equations. Second, we show that the reported perpetual fluctuations in the population proportions of the two strategies in an exclusive population with anti-diagonal anticoordination payoff matrices converge to zero with probability one–**Theorem 1** and **Corollary 1**. We obtain a similar result for an exclusive population with diagonal coordination payoff matrices–**Theorem 2**.

The main focus of the paper is on a population with anti-diagonal anticoordination payoff matrices. In Section VI, we, in brief, take the similar steps for a population with diagonal coordination payoff matrices.

Notations

In this paper, we use the following notations. Scalars are denoted by non-boldface letters. The calligraphic font $\mathcal{X}$ is used to denote a set. By the notation $\langle x_k \rangle$, we mean a sequence of variables $x_0, x_1, x_2, \ldots$. The floor function is denoted by $\lfloor x \rfloor$. The notation $|x|$ refers to the norm-1 if x is a vector and refers to the set cardinality if x is a set. The notation $[a, b] - c$ implies $[a - c, b - c]$. We use the notation $2^{\mathcal{X}}$ to denote the set of all subsets of the set $\mathcal{X}$. The notation $[k]$ for a positive integer k means $\{1, 2, \ldots, k\}$. The support of a random variable X is denoted by $\mathcal{R}_X$. The function $\mathtt{1}(\cdot)$ equals one for a positive argument and zero otherwise. A vector with all elements equal to 1 (resp. 0) with an appropriate dimension is denoted by $\mathbf{1}$ (resp. $\mathbf{0}$). The notation $\frac{1}{n}\mathbb{Z}^m$ denotes the set of m-dimensional vectors whose components, when multiplied by n, are integers.

II. Problem Formulation

We consider a population of N agents, labeled by $1, 2, \ldots, \mathsf{N}$. The agents repeatedly play a two-strategy game and earn an accumulated payoff. A 2×2 payoff matrix summarizes the four possible payoff gains of agent i against another agent:

$$\pi_i = \begin{array}{c} \\ \mathtt{A} \\ \mathtt{B} \end{array} \begin{array}{c} \begin{array}{cc} \mathtt{A} & \mathtt{B} \end{array} \\ \begin{pmatrix} w_i & q_i \\ y_i & z_i \end{pmatrix} \end{array}, \tag{1}$$

where w_i, q_i, y_i, and z_i correspond to strategy pair $\mathtt{A}$–against–$\mathtt{A}$, $\mathtt{A}$–against–$\mathtt{B}$, $\mathtt{B}$–against–$\mathtt{A}$, and $\mathtt{B}$–against–$\mathtt{B}$, respectively, where for example $\mathtt{A}$–against–$\mathtt{B}$ means player i plays strategy $\mathtt{A}$ and her opponent plays strategy $\mathtt{B}$. Over a discrete time sequence $t \in \frac{1}{\mathsf{N}}\mathbb{Z}_{\geq 0}$, which is indexed by k where $k = \mathsf{N}t$, agents become activate randomly and choose either strategy $\mathtt{A}$ or $\mathtt{B}$ and accordingly receive an accumulated payoff. We assume a well-mixed population where each agent plays against all other agents including herself. At time index k when the population proportion of strategy-$\mathtt{A}$ players equals x^{N}, the accumulated payoff or *utility* of agent i playing strategy $s_i(k) \in \{\mathtt{A}, \mathtt{B}\}$ is

$$\Pi_i(k, x^{\mathsf{N}}) = \begin{cases} (w_i - q_i)\mathsf{N}x^{\mathsf{N}} + q_i\mathsf{N} & \text{if } s_i(k) = \mathtt{A}, \\ (y_i - z_i)\mathsf{N}x^{\mathsf{N}} + z_i\mathsf{N} & \text{if } s_i(k) = \mathtt{B}. \end{cases} \tag{2}$$

At each time index k only one agent becomes active and gets to revise her strategy according to *imitation* update rule, that is, she switches to the strategy which is played by the highest earners. More specifically, at time index $k + 1$, the strategy of agent i active at time index k will be

$$s_i(k+1) = \begin{cases} s_i(k) & \text{if } s_i(k) \in \mathcal{M}(k), \\ \neg s_i(k) & \text{otherwise} \end{cases} \tag{3}$$

where $\neg\mathtt{A}$ (resp. $\neg\mathtt{B}$) is $\mathtt{B}$ (resp. $\mathtt{A}$) and

$$\mathcal{M}(k) = \{s_j(k) | j \in \arg \max_{i \in \{1,2,\ldots,\mathsf{N}\}} \Pi_i(k, x^{\mathsf{N}})\} \tag{4}$$

is the set of strategies which are played by the highest earners at time index k. The set of strategies of the highest earners $\mathcal{M}(k)$ equals $\{\mathtt{A}\}$ (resp. $\{\mathtt{B}\}$) if the maximum utility of $\mathtt{A}$-players at time index k (resp. $\mathtt{B}$-players) in the population exceeds that of $\mathtt{B}$-players ($\mathtt{A}$-players). If both strategies $\mathtt{A}$ and $\mathtt{B}$ are played by the highest earners at time index k, then $\mathcal{M}(k)$ equals $\{\mathtt{A}, \mathtt{B}\}$. The imitation update rule (3) dictates the same preferred strategies for all agents.

It is assumed that the payoff matrices of the population are of anti-diagonal *anticoordination* type [13], i.e., for all $i \in [\mathsf{N}]$

$$w_i < q_i \text{ and } z_i < y_i \text{ with } w_i = z_i = 0. \tag{5}$$

Agents who have the same payoff matrix build up a subpopulation, and there are altogether p non-empty subpopulations. The distribution of the population proportions over the total p subpopulations is shown by $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_{\mathsf{p}})$ where $\rho_p \geq \frac{1}{\mathsf{N}}$ equals the population proportion of type p, i.e., the number of agents in subpopulation p divided by the population size N. At each index k, the *population state*, $\boldsymbol{x}^{\mathsf{N}}(k)$, is defined as the distribution of the $\mathtt{A}$-players over the p subpopulations, i.e., $\boldsymbol{x}^{\mathsf{N}} = (x_1^{\mathsf{N}}, \ldots, x_{\mathsf{p}}^{\mathsf{N}})$, where x_p^{N} represents the proportion of $\mathtt{A}$-players in subpopulation p, i.e., the number of $\mathtt{A}$-players divided by the total population size N. The state space then equals $\boldsymbol{\mathcal{X}}_{ss} \cap \frac{1}{\mathsf{N}}\mathbb{Z}^{\mathsf{p}}$ where $\boldsymbol{\mathcal{X}}_{ss} = \prod_{j=1}^{\mathsf{p}}[0, \rho_j]$.

In view of (4), when no agents in subpopulation p play strategy $\mathtt{A}$ (resp. $\mathtt{B}$), i.e., $x_p^{\mathsf{N}} = 0$ (resp. $x_p^{\mathsf{N}} = \rho_p$), the utility of subpopulation p for playing strategy $\mathtt{A}$ (resp. $\mathtt{B}$), $-q_p\mathsf{N}x^{\mathsf{N}} + q_p\mathsf{N}$ (resp. $y_p\mathsf{N}x^{\mathsf{N}}$), is not accounted in determining the highest earner. Thus, the set of strategies of the highest earners given the population state $\boldsymbol{x}^{\mathsf{N}}$ equals $\{\mathtt{A}\}$ if

$$\max_p(-q_p\mathsf{N}x^{\mathsf{N}} + q_p\mathsf{N})\mathtt{1}(x_p^{\mathsf{N}}) > \max_p y_p\mathsf{N}x^{\mathsf{N}}\mathtt{1}(\rho_p - x_p^{\mathsf{N}}), \tag{6}$$

equals $\{\mathtt{B}\}$ if

$$\max_p(-q_p\mathsf{N}x^{\mathsf{N}} + q_p\mathsf{N})\mathtt{1}(x_p^{\mathsf{N}}) < \max_p y_p\mathsf{N}x^{\mathsf{N}}\mathtt{1}(\rho_p - x_p^{\mathsf{N}}), \tag{7}$$

and equals $\{\mathtt{A}, \mathtt{B}\}$ otherwise. Rearranging (6) (resp. (7)) yields an equivalent condition for $\mathtt{A}$ (resp. $\mathtt{B}$) to be the preferred strategy of the population at the state $\boldsymbol{x}^{\mathsf{N}}$, that is, $x^{\mathsf{N}} < \alpha(\boldsymbol{x}^{\mathsf{N}})$ (resp. $x^{\mathsf{N}} > \alpha(\boldsymbol{x}^{\mathsf{N}})$) where

$$\alpha(\boldsymbol{x}^{\mathsf{N}}) = \frac{\max_p q_p \mathbb{1}(x_p^{\mathsf{N}})}{\max_p q_p \mathbb{1}(x_p^{\mathsf{N}}) + \max_p y_p \mathbb{1}(\rho_p - x_p^{\mathsf{N}})}. \quad (8)$$

The *population dynamics* are defined by the evolution of the population state $\boldsymbol{x}^{\mathsf{N}}$ over time governed by update rule (3) and the activation sequence of the agents. The activation sequence is assumed to be generated by a sequence of random variables $\langle A_k \rangle$ where A_k is the active agent at time index k with support $\mathcal{R}_{A_k} = [\mathsf{N}]$ and the distribution $\mathbb{P}[A_k = i] = \frac{1}{\mathsf{N}}$. The discrete imitation anticoordination population dynamics are then defined as follows:

Definition 1: The following discrete time stochastic equation

$$\boldsymbol{x}^{\mathsf{N}}(k+1) = \boldsymbol{x}^{\mathsf{N}}(k) + \frac{1}{\mathsf{N}}\big(S_k - u(\boldsymbol{x}^{\mathsf{N}})\big)\mathbb{1}\big(u(\boldsymbol{x}^{\mathsf{N}})\big)\mathbf{e}_{P_k} \quad (9)$$

defines the ***discrete anticoordination imitation population dynamics*** where P_k and S_k are random variables with supports $\mathcal{R}_{P_k} = [\mathsf{p}]$ and $\mathcal{R}_{S_k} = \{1, 2\}$, and distributions $\mathbb{P}[P_k = p] = \rho_p$ and $\mathbb{P}[S_k = 1 | P_k = p] = x^{\mathsf{N}}/\rho_p$, respectively, and the function $u(\boldsymbol{x}^{\mathsf{N}})$ returns 1 (resp. 2) if A (resp. B) is the only preferred strategy of the population at state $\boldsymbol{x}^{\mathsf{N}}$ and 0 otherwise:

$$u(\boldsymbol{x}^{\mathsf{N}}) = \begin{cases} 1 & \text{if } x^{\mathsf{N}} < \alpha(\boldsymbol{x}^{\mathsf{N}}), \\ 0 & \text{if } x^{\mathsf{N}} = \alpha(\boldsymbol{x}^{\mathsf{N}}), \\ 2 & \text{if } x^{\mathsf{N}} > \alpha(\boldsymbol{x}^{\mathsf{N}}), \end{cases} \quad (10)$$

and the vector $\mathbf{e}_{P_k}$ is the P_kth column of the identity matrix of size p.

In Definition 1, the random variable P_k is the label of the active agent's subpopulation at time index k, and the random variable S_k equals 1 (resp. 2) if the strategy of the active agent at time index k equals A (resp. B).

It has been shown that a finite population of imitators with anti-diagonal anticoordination payoff matrices may undergo perpetual fluctuations [12], i.e., $\exists \boldsymbol{\mathcal{Y}}^{\mathsf{N}} \subseteq \boldsymbol{\mathcal{X}}_{ss} \cap \frac{1}{\mathsf{N}}\mathbb{Z}^{\mathsf{p}}$ s.t. $\forall \boldsymbol{y}^{\mathsf{N}} \in \boldsymbol{\mathcal{Y}}^{\mathsf{N}}, \forall k > 0 \exists T \geq k$ s.t. $\boldsymbol{x}^{\mathsf{N}}(k+T) = \boldsymbol{y}^{\mathsf{N}}$. But whether these perpetual fluctuations persist as the population size approaches infinity?

III. Link to the Semicontinuous Dynamics

We are interested in the asymptotic behaviour of the discrete imitation population dynamics as the population size N approaches infinity. If we show that the dynamics define a Markov chain, and the collection of these Markov chains indexed by the population size is a GSAP for a GUS differential inclusion, then, in view of [28, Theorem 3.5 and Corollary 3.9], repeated as Theorem A 1, the Birkhoff center associated with the semicontinuous differential inclusion contains the support of the limit of the invariant probability measures of the GSAPs. Hence, the perpetual fluctuations in the population proportion of A-players vanish with probability one if the Birkhoff center consists of isolated points.

In view of the discrete imitation population dynamics (9), the population state at index $k+1$ is completely determined by the population state and the active agent at time index k. The sequence $\langle \boldsymbol{x}^{\mathsf{N}}(k) \rangle_k$ hence defines a Markov chain.

Definition 2: The ***anticoordination imitation population dynamics Markov chain*** is defined as the Markov chain $\langle \mathbf{X}_k^{\frac{1}{\mathsf{N}}} \rangle_k$ with transition probabilities

$$\Pr_{\boldsymbol{x}^{\mathsf{N}}, \boldsymbol{y}^{\mathsf{N}}} = \quad (11)$$
$$\begin{cases} (\rho_p - x_p^{\mathsf{N}})(2 - u(\boldsymbol{x}^{\mathsf{N}}))\mathbb{1}\big(u(\boldsymbol{x}^{\mathsf{N}})\big) & \text{if } \boldsymbol{y}^{\mathsf{N}} = \frac{1}{\mathsf{N}}\mathbf{e}_p + \boldsymbol{x}^{\mathsf{N}}, \\ x_p^{\mathsf{N}}(u(\boldsymbol{x}^{\mathsf{N}}) - 1)\mathbb{1}\big(u(\boldsymbol{x}^{\mathsf{N}})\big) & \text{if } \boldsymbol{y}^{\mathsf{N}} = -\frac{1}{\mathsf{N}}\mathbf{e}_p + \boldsymbol{x}^{\mathsf{N}}, \\ 1 - \Big(\sum_{p=1}^{\mathsf{p}}(\rho_p - x_p^{\mathsf{N}})(2 - u(\boldsymbol{x}^{\mathsf{N}})) & \text{if } \boldsymbol{y}^{\mathsf{N}} = \boldsymbol{x}^{\mathsf{N}}, \\ \quad + x_p^{\mathsf{N}}(u(\boldsymbol{x}^{\mathsf{N}}) - 1)\mathbb{1}\Big)\mathbb{1}\big(u(\boldsymbol{x}^{\mathsf{N}})\big) & \\ 0 & \text{otherwise}, \end{cases}$$

the state space $\boldsymbol{\mathcal{X}}_{ss} \cap \frac{1}{\mathsf{N}}\mathbb{Z}^{\mathsf{p}}$, and the initial state $\mathbf{X}_0^{\frac{1}{\mathsf{N}}} = \boldsymbol{x}^{\mathsf{N}}(0)$, where and the vector $\mathbf{e}_p$ is the p^{th} column of the identity matrix of size p.

Proposition 1: The sequence $\langle \boldsymbol{x}^{\mathsf{N}}(k) \rangle_k$ is a realization of the Markov chain $\langle \mathbf{X}_k^{\frac{1}{\mathsf{N}}} \rangle_k$.

We skip the proof of some results due to space limitation. The next step is to show that the collection of $\langle \langle \mathbf{X}_k^{\frac{1}{\mathsf{N}}} \rangle_k \rangle_{\mathsf{N} \in \mathcal{N}}$ is a GSAP for a GUS differential inclusion where $\mathcal{N}$ is the set of valid population size N such that the distribution of the population proportions $\boldsymbol{\rho}$ remains unchanged. We claim that $\langle \langle \mathbf{X}_k^{\frac{1}{\mathsf{N}}} \rangle_k \rangle_{\mathsf{N} \in \mathcal{N}}$ is a GSAP for the following differential inclusion:

Definition 3: The ***semicontinuous anticoordination imitation population dynamics*** is defined by $\dot{\boldsymbol{x}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{x})$, where $\boldsymbol{\mathcal{V}} : \boldsymbol{\mathcal{X}}_{ss} \to 2^{\boldsymbol{\mathcal{X}}_{ss}}$ and for all $p \in [\mathsf{p}]$

$$\mathcal{V}_p(\boldsymbol{x}) = \begin{cases} \{\rho_p - x_p\} & \text{if } x < \alpha(\boldsymbol{x}), \\ [0, \rho_p] - x_p & x = \alpha(\boldsymbol{x}), \\ \{-x_p\} & \text{otherwise}, \end{cases} \quad (12)$$

where $x = \boldsymbol{x}^\top \mathbf{1}$.

When $\boldsymbol{x}$ belongs to the interior of $\boldsymbol{\mathcal{X}}_{ss}$, where in every subpopulation both A-players and B-players exist, i.e., $\boldsymbol{x} \in \prod_{j=1}^{\mathsf{p}}(0, \rho_j)$, we have $\mathbb{1}(\rho_p - x_p^{\mathsf{N}}) = \mathbb{1}(x_p^{\mathsf{N}}) = 1$, for all $p \in [\mathsf{p}]$, and, in turn, $\alpha(\boldsymbol{x})$ equals to

$$\alpha = \frac{\max_p q_p}{\max_p q_p + \max_p y_p}. \quad (13)$$

It can be shown that the semicontinuous population dynamics (12) is GUS under the following assumption.

Assumption 1: The subpopulation with the largest value of q_p also has the largest value of y_p and its population proportion, denoted by ρ_p, satisfies the following inequality

$$\rho_p > \max\{\frac{\max_i q_i}{\max_i q_i + \min_i y_i}, \frac{\max_i y_i}{\min_i q_i + \max_i y_i}\}. \quad (14)$$

Lemma 1: Under Assumption 1, the collection of $\langle \langle \mathbf{X}_k^{\frac{1}{\mathsf{N}}} \rangle_k \rangle_{\mathsf{N} \in \mathcal{N}}$ is a GSAP for (12).

Now, the next step is to analyze the semicontinuous population dynamics and obtain the Birkhoff center.

IV. The Analysis of the Semicontinuous Dynamics

The next lemma reveals the steady state behaviour of the semicontinuous population dynamics if the initial condition $\boldsymbol{x}_0$ is not at the extreme point, i.e., $\boldsymbol{x}_0 \notin \{\mathbf{0}, \boldsymbol{\rho}\}$.

Lemma 2: The semicontinuous anticoordination imitation population dynamics (12) converge to $\alpha\boldsymbol{\rho}$ provided that the initial condition $\boldsymbol{x}_0$ is not at the extreme points, i.e., $\boldsymbol{x}_0 \notin \{\mathbf{0}, \boldsymbol{\rho}\}$. Otherwise, the population dynamics will converge to either $\boldsymbol{\rho}$, $\mathbf{0}$, or $\alpha\boldsymbol{\rho}$.

Proof: Due to the space limitation, we just prove the first part, i.e., $\boldsymbol{x}_0 \notin \{\mathbf{0}, \boldsymbol{\rho}\}$. Depending on the initial condition $\boldsymbol{x}_0$, two cases might happen: *Case 1*, where $\boldsymbol{x}_0 \in \prod_{j=1}^{\mathsf{p}}(0, \rho_j)$, and *Case 2*, where $\boldsymbol{x}_0 \notin \prod_{j=1}^{\mathsf{p}}(0, \rho_j)$. We split the proof for *Case 1* into two parts. *Case 1 Part A.* As long as $\boldsymbol{x}(t) \in \prod_{j=1}^{\mathsf{p}}(0, \rho_j)$, $\alpha(\boldsymbol{x}(t))$ equals α defined in (13). Accordingly, the population dynamics read as $\dot{x}_p = \rho_p - x_p$ if $x < \alpha$, $\dot{x}_p = -x_p$ if $x > \alpha$, and $\dot{x}_p \in [0, \rho_p] - x_p$ otherwise, for $p \in [\mathsf{p}]$. Assume that $x_0 = \boldsymbol{x}_0^\top \mathbf{1} < \alpha$ (resp. $x_0 = \boldsymbol{x}_0^\top \mathbf{1} > \alpha$). Then we have $\dot{x}_p = \rho_p - x_p$ (resp. $\dot{x}_p = -x_p$), for $p \in [\mathsf{p}]$, and, in turn, $\dot{x} = 1 - x$ (resp. $\dot{x} = -x$). The evolution of $x(t)$ will be equal to $x(t) = (x_0 - 1)\exp(-t) + 1$ (resp. $x(t) = x_0 \exp(-t)$). Consequently, the value of Σx_p reaches α in finite time $t_1 = \ln((1-\alpha)/(1-x_0))$ (resp. $t_1 = \ln(x_0/\alpha)$). It is obvious that in the finite interval $t \in [0, t_1]$, the value of $x_p(t)$ remains in the open interval $(0, \rho_p)$, and we have $\alpha(\boldsymbol{x}(t)) = \alpha$. Therefore, in this case the semicontinuous population dynamics are piece-wise continuous with the boundary $x = \alpha$.

Case 1. Part B. Define $h(\boldsymbol{x}) = \boldsymbol{x}^\top \mathbf{1} - \alpha$ and $\Sigma = \{x \in \mathbb{R}^{\mathsf{p}} | h(\boldsymbol{x}) = 0\}$. Based on (12), the dynamics for $h(\boldsymbol{x}) < 0$ (resp. $h(\boldsymbol{x}) > 0$) is $\dot{\boldsymbol{x}} = \boldsymbol{\rho} - \boldsymbol{x}$ (resp. $\dot{\boldsymbol{x}} = -\boldsymbol{x}$). Denote the normal vector of $h(\boldsymbol{x})$ by $\boldsymbol{n}_h$, which is equal to $\mathbf{1}$. The sign of $\boldsymbol{n}_h^\top(\boldsymbol{\rho} - \boldsymbol{x}) = 1 - \alpha$ is positive, and the sign of $\boldsymbol{n}_h^\top(-\boldsymbol{x}) = -\alpha$ is negative, implying that $h(\boldsymbol{x}) = 0$ is an attracting sliding surface for the population dynamics (12), i.e., once the state reaches $h(\boldsymbol{x})$, it cannot leave it [31]. Up to now, it has been shown that provided $\boldsymbol{x}_0 \in \prod_{j=1}^{\mathsf{p}}(0, \rho_j)$, the population dynamics (12) converge to $h(\boldsymbol{x}) = 0$ and remain there afterwards. For the trajectory to remain at $h(\boldsymbol{x}) = 0$, the value of $\boldsymbol{n}_h^\top \dot{\boldsymbol{x}}(t)$ must be equal to zero. The population dynamics at $h(\boldsymbol{x})$ read as $\dot{\boldsymbol{x}} \in [1 - c(\boldsymbol{x})]\boldsymbol{\rho} - \boldsymbol{x}$ where $c(\boldsymbol{x}) \in [0, 1]$ satisfies $\boldsymbol{n}_h^\top \dot{\boldsymbol{x}} = 0$. This yields $c(\boldsymbol{x}) = \frac{\boldsymbol{n}_h^\top(\boldsymbol{\rho} - \boldsymbol{x})}{\boldsymbol{n}_h^\top(\boldsymbol{\rho})}$. Substituting $\boldsymbol{n}_h^\top$ with $\mathbf{1}$ results in $c(\boldsymbol{x}) = 1 - \alpha$, and, in turn, $\dot{\boldsymbol{x}} = \alpha\boldsymbol{\rho} - \boldsymbol{x}$. The evolution of the population dynamics reads as $x_p(t) = \big(x_p(t_1) - \alpha\rho_p\big)\exp\big(-(t - t')\big) + \alpha\rho_p$ for $p \in [\mathsf{p}]$, which implies that $\boldsymbol{x}(t)$ converges to $\alpha\boldsymbol{\rho}$.

Case 2. Depending on the value of $\alpha(\boldsymbol{x}_0)$, several cases might happen. *Case 2.1.* $\alpha(x_0) = \alpha$. A similar reasoning provided in *Case 1* can be applied here. *Case 2.2.* $\alpha(x_0) \neq \alpha$. If $x < \alpha(\boldsymbol{x}_0)$ (resp. $x > \alpha(\boldsymbol{x}_0)$), the population dynamics read as $\dot{x}_p = \rho_p - x_p$ (resp. $\dot{x}_p = -x_p$), for $p \in [\mathsf{p}]$. Hence, for p in $[\mathsf{p}]$ where $x_p(0) = 0$ (resp. $x_p(0) = \rho_p$), the value of $x_p(t)$ will be greater than 0 (resp. lower than ρ_p) for $t > \epsilon$, where ϵ is an arbitrarily small positive value. But for the remaining p in $[\mathsf{p}]$ where $x_p(0) = \rho_p$ (resp. $x_p(0) = 0$), we have $x_p(\epsilon) = 0$ (resp. $x_p(\epsilon) = \rho_p$). Therefore, depending on the value of $\boldsymbol{x}_0$, we either have $\alpha(\boldsymbol{x}(\epsilon)) = \alpha$ or not. If $\alpha(\boldsymbol{x}(\epsilon)) = \alpha$, then the reasoning provided in *Case 1* can also be applied here. Otherwise, similar to the steps taken in *Case 1 Part A*, it can be shown that the population dynamics converge to $h'(\boldsymbol{x}) = x - \alpha(\boldsymbol{x}_0) = 0$ in some finite time, say t_1'. At time t_1', the sign of $\boldsymbol{n}_{h'}^\top(\boldsymbol{\rho} - \boldsymbol{x}) = 1 - \alpha(\boldsymbol{x}_0)$ is positive, and the sign of $\boldsymbol{n}_{h'}^\top(-\boldsymbol{x}) = -\alpha(\boldsymbol{x}_0)$ is negative. This implies that the hyperplane $h'(\boldsymbol{x}) = 0$ is an attracting sliding surface, and once the state reaches $h'(\boldsymbol{x})$, it cannot leave it. Therefore, similar to *Case 1 Part B*, the population dynamics at $h'(\boldsymbol{x}) = 0$ read as $\dot{\boldsymbol{x}} = \alpha(\boldsymbol{x}_0)\boldsymbol{\rho} - \boldsymbol{x}$. This implies that $x_p(t) = (x_p(t_1') - \alpha(\boldsymbol{x}_0)\rho_p)\exp(-(t - t_1')) + \alpha(\boldsymbol{x}_0)\rho_p$ and consequently $x_p(t_1' + \varepsilon)$ will belong to $(0, \rho_p)$ for all $p \in [\mathsf{p}]$. As a result, $\alpha(\boldsymbol{x}(t_1' + \varepsilon))$ will be equal to α, and, in turn, the dynamics read as $\dot{\boldsymbol{x}} = \boldsymbol{\rho} - \boldsymbol{x}$ if $\alpha(\boldsymbol{x}_0) < \alpha$ or $\dot{\boldsymbol{x}} = -\boldsymbol{x}$ if $\alpha(\boldsymbol{x}_0) > \alpha$. In either case, the states will move from the hyperplane $h'(\boldsymbol{x}) = 0$ toward the hyperplane $\boldsymbol{x}^\top \mathbf{1} - \alpha$. The rest of the proof will be similar to the steps taken in *Case 1*. This completes the proof. ■

Proposition 2: The Birkhoff center of the semicontinuous population dynamics equals $\boldsymbol{\mathcal{Q}} = \{\mathbf{0}, \boldsymbol{\rho}, \alpha\boldsymbol{\rho}\}$.

V. The Asymptotic Behaviour of the Discrete Population Dynamics

The main result of this paper is summarized in the following theorem.

Theorem 1: Consider the discrete anticoordination imitation population dynamics (9) for a population of size N. Let $\mu^{\frac{1}{\mathsf{N}}}$ be an invariant probability measure of the associated anticoordination imitation population dynamics Markov chain (11). If Assumption 1 holds, then for every sequence $\langle\frac{1}{\mathsf{N}}\rangle_{\mathsf{N}\in\mathcal{N}}$ approaching zero, and for any open set containing $\boldsymbol{\mathcal{Q}}$ we have $\lim_{\frac{1}{\mathsf{N}}\to 0}\mu^{\frac{1}{\mathsf{N}}}(\boldsymbol{\mathcal{O}}) = 1$.

Corollary 1: Under the conditions of Theorem 1 and starting from a specific initial condition, with probability one the length of the fluctuations in the population proportion of strategy `A` players converges to zero, when the size of the population approaches infinity.

VI. Analysis of a Population with Coordination Payoff Matrices

In the preceding sections we analyzed the asymptotic behaviour of a population of imitators with anti-diagonal anticoordination payoff matrices. In this section, we would like to investigate that of a population of imitators with diagonal *coordination payoff* matrices, i.e.,

$$w_i > q_i \text{ and } z_i > y_i \text{ with } q_i = y_i = 0. \tag{15}$$

In [13, Theorem 6], it has been proved that every discrete imitation population dynamics with coordination payoff matrices will equilibrate. Hence, we expect that Theorem A 1 will lead us to the same conclusion when the population size approaches infinity.

For a population with diagonal coordination payoff matrices, Equations (6) and (7) respectively change to the following:

$$\max_p w_p \mathsf{N} x^{\mathsf{N}} \mathtt{1}(x_p^{\mathsf{N}}) > \max_p (-z_p \mathsf{N} x^{\mathsf{N}} + \mathsf{N} z_p) \mathtt{1}(\rho_p - x_p^{\mathsf{N}}), \tag{16}$$

$$\max_p w_p \mathsf{N} x^{\mathsf{N}} \mathtt{1}(x_p^{\mathsf{N}}) < \max_p (-z_p \mathsf{N} x^{\mathsf{N}} + \mathsf{N} z_p) \mathtt{1}(\rho_p - x_p^{\mathsf{N}}). \tag{17}$$

In view of (16) and (17), the preferred strategy in a population with coordination payoff matrices is A (resp. B) whenever the population proportion of strategy-A players exceeds (resp. falls short of) $\alpha'(\boldsymbol{x}^{\mathsf{N}})$ where

$$\alpha'(\boldsymbol{x}^{\mathsf{N}}) = \frac{\max_p z_p \mathtt{1}(\rho_p - x_p^{\mathsf{N}})}{\max_p z_p \mathtt{1}(\rho_p - x_p^{\mathsf{N}}) + \max_p w_p \mathtt{1}(x_p^{\mathsf{N}})}. \tag{18}$$

Definition 4: The following discrete time stochastic equation

$$\boldsymbol{x}^{\mathsf{N}}(k+1) = \boldsymbol{x}^{\mathsf{N}}(k) + \frac{1}{\mathsf{N}}\big(S_k - u'(\boldsymbol{x}^{\mathsf{N}})\big)\mathtt{1}\big(u'(\boldsymbol{x}^{\mathsf{N}})\big)\mathbf{e}_{P_k} \tag{19}$$

defines the ***discrete coordination imitation population dynamics*** where P_k and S_k are defined in Definition 1, and the function $u'(\boldsymbol{x}^{\mathsf{N}})$ returns 1 (resp. 2) if A (resp. B) is the only preferred strategy of the population at state $\boldsymbol{x}^{\mathsf{N}}$ and 0 otherwise:

$$u'(\boldsymbol{x}^{\mathsf{N}}) = \begin{cases} 2 & \text{if } x^{\mathsf{N}} < \alpha'(\boldsymbol{x}^{\mathsf{N}}), \\ 0 & \text{if } x^{\mathsf{N}} = \alpha'(\boldsymbol{x}^{\mathsf{N}}), \\ 1 & \text{if } x^{\mathsf{N}} > \alpha'(\boldsymbol{x}^{\mathsf{N}}). \end{cases} \tag{20}$$

The population dynamics Markov chain (11) remains unchanged, except that $u(\boldsymbol{x}^{\mathsf{N}})$ is replaced by $u'(\boldsymbol{x}^{\mathsf{N}})$.

Definition 5: The ***coordination imitation population dynamics Markov chain*** is defined as the Markov chain $\langle \mathbf{X}_k^{\frac{1}{\mathsf{N}}} \rangle_k$ with transition probabilities

$$\mathrm{Pr}'_{\boldsymbol{x}^{\mathsf{N}},\boldsymbol{y}^{\mathsf{N}}} = \tag{21}$$

$$\begin{cases} (\rho_p - x_p^{\mathsf{N}})(2-u'(\boldsymbol{x}^{\mathsf{N}}))\mathtt{1}\big(u'(\boldsymbol{x}^{\mathsf{N}})\big) & \text{if } \boldsymbol{y}^{\mathsf{N}} = \frac{1}{\mathsf{N}}\mathbf{e}_p + \boldsymbol{x}^{\mathsf{N}}, \\ x_p^{\mathsf{N}}(u'(\boldsymbol{x}^{\mathsf{N}})-1)\mathtt{1}\big(u'(\boldsymbol{x}^{\mathsf{N}})\big) & \text{if } \boldsymbol{y}^{\mathsf{N}} = -\frac{1}{\mathsf{N}}\mathbf{e}_p + \boldsymbol{x}^{\mathsf{N}}, \\ 1 - \Big(\sum_{p=1}^{\mathsf{p}}(\rho_p - x_p^{\mathsf{N}})(2-u'(\boldsymbol{x}^{\mathsf{N}})) & \text{if } \boldsymbol{y}^{\mathsf{N}} = \boldsymbol{x}^{\mathsf{N}}, \\ \quad + x_p^{\mathsf{N}}(u'(\boldsymbol{x}^{\mathsf{N}})-1)\mathtt{1}\Big)\mathtt{1}\big(u'(\boldsymbol{x}^{\mathsf{N}})\big) & \\ 0 & \text{otherwise,} \end{cases}$$

the state space $\boldsymbol{\mathcal{X}}_{ss} \cap \frac{1}{\mathsf{N}}\mathbb{Z}^{\mathsf{p}}$, and the initial state $\mathbf{X}_0^{\frac{1}{\mathsf{N}}} = \boldsymbol{x}^{\mathsf{N}}(0)$. As for semicontinuous population dynamics, we have the following definition

Definition 6: The ***semicontinuous coordination imitation population dynamics*** is defined by $\dot{x} \in \boldsymbol{\mathcal{V}}'(\boldsymbol{x})$, where $\boldsymbol{\mathcal{V}}' : \boldsymbol{\mathcal{X}}_{ss} \to 2^{\boldsymbol{\mathcal{X}}_{ss}}$ and for all $p \in [\mathsf{p}]$

$$\mathcal{V}_p'(\boldsymbol{x}) = \begin{cases} \{-x_p\} & \text{if } x < \alpha'(\boldsymbol{x}), \\ [0, \rho_p] - x_p & \text{if } x = \alpha'(\boldsymbol{x}) \\ \{\rho_p - x_p\} & \text{otherwise,} \end{cases} \tag{22}$$

where $x = \boldsymbol{x}^\top \mathbf{1}$.

It can be shown that the semicontinuous population dynamics (12) is GUS under the following assumption.

Assumption 2: The subpopulation with the largest value of w_p also has the largest value of z_p and its population proportion, denoted by ρ_p, satisfies the following inequality

$$\rho_p > \max\{\frac{\max_i z_i}{\max_i z_i + \min_i w_i}, \frac{\max_i w_i}{\min_i z_i + \max_i w_i}\}. \tag{23}$$

Similar to Lemma 1, we have the following result.

Lemma 3: Under Assumption 2, the collection of $\langle\langle \mathbf{X}_k^{\frac{1}{\mathsf{N}}} \rangle_k \rangle_{\mathsf{N} \in \mathcal{N}}$ with transition probabilities (21) is a GSAP for (22).

We now move on to determine the steady-state behavior of the population dynamics (22).

Lemma 4: Consider the semicontinuous coordination imitation population dynamics (22). The population dynamics converge to either $\mathbf{0}$ or $\boldsymbol{\rho}$, provided that the initial condition $\boldsymbol{x}_0$ satisfies $\boldsymbol{x}_0^\top \mathbf{1} \neq \alpha'$ where

$$\alpha' = \frac{\max_p z_p}{\max_p z_p + \max_p w_p}. \tag{24}$$

Otherwise, the population dynamics will converge to either $\boldsymbol{\rho}$, $\mathbf{0}$, or $\alpha' \boldsymbol{\rho}$.

We accordingly have the following proposition.

Proposition 3: The Birkhoff center of the semicontinuous coordination imitation population dynamics equals $\boldsymbol{\mathcal{Q}}' = \{\mathbf{0}, \boldsymbol{\rho}\alpha', \boldsymbol{\rho}\}$.

The following theorem summarizes the results.

Theorem 2: Consider the discrete coordination imitation population dynamics (19) for a population of size N. Let $\mu^{\frac{1}{\mathsf{N}}}$ be an invariant probability measure of the associated coordination imitation population dynamics Markov chain (21). If Assumption 2 holds, then for every sequence $\langle \frac{1}{\mathsf{N}} \rangle_{\mathsf{N} \in \mathcal{N}}$ approaching zero, and for any open set containing $\boldsymbol{\mathcal{Q}}'$ we have $\lim_{\frac{1}{\mathsf{N}} \to 0} \mu^{\frac{1}{\mathsf{N}}}(\boldsymbol{\mathcal{O}}) = 1$.

Based on Theorem 2 and similar to anticoordination case, the length of the fluctuations in the population proportion of strategy-A player will be expected to vanish as the size of a population of agents with diagonal coordination payoff matrices approaches infinity, which is in line with the existing result on the steady-state behaviour of finite populations with coordination payoff matrices [13].

VII. Concluding Remarks

It has been previously shown that a finite population of imitators involving in an asynchronous matrix-game may equilibrate or undergo perpetual fluctuations. Here, we utilized the available results in the stochastic approximation theory and then showed that the lengths of the reported fluctuations in the population proportion of strategy-A players will converge to zero with probability one. We also proved that no fluctuations will be expected as the size of a population of imitators with diagonal coordination payoff matrices approaches infinity–consistent with the existing result on the steady-state behaviour of finite populations with coordination payoff matrices. Whether the same conclusion can be made for a heterogeneous population of imitators with coordination and anticoordination payoff matrices remains a topic for future research.

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APPENDIX

Definition 7 ([28]): Let $\boldsymbol{\Phi}$ be the set of all solutions of the differential inclusion $\dot{\boldsymbol{x}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{x})$. The recurrent points of $\boldsymbol{\Phi}$ are defined as $\boldsymbol{\mathcal{C}}_{\boldsymbol{\Phi}} = \left\{\boldsymbol{x}_0 \middle| \boldsymbol{x}_0 \in \bigcup_{\boldsymbol{y}\in\boldsymbol{\mathcal{S}}_{\boldsymbol{x}_0}} \bigcap_{t\geq 0} \mathrm{cl}(\boldsymbol{y}[t,\infty])\right\}$, where $\boldsymbol{\mathcal{S}}_{\boldsymbol{x}_0}$ is the set of solutions of the differential inclusion with the initial condition $\boldsymbol{x}_0$. The *Birkhoff center* of $\boldsymbol{\Phi}$ is defined as the closure of $\boldsymbol{\mathcal{C}}_{\boldsymbol{\Phi}}$.

Definition 8 ([28]): Let $\dot{\boldsymbol{x}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{x})$ be a GUS differential inclusion over the state space $\boldsymbol{\mathcal{X}}_0$ which is convex and compact. Consider a sequence of positive scalar values $\langle \epsilon_k \rangle$ that converges to 0. Let $\boldsymbol{U}^\epsilon = \langle \boldsymbol{U}_k^\epsilon \rangle_k$ be a sequence of $\mathbb{R}^n$-valued random variables and $\langle \boldsymbol{\mathcal{V}}^\epsilon \rangle$ be a family of set-valued maps on $\mathbb{R}^n$. We say that $\langle \langle \mathbf{X}_k^\epsilon \rangle_k \rangle_{\epsilon>0}$ is a GSAP for the differential inclusion $\dot{\boldsymbol{x}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{x})$ if the conditions listed below are met:

1) $\mathbf{X}_k^\epsilon \in \boldsymbol{\mathcal{X}}_0$ for all $k \geq 0$,
2) $\mathbf{X}_{k+1}^\epsilon - \mathbf{X}_k^\epsilon - \epsilon \boldsymbol{U}_{k+1}^\epsilon \in \epsilon \boldsymbol{\mathcal{V}}^\epsilon(\mathbf{X}_k^\epsilon)$,
3) $\forall \delta > 0 \exists \epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$ and $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}_0$
$$\boldsymbol{\mathcal{V}}^\epsilon(\boldsymbol{x}) \subset \{\boldsymbol{z} \in \mathbb{R}^n \,|\, \exists \boldsymbol{y} : |\boldsymbol{x}-\boldsymbol{y}| < \delta, \inf_{\boldsymbol{v}\in\boldsymbol{\mathcal{V}}(\boldsymbol{y})} |\boldsymbol{z}-\boldsymbol{v}| < \delta\},$$
4) for all $T > 0$ and for all $\alpha > 0$,
$$\lim_{\epsilon\to 0} \mathbb{P}\left[\max_{k\leq \frac{T}{\epsilon}} \left|\sum_{i=1}^{k} \epsilon \boldsymbol{U}_i^\epsilon\right| > \alpha \mid \mathbf{X}_0^\epsilon = \boldsymbol{x}\right] = 0$$
uniformly in $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}_0$.

Theorem A 1 (Theorem 3.5 and Corollary 3.9 [28]): Consider a sequence of positive scalar values $\langle \epsilon_k \rangle$ that converges to 0. Let $\langle \langle \mathbf{X}_k^\epsilon \rangle_k \rangle_{\epsilon>0}$ be GSAPs for a GUS differential inclusion $\dot{\boldsymbol{x}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{x})$. Assume that $\langle \mathbf{X}_k^\epsilon \rangle_k$ is a Markov chain for each ϵ. Let μ^ϵ be an invariant probability measure of $\langle \mathbf{X}_k^\epsilon \rangle_k$. Denote a limit point of $\langle \mu^\epsilon \rangle_{\epsilon>0}$ in the topology of weak convergence by μ. Then the support of μ is contained in the Birkhoff center of the dynamical system induced by $\dot{\boldsymbol{x}} \in \boldsymbol{\mathcal{V}}(\boldsymbol{x})$.