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Boundary Stabilization for Mixed Traffic Flow in the Presence of Autonomous Vehicle Platooning

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Abstract— **This paper investigates the boundary stabilization problem for a mixed traffic flow system composed of the traditional human-driven traffic flow and a platoon of autonomous vehicles. Firstly, we employ the first-order LWR model to describe the mixed traffic flow, which is with a bilateral moving spatial domain governed by the platoon. In order to stabilize the mixed traffic flow into the desired density and platoon length, a downstream boundary controller is designed based on the information of upstream density and platoon length. To facilitate the well-posedness and stabilization analysis, we transform the system into a coupled PDE-ODE system with fixed spatial domain. Then we prove the well-posedness of the system, and we further derive sufficient conditions for ensuring the local exponential stability of the system by employing the Lyapunov function method. Finally, numerical simulations are provided to validate the theoretical results.**

I. INTRODUCTION

Macroscopic traffic flow models are commonly used for studying traffic problems due to their capability in capturing traffic features and efficiency in addressing traffic control problems with low computation cost [1]. The evolution of traffic state can usually be described by hyperbolic Partial Differential Equations (PDEs), such as the first-order Lighthill-Whitham-Richards (LWR) model [2], [3] and the second-order Aw-Rascle-Zhang (ARZ) model [4]. In recent decades, extensive studies have been devoted to the boundary feedback control of PDEs in LWR or ARZ traffic flow models [5], [6].

Most of these macroscopic model based studies merely focus on the traditional human-driven traffic flow. However, connected autonomous vehicles (CAVs) have achieved great progress in industrial applications, and they have also attracted increasing attention from researchers in both disciplines of automatic control and transportation. The platooning of CAVs is an effective way to improve road safety and traffic efficiency as well as reduce energy consumption [7]. Considering the fact that CAVs would not replace human-driven vehicles completely in the near future, a mixed traffic system composed of the human-driven traffic flow and CAV platoons is becoming a more common scenario [8], where the emergence of CAV platooning would change the traditional traffic features. Therefore, it is pressing to come up with new traffic flow models, and the research on the stabilization problem for mixed traffic flow is of greater practical significance.

More related to the topic of this paper, a macroscopic model was constructed in [9] for the vehicular traffic flow with a slower vehicle moving inside, which was represented by a coupled system formed by the LWR model describing the evolution of traffic density and an Ordinary Differential Equation (ODE) describing the position of the slower vehicle. The LWR model was extended in [10] to describe a traffic flow in the presence of a vehicle platoon, which is with moving upstream and downstream endpoints and acts as a road capacity reduction. In [11], a novel macroscopic model was proposed for the mixed traffic flow by employing the mean field game model and the second-order ARZ model to describe the behaviors of CAVs and the human-driven traffic flow, respectively. In [12], a coupled PDE-ODE model describing the interaction between the bulk traffic flow and a vehicle platoon was adopted to develop a controller for reducing the fuel consumption of the whole traffic flow.

To the best of authors' knowledge, the boundary control problem for the mixed traffic flow in the presence of an autonomous vehicle platoon based on macroscopic models is still open. In this paper, we employ the LWR model to describe a mixed traffic flow in the presence of a CAV platoon. Considering the bilateral moving spatial domain governed by the platoon, the established model is a coupled PDE-ODE system. In order to stabilize the mixed traffic flow into the desired density and platoon length, a downstream boundary controller is designed by using the information of upstream density and platoon length. To facilitate the well-posedness and stabilization analysis, we transform the system into a coupled PDE-ODE system with fixed spatial domain. Then, the well-posedness and the local exponential stability of the system are proved by employing the Lyapunov function method. Sufficient conditions for ensuring the local exponential stability of the system are derived in terms of a matrix inequality. Finally, numerical simulations are provided to validate the theoretical results.

The remainder of this paper is organized as follows. The problem formulation is presented in Section II. The well-posedness and the local exponential stability analyses are given in Section III. Numerical simulation results are given in Section IV, and Section V concludes the whole paper.

Notations: $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ are the sets of n-order vectors and $m\times n$-order matrices, respectively. I_n is the identity matrix of order n. Given a matrix A, $A^\top$ denotes the transpose of

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A, $A < (\leq)\ 0$ denotes A is negative definite (semi-definite). For a partitioned symmetric matrix, the symbol $\star$ stands for the symmetric block with appropriate dimensions. For a vector $\xi = (\xi_1, \ldots, \xi_n)^\top \in C^0([0, D]; \mathbb{R}^n)$, denote $|\xi|_\sigma = \max\{|\xi(x)|_\sigma, x \in [0, D]\}$ with $|\xi(x)|_\sigma = \max\{|\xi_i(x)|, i = 1, \ldots, n\}$. Given a function $g : [0, L] \to \mathbb{R}^n$, define $\|g\|_{H^1}^2 = \int_0^L (|g|^2 + |g_x|^2) dx < \infty$ with $|\cdot|$ being the Euclidean norm in $\mathbb{R}^n$.

II. Problem Formulation

In this paper, we study a macroscopic model for the freeway traffic in the presence of a platoon of CAVs. Obviously, the platoon location is moving, and we denote the downstream and upstream endpoints of the platoon by $x_d(t)$ and $x_u(t)$, respectively. Hence, the considered freeway traffic with the vehicle platoon is defined on interval $[x_u(t), x_d(t)]$ with bilateral moving boundaries.

Assuming the freeway is with a given number of lanes, the freeway capacity is reduced proportionally to the number of lanes occupied by the platoon, and thus the platoon acts as a flux constraint on interval $[x_u(t), x_d(t)]$. Let $\rho = \rho(t, x)$ be the traffic density, and define the associated flux function (see [10], [12]) as

$$f_\alpha(\rho) = v_f \rho \left(1 - \frac{\rho}{\alpha \rho_m}\right), \tag{1}$$

where v_f is the maximum speed of the traffic flow, ρ_m is the maximum density, and $\alpha \in (0, 1)$ represents the remaining capacity ratio, i.e. the ratio of lanes not occupied by the platoon. Let $v = v(\rho)$ be the average speed of the traffic flow, and considering the reduced flux function (1), we have $v(\rho) = v_f - a\rho(t, x)$ with $a = v_f/(\alpha \rho_m)$.

We adopt the LWR model to describe the freeway traffic in the presence of an autonomous vehicle platoon as

$$\partial_t \rho + \partial_x f_\alpha(\rho) = 0,\ x \in [x_u(t), x_d(t)], \tag{2a}$$

$$\dot{x}_u(t) = v_u(t), \tag{2b}$$

$$\dot{x}_d(t) = v_d(t), \tag{2c}$$

$$\rho(0, x) = \rho_0(x),\ x_u(0) = x_u^0,\ x_d(0) = x_d^0, \tag{2d}$$

where the speeds of the upstream and downstream endpoints are given by $v_u(t) = v_f - a\rho(t, x_u(t))$ and $v_d(t) = v_f - a\rho(t, x_d(t))$, respectively. The length of the platoon is denoted by $l(t)$, which varies in time and depends on both speeds of the upstream and downstream endpoints of the platoon, i.e.

$$\begin{aligned} \dot{l}(t) &= v_d(t) - v_u(t) \\ &= -a\big(\rho(t, x_d(t)) - \rho(t, x_u(t))\big). \end{aligned} \tag{3}$$

Our goal is to stabilize the traffic flow within $[x_u(t), x_d(t)]$ to the desired steady state (ρ^*, l^*) by designing $\rho(t, x_d(t))$. Let the boundary control input be

$$\rho(t, x_d(t)) = \rho^* + k_\rho(\rho(t, x_u(t)) - \rho^*) + k_l(l(t) - l^*), \tag{4}$$

where k_ρ and k_l are controller gains for the density and the platoon length, respectively.

Denote the deviations of traffic density ρ and platoon length l from the steady states ρ^* and l^* by $\tilde{\rho} = \rho - \rho^*$ and $\tilde{l} = l - l^*$, respectively. Then we get the linearized LWR model

$$\partial_t \tilde{\rho} + \lambda \partial_x \tilde{\rho} = 0 \tag{5}$$

with $\lambda = v_f - 2a\rho^* < 0$ and the boundary condition

$$\tilde{\rho}(t, x_d(t)) = k_\rho \tilde{\rho}(t, x_u(t)) + k_l \tilde{l}(t). \tag{6}$$

Substituting (6) into (3), we have

$$\begin{aligned} \dot{\tilde{l}}(t) &= -a\big(k_\rho \tilde{\rho}(t, x_u(t)) + k_l \tilde{l}(t) - \tilde{\rho}(t, x_u(t))\big) \\ &= -ak_l \tilde{l}(t) - a(k_\rho - 1)\tilde{\rho}(t, x_u(t)). \end{aligned} \tag{7}$$

In order to facilitate the stability analysis, we transform the bilateral moving spatial domain $x \in [x_u(t), x_d(t)]$ into a bilateral fixed spatial domain $y \in [0, l^*]$, and thus

$$y = \frac{l^*}{l(t)} \left(x - \left(\int_0^t v_u(\tau) d\tau + x_u^0\right)\right). \tag{8}$$

Suppose that applying the spatial coordinate change to $\tilde{\rho}(t, x)$ gives $\check{\rho}(t, y)$. Then the space and time differentiation relations for $\tilde{\rho}(t, x)$ and $\check{\rho}(t, y)$ are

$$\begin{aligned} \frac{\partial \tilde{\rho}(t, x)}{\partial x} &= \frac{\partial \check{\rho}(t, y)}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial \check{\rho}(t, y)}{\partial t} \cdot \frac{\partial t}{\partial x} \\ &= \frac{\partial \check{\rho}(t, y)}{\partial y} \cdot \frac{l^*}{l(t)}, \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial \tilde{\rho}(t, x)}{\partial t} &= \frac{\partial \check{\rho}(t, y)}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial \check{\rho}(t, y)}{\partial t} \\ &= -\frac{\partial \check{\rho}(t, y)}{\partial y} \cdot \frac{l^* v_f - l^* a \check{\rho}(t, 0) + y\dot{l}(t)}{l(t)} \\ &\quad + \frac{\partial \check{\rho}(t, y)}{\partial t}. \end{aligned} \tag{10}$$

Substituting (9) and (10) into (5), system (5)-(7) is transformed into y-coordinate as

$$\partial_t \check{\rho} + F(\tilde{l}(t), \dot{\tilde{l}}(t), y, \check{\rho}_{out}(t)) \partial_y \check{\rho} = 0, \tag{11}$$

$$\dot{\tilde{l}}(t) = -ak_l \tilde{l}(t) - a(k_\rho - 1)\check{\rho}_{out}(t), \tag{12}$$

with the boundary condition

$$\check{\rho}_{in}(t) = k_\rho \check{\rho}_{out}(t) + k_l \tilde{l}(t), \tag{13}$$

where

$$F(\tilde{l}(t), \dot{\tilde{l}}(t), y, \check{\rho}_{out}(t)) = \frac{l^*\big(\lambda - v_f + a\check{\rho}_{out}(t)\big) - y\dot{\tilde{l}}(t)}{\tilde{l}(t) + l^*}, \tag{14}$$

and $\check{\rho}_{in}(t) = \check{\rho}(t, l^*),\ \check{\rho}_{out}(t) = \check{\rho}(t, 0)$.

Till now, system (2) is equivalently transformed into a semi-linear hyperbolic PDE-ODE coupled system (11)-(13) under the bilateral fixed spatial domain $[0, l^*]$. We will study the well-posedness and stabilization problems for the mixed traffic flow by considering the coupled hyperbolic PDE-ODE system (11)-(13) hereinafter in this paper.

III. Well-posedness and stabilization analysis

The well-posedness analysis of the coupled PDE-ODE system (11)-(13) can be conducted following the arguments in [13, Appendix A] by carefully estimating the related norms of the solution along the characteristic curves. Furthermore, the C^1 regularity of $\tilde{l}(t)$ is obtained directly from (12). The resulting well-posedness of the coupled PDE-ODE system (11)-(13) and the existence of unique solutions are stated in the following lemma.

Lemma 1: For any $T > 0$, if there exists $\epsilon_1(T) > 0$ such that

$$\|\check{\rho}_0\|_{H^1} + |\tilde{l}_0| \leq \epsilon_1(T) \tag{15}$$

for every initial conditions $\check{\rho}_0(y) = \check{\rho}(0, y) \in H^1([0, l^*]; \mathbb{R})$ and $\tilde{l}_0 = \tilde{l}(0) \in \mathbb{R}$ as well as the compatibility conditions driven from (13), the coupled PDE-ODE system (11)-(13) has unique solutions

$$\tilde{l}(t) \in C^1([0, T]; \mathbb{R}), \tag{16}$$

and

$$\check{\rho}(t, y) \in C^0\big([0, T]; H^1([0, l^*]; \mathbb{R})\big). \tag{17}$$

Moreover, there exists $\beta(T) > 0$ such that

$$\|\check{\rho}(t, \cdot)\|_{H^1} + |\tilde{l}(t)| \leq \beta(T)(\|\check{\rho}_0\|_{H^1} + |\tilde{l}_0|) \tag{18}$$

for all $t \in [0, T]$.

Before giving the main result of this paper, we introduce the definition of the locally exponential stability for the coupled PDE-ODE system (11)-(13) as follows.

Definition 1: The coupled PDE-ODE system (11)-(13) is locally exponentially stable with decay rate α_1, if there exist $\epsilon_1 > 0$ and $\beta_1 > 0$ such that

$$\|\check{\rho}(t, \cdot)\|_{H^1}^2 + |\tilde{l}(t)|^2 \leq \beta_1 e^{-\alpha_1 t}\big(\|\check{\rho}_0\|_{H^1}^2 + |\tilde{l}_0|^2\big), \forall t \geq 0 \tag{19}$$

for any initial conditions satisfying $\|\check{\rho}_0\|_{H^1} + |\tilde{l}_0| \leq \epsilon_1$ and the compatibility condition derived from (13).

Then we derive sufficient conditions in terms of matrix inequalities w.r.t. controller gains and traffic parameters for stabilizing the traffic flow within $[x_u(t), x_d(t)]$ to the desired steady state (ρ^*, l^*), by employing the Lyapunov function method. The main result of this paper is given in the following theorem.

Theorem 1: The coupled PDE-ODE system (11)-(13) is locally exponentially stable, if there exist constants $p_1 > 0, p_3 > 0, p_2$ and $\mu > 0$ such that $p_1 p_3 > p_2^2$ and the following matrix inequality

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ \star & m_{22} & m_{23} & m_{24} \\ \star & \star & m_{33} & m_{34} \\ \star & \star & \star & m_{44} \end{bmatrix} < 0, \tag{20}$$

holds for all $y \in [0, l^*]$, where

$$\begin{aligned}
m_{11} &= -\mu\gamma\hat{p}_1(y), \\
m_{12} &= -\left(\mu\gamma + ak_l\right)\hat{p}_2(y), \\
m_{13} &= -a(k_\rho - 1)\hat{p}_2(y), \\
m_{22} &= \frac{e^{\mu l^*}\gamma k_l^2}{l^*}\left(1 + a^2 k_l^2\right)p_1 + 2\frac{e^{\mu l^*}\gamma k_l}{l^*}p_2 - 2ak_l p_3, \\
m_{23} &= \frac{e^{\mu l^*}\gamma}{l^*}\left(k_\rho k_l p_1 + a^2 k_l^3 (k_\rho - 1)p_1 + k_\rho p_2\right) \\
&\quad - \frac{\gamma}{l^*}p_2 - a(k_\rho - 1)p_3, \\
m_{24} &= -\frac{e^{\mu l^*}\gamma a}{l^*}k_l^2 k_\rho p_1, \\
m_{33} &= \frac{e^{\mu l^*}\gamma}{l^*}\left(k_\rho^2 + a^2(k_\rho - 1)^2 k_l^2\right)p_1 - \gamma p_1/l^*, \\
m_{34} &= -e^{\mu l^*}\gamma a(k_\rho - 1)k_l k_\rho p_1/l^*, \\
m_{44} &= \gamma p_1\left(e^{\mu l^*}k_\rho^2 - 1\right)/l^*,
\end{aligned}$$

with $\gamma = 2a\rho^*$, $\hat{p}_1(y) = e^{\mu y}p_1$ and $\hat{p}_2(y) = e^{\mu y}p_2$.

Proof: We tentatively assume that $\check{\rho}(t, y)$ is of class C^2 on $[0, T) \times [0, l^*]$. Define a Lyapunov candidate function as

$$W(t) = V(t) + S(t) \tag{21}$$

with

$$V(t) := \int_0^{l^*} \begin{bmatrix} \check{\rho} \\ \tilde{l} \end{bmatrix}^\top \begin{bmatrix} \hat{p}_1(y) & \hat{p}_2(y) \\ \hat{p}_2(y) & p_3 \end{bmatrix} \begin{bmatrix} \check{\rho} \\ \tilde{l} \end{bmatrix} dy, \tag{22}$$

$$S(t) := \int_0^{l^*} \hat{p}_1(y)\check{\rho}_t^2 dy. \tag{23}$$

For clarity, we introduce a notation to deal with the estimates of the higher-order terms. We denote the higher-order terms by $\mathcal{O}(X; Y)$ with $X \geqslant 0$ and $Y \geqslant 0$ such that there exist $\kappa > 0$ and $\nu > 0$, satisfying

$$(Y \leq \kappa) \Rightarrow (|\mathcal{O}(X; Y)| \leq \nu X). \tag{24}$$

• *The time derivative of $V(t)$*

Taking the time derivative of $V(t)$ along with system (11)-(13) and using the integration by parts, we have

$$\begin{aligned}
\dot{V}(t) = &\int_0^{l^*} \big(\check{\rho}_t\hat{p}_1(y)\check{\rho} + \check{\rho}\hat{p}_1(y)\check{\rho}_t\big)dy \\
&+ 2\int_0^{l^*} \left(\check{\rho}_t\hat{p}_2(y)\tilde{l} + \check{\rho}\hat{p}_2(y)\dot{\tilde{l}} + \tilde{l}p_3\dot{\tilde{l}}\right)dy \\
= &\int_0^{l^*} \big(F\hat{p}_1'(y)\check{\rho}^2 - (F\hat{p}_1(y)\check{\rho}^2)_y + F_y\hat{p}_1(y)\check{\rho}^2\big)dy \\
&+ 2\int_0^{l^*} \check{\rho}(F\hat{p}_2'(y) - ak_l\hat{p}_2(y))\tilde{l}dy \\
&- 2\int_0^{l^*} (\check{\rho}F\hat{p}_2(y)\tilde{l})_y dy + 2\int_0^{l^*} \check{\rho}F_y\hat{p}_2(y)\tilde{l}dy \\
&+ 2\int_0^{l^*} \Big(-a(k_\rho - 1)\check{\rho}\hat{p}_2(y)\check{\rho}_{out} - ak_l p_3\tilde{l}^2 \\
&\qquad - a(k_\rho - 1)\tilde{l}p_3\check{\rho}_{out}\Big)dy \\
= &V_1 + V_2 + V_3,
\end{aligned} \tag{25}$$

in which

$$V_1 = \int_0^{l^*} F\hat{p}_1'(y)\check{\rho}^2 dy + 2\int_0^{l^*} \check{\rho}(F\hat{p}_2'(y) - ak_l\hat{p}_2(y))\tilde{l}dy$$

$$+\int_0^{l^*}\Big(F_y\hat{p}_1(y)\check{\rho}^2+2\check{\rho}F_y\hat{p}_2(y)\tilde{l}\Big)dy, \tag{26}$$

$$V_2=-\int_0^{l^*}\Big((F\hat{p}_1(y)\check{\rho}^2)_y+2(\check{\rho}F\hat{p}_2(y)\tilde{l})_y\Big)dy, \tag{27}$$

$$\begin{aligned}V_3=&2\int_0^{l^*}\Big(-a(k_\rho-1)\check{\rho}\hat{p}_2(y)\check{\rho}_{out}-ak_lp_3\tilde{l}^2\\&-a(k_\rho-1)\tilde{l}p_3\check{\rho}_{out}\Big)dy.\end{aligned} \tag{28}$$

Inspired by [14], the linear approximation technique is introduced to estimate the time derivatives $\dot{V}(t)$. Therefore, we linearize F in (14) around the desired steady state $\tilde{l}(t)=0,\ \check{\rho}(t,y)=0$ as

$$F=F\big|_{\tilde{l}=0,\check{\rho}_{out}=0}+\tilde{F}=-2a\rho^*+\tilde{F}, \tag{29}$$

where $\tilde{F}$ is the error of linear approximates satisfying

$$\begin{aligned}\tilde{F}=&\frac{\partial\tilde{F}}{\partial\check{\rho}_{out}}\bigg|_{\tilde{l}=0,\check{\rho}_{out}=0}\check{\rho}_{out}+\frac{\partial\tilde{F}}{\partial\tilde{l}}\bigg|_{\tilde{l}=0,\check{\rho}_{out}=0}\tilde{l}\\=&\frac{al^*+a(k_\rho-1)y}{l^*}\cdot\check{\rho}_{out}+\frac{2a\rho^*+ak_ly}{l^*}\cdot\tilde{l}.\end{aligned} \tag{30}$$

By substituting (29) into (26), we have

$$\begin{aligned}V_1=&-\int_0^{l^*}2a\rho^*\hat{p}_1'(y)\check{\rho}^2dy\\&-2\int_0^{l^*}(2a\rho^*\hat{p}_2'(y)+ak_l\hat{p}_2(y))\check{\rho}\tilde{l}dy\\&+\int_0^{l^*}\Big(\tilde{F}\hat{p}_1'(y)\check{\rho}^2+2\check{\rho}\tilde{F}\hat{p}_2'(y)\tilde{l}\Big)dy\\&+\int_0^{l^*}\Big(\tilde{F}_y\hat{p}_1(y)\check{\rho}^2+2\check{\rho}\tilde{F}_y\hat{p}_2(y)\tilde{l}\Big)dy\\=&-\int_0^{l^*}2\mu a\rho^*\hat{p}_1(y)\check{\rho}^2dy\\&-2\int_0^{l^*}(2\mu a\rho^*\hat{p}_2(y)+ak_l\hat{p}_2(y))\check{\rho}\tilde{l}dy+\check{V}_1,\end{aligned} \tag{31}$$

with the higher order term

$$\check{V}_1\approx\mathcal{O}\Big(\int_0^{l^*}(|\check{\rho}_{out}|+|\tilde{l}|)(|\check{\rho}|^2+|\check{\rho}||\tilde{l}|)dy;|\check{\rho}|_\sigma+|\tilde{l}|_\sigma\Big). \tag{32}$$

Following the boundary condition (13) and (29), we have

$$\begin{aligned}V_2=&-F\hat{p}_1\check{\rho}^2\big|_0^{l^*}-2\check{\rho}F\hat{p}_2\tilde{l}\big|_0^{l^*}\\=&F(0)\hat{p}_1(0)\check{\rho}^2(t,0)-F(l^*)\hat{p}_1(l^*)\check{\rho}^2(t,l^*)\\&+2\big(\check{\rho}(t,0)F(0)\hat{p}_2(0)\tilde{l}-\check{\rho}(t,l^*)F(l^*)\hat{p}_2(l^*)\tilde{l}\big)\\=&\Big(2e^{\mu l^*}a\rho^*p_1k_\rho^2-2a\rho^*p_1\Big)\check{\rho}_{out}^2\\&+4a\rho^*\Big(e^{\mu l^*}p_1k_\rho k_l+e^{\mu l^*}p_2k_\rho-p_2\Big)\check{\rho}_{out}\tilde{l}\\&+\Big(2e^{\mu l^*}a\rho^*p_1k_l^2+4e^{\mu l^*}a\rho^*p_2k_l\Big)\tilde{l}^2+\check{V}_2,\end{aligned} \tag{33}$$

with the higher order term

$$\check{V}_2\approx\mathcal{O}\big((|\check{\rho}_{out}|+|\tilde{l}|)(|\check{\rho}_{out}|^2+|\check{\rho}_{out}||\tilde{l}|);|\check{\rho}|_\sigma+|\tilde{l}|_\sigma\big). \tag{34}$$

Substituting (28), (31) and (33) into (25), we have

$$\begin{aligned}\dot{V}=&V_1+V_2+V_3\\=&-\int_0^{l^*}2\mu a\rho^*\hat{p}_1(y)\check{\rho}^2dy\\&-2\int_0^{l^*}\Big(2\mu a\rho^*\hat{p}_2(y)+ak_l\hat{p}_2(y)\Big)\check{\rho}\tilde{l}dy\\&+2\int_0^{l^*}\Big(-a(k_\rho-1)\hat{p}_2(y)\check{\rho}\check{\rho}_{out}-ak_lp_3\tilde{l}^2\\&\qquad-a(k_\rho-1)p_3\tilde{l}\check{\rho}_{out}\Big)dy\\&+\Big(2e^{\mu l^*}a\rho^*p_1k_\rho^2-2a\rho^*p_1\Big)\check{\rho}_{out}^2\\&+4a\rho^*\Big(e^{\mu l^*}p_1k_\rho k_l+e^{\mu l^*}p_2k_\rho-p_2\Big)\check{\rho}_{out}\tilde{l}\\&+\Big(2e^{\mu l^*}a\rho^*p_1k_l^2+4e^{\mu l^*}a\rho^*p_2k_l\Big)\tilde{l}^2+\check{V},\end{aligned} \tag{35}$$

with $\check{V}=\check{V}_1+\check{V}_2$.

• *The time derivative of* $S(t)$

By taking the time differentiation of (11)-(13), the dynamic of $\check{\rho}_{tt}$ is given as

$$\check{\rho}_{tt}+F_t\check{\rho}_y+F\check{\rho}_{yt}=0, \tag{36}$$

with the boundary condition

$$\dot{\check{\rho}}_{in}(t)=k_\rho\dot{\check{\rho}}_{out}(t)-ak_l^2\tilde{l}(t)-ak_l(k_\rho-1)\check{\rho}_{out}(t). \tag{37}$$

Taking the time derivative of $S(t)$ along with the solutions of (36) and using the integration by parts, we obtain

$$\begin{aligned}\dot{S}=&-2\int_0^{l^*}(\check{\rho}_{yt}F\hat{p}_1(y)\check{\rho}_t+\check{\rho}_yF_t\hat{p}_1(y)\check{\rho}_t)dy\\=&\int_0^{l^*}F\hat{p}_1'(y)\check{\rho}_t^2dy-2\int_0^{l^*}\check{\rho}_yF_t\hat{p}_1(y)\check{\rho}_tdy\\&-\int_0^{l^*}(F\hat{p}_1(y)\check{\rho}_t^2)_ydy+\int_0^{l^*}\check{\rho}_tF_y\hat{p}_1(y)\check{\rho}_tdy\\=&S_1+S_2,\end{aligned} \tag{38}$$

in which

$$S_1=\int_0^{l^*}(F\hat{p}_1'(y)+F_y\hat{p}_1(y))\check{\rho}_t^2dy-2\int_0^{l^*}\check{\rho}_yF_t\hat{p}_1(y)\check{\rho}_tdy,$$

$$S_2=-\int_0^{l^*}(F\hat{p}_1(y)\check{\rho}_t^2)_ydy.$$

For S_1, according to (29), we have

$$S_1=-2\int_0^{l^*}a\rho^*\mu\hat{p}_1(y)\check{\rho}_t^2dy+\check{S}_1, \tag{39}$$

with the higher order term

$$\check{S}_1\approx\mathcal{O}\Big(\int_0^{l^*}(|\check{\rho}_{out}|+|\tilde{l}|)|\check{\rho}_t|^2dy;|\check{\rho}|_\sigma+|\tilde{l}|_\sigma\Big). \tag{40}$$

For S_2, taking similar procedures as (33) and using the boundary condition (37), we have

$$\begin{aligned}S_2=&-F\hat{p}_1(y)\check{\rho}_t^2\big|_0^{l^*}\\=&F(0)\hat{p}_1(0)\check{\rho}_t^2(t,0)-F(l^*)\hat{p}_1(l^*)\check{\rho}_t^2(t,l^*)\end{aligned}$$

$$
\begin{aligned}
&= \left(2e^{\mu l^*} a\rho^* p_1 k_\rho^2 - 2a\rho^* p_1\right) \dot{\check{\rho}}_{out}^2 \\
&\quad + e^{\mu l^*} \left(2a^2 k_l^4 a\rho^* p_1 \tilde{l}^2 + 2a\rho^* p_1 a^2 (k_\rho - 1)^2 k_l^2 \check{\rho}_{out}^2\right) \\
&\quad + 4e^{\mu l^*} a\rho^* \left(a^2 k_l^3 (k_\rho - 1) p_1 \tilde{l} \check{\rho}_{out} - a k_l^2 k_\rho p_1 \tilde{l} \dot{\check{\rho}}_{out}\right) \\
&\quad - 4e^{\mu l^*} a^2 (k_\rho - 1) k_l k_\rho \rho^* p_1 \check{\rho}_{out} \dot{\check{\rho}}_{out} + \check{S}_2,
\end{aligned} \tag{41}
$$

with the higher order term

$$
\check{S}_2 \approx \mathcal{O}\big((|\check{\rho}_{out}| + |\tilde{l}|)|\dot{\check{\rho}}_{out}|^2); |\check{\rho}|_\sigma + |\tilde{l}|_\sigma\big). \tag{42}
$$

Substituting (39) and (41) into (38), we have

$$
\begin{aligned}
\dot{S} =& - 2\int_0^{l^*} a\rho^* \mu \hat{p}_1(y) \check{\rho}_t^2 dy \\
&+ \left(2e^{\mu l^*} a\rho^* p_1 k_\rho^2 - 2a\rho^* p_1\right) \dot{\check{\rho}}_{out}^2 \\
&+ e^{\mu l^*} \left(2a^2 k_l^4 a\rho^* p_1 \tilde{l}^2 + 2a\rho^* p_1 a^2 (k_\rho - 1)^2 k_l^2 \check{\rho}_{out}^2\right) \\
&+ 4e^{\mu l^*} a\rho^* \left(a^2 k_l^3 (k_\rho - 1) p_1 \tilde{l} \check{\rho}_{out} - a k_l^2 k_\rho p_1 \tilde{l} \dot{\check{\rho}}_{out}\right) \\
&- 4e^{\mu l^*} a^2 (k_\rho - 1) k_l k_\rho \rho^* p_1 \check{\rho}_{out} \dot{\check{\rho}}_{out} + \check{S}_1 + \check{S}_2.
\end{aligned} \tag{43}
$$

- *The exponential convergence*

Define $\Phi(t,y) = [\check{\rho}(t,y), \tilde{l}(t), \check{\rho}_{out}(t), \dot{\check{\rho}}_{out}(t)]^\top$. By combining (35) and (43), the time derivative of $W(t)$ along with the PDE-ODE coupled system (11)-(13) can be reorganized into the following compact form as

$$
\dot{W}(t) = \int_0^{l^*} \Phi^\top M \Phi dy + \int_0^{l^*} m_{11} \check{\rho}_t^2 dy + \check{W}, \tag{44}
$$

where M is given in (20) and the higher order term

$$
\begin{aligned}
\check{W} =& \mathcal{O}\Big(\int_0^{l^*} (|\check{\rho}_{out}|+|\tilde{l}|)(|\check{\rho}|^2+|\check{\rho}||\tilde{l}|+|\check{\rho}_t|^2)dy; |\check{\rho}|_\sigma+|\tilde{l}|_\sigma\Big) \\
&+\mathcal{O}\big((|\check{\rho}_{out}|+|\tilde{l}|)(|\check{\rho}_{out}|^2+|\check{\rho}_{out}||\tilde{l}|+|\dot{\check{\rho}}_{out}|^2); |\check{\rho}|_\sigma+|\tilde{l}|_\sigma\big).
\end{aligned} \tag{45}
$$

Following the definition of $W(t)$, there always exists β_2 large enough such that

$$
\begin{aligned}
\frac{1}{\beta_2}\Big(\int_0^{l^*} |\check{\rho}|^2 + |\check{\rho}_y|^2 dy + |\tilde{l}|^2\Big) &\le W(t) \\
&\le \beta_2\Big(\int_0^{l^*} |\check{\rho}|^2 + |\check{\rho}_y|^2 dy + |\tilde{l}|^2\Big).
\end{aligned} \tag{46}
$$

The negative definiteness of M as given in (20) implies that $m_{ii} < 0$ for all $i = 1, ..., 4$, which further gives that $\int_0^{l^*} m_{11}\check{\rho}^2 dy$, $\int_0^{l^*} m_{22}\tilde{l}^2 dy$, $\int_0^{l^*} m_{33}\check{\rho}_{out}^2 dy$, $\int_0^{l^*} m_{44}\dot{\check{\rho}}^2 dy$ and $\int_0^{l^*} m_{11}\check{\rho}_t^2 dy$ are all negative. Then, following (44) and the Young's inequality in (45), there always exist ε_1, α_2, η_j, κ_j and ν_j, $j = 1, 2$ such that

$$
\begin{aligned}
\dot{W}(t) <& - \kappa_1 W(t) - \kappa_2(|\check{\rho}_{out}(t)|^2 + |\dot{\check{\rho}}_{out}(t)|^2) \\
&+l^*(|\check{\rho}|_\sigma+|\tilde{l}|_\sigma)\mathcal{O}\Big(\int_0^{l^*} |\check{\rho}|^2+|\check{\rho}||\tilde{l}|+|\check{\rho}_t|^2 dy; |\check{\rho}|_\sigma+|\tilde{l}|_\sigma\Big) \\
&+(|\check{\rho}|_\sigma+|\tilde{l}|_\sigma)\mathcal{O}\big(|\check{\rho}_{out}|^2+|\check{\rho}_{out}||\tilde{l}|+|\dot{\check{\rho}}_{out}|^2; |\check{\rho}|_\sigma+|\tilde{l}|_\sigma\big) \\
<& l^*(|\check{\rho}|_\sigma + |\tilde{l}|_\sigma)\mathcal{O}\Big(\int_0^{l^*} |\check{\rho}|^2 + \eta_1|\check{\rho}|^2 + |\check{\rho}_t|^2 dy \\
&\quad + \frac{1}{\eta_1}|\tilde{l}|^2); |\check{\rho}|_\sigma + |\tilde{l}|_\sigma\Big) \\
&+ (|\check{\rho}|_\sigma + |\tilde{l}|_\sigma)\mathcal{O}\big(|\check{\rho}_{out}|^2 + \eta_2|\check{\rho}_{out}| + \frac{1}{\eta_2}|\tilde{l}|^2 \\
&\quad + |\dot{\check{\rho}}_{out}|^2; |\check{\rho}|_\sigma + |\tilde{l}|_\sigma\big) \\
<& \big(- \kappa_1 + \nu_1(|\check{\rho}|_\sigma + |\tilde{l}|_\sigma)\big)W(t) \\
&\big(- \kappa_2 + \nu_2(|\check{\rho}|_\sigma + |\tilde{l}|_\sigma)\big)\big(|\check{\rho}_{out}|^2 + |\dot{\check{\rho}}_{out}|^2\big) \\
\le& - \alpha_2 W(t),
\end{aligned} \tag{47}
$$

for every $|\check{\rho}|_\sigma + |\tilde{l}|_\sigma \le \varepsilon_1$ with sufficiently small ε_1.

Although we have assumed the solutions $\check{\rho}(t,y)$ are of class C^2, we notice that the selections of β_2 and α_2 in (46) and (47) depend only on the $C^0([0,T), H^1([0,l^*];\mathbb{R}^2))$-norm of $\check{\rho}$. Hence, using the arguments in [15, Comment 4.6, page 127], the conditions (46) and (47) remain valid in the distributed sense with $\check{\rho}$ being only of class C^0.

Following the Sobolev inequality (see Theorem 8.8 in [16]), there exists $\theta_1 > 0$ such that

$$
|\check{\rho}|_\sigma \le \theta_1\Big(\int_0^{l^*} |\check{\rho}|^2 + |\check{\rho}_y|^2 dy\Big)^{\frac{1}{2}}, \tag{48}
$$

for every $\check{\rho}$ in the Sobolev space $H^1[0,l^*]$. When choosing $\theta_1 \ge 1$, we have

$$
|\tilde{l}|_\sigma \le \theta_1|\tilde{l}|. \tag{49}
$$

Let us introduce

$$
\delta \triangleq \min\left\{\frac{\varepsilon_1}{\beta_2}, \frac{\varepsilon_2}{2\theta_1\beta_2}\right\}, \tag{50}
$$

where $\beta_2 \ge 1$ and $\varepsilon_2 \le \varepsilon_1$. By combining (46)-(50),

$$
\|\check{\rho}\|_{H^1} + |\tilde{l}| \le \delta \tag{51}
$$

gives that

$$
\|\check{\rho}\|_\sigma + |\tilde{l}|_\sigma \le \frac{\varepsilon_2}{2},\ W(t) \le \beta_2\varepsilon_1^2, \tag{52}
$$

which further implies that

$$
\|\check{\rho}\|_{H^1} + |\tilde{l}| \le \varepsilon_1,\ \dot{W}(t) \le 0 \tag{53}
$$

for every $t \in [0,T)$.

Suppose the initial conditions $\check{\rho}_0 \in ([0,l^*];\mathbb{R}^2)$ and $|\tilde{l}_0| \in \mathbb{R}$ satisfy the compatibility conditions (13) and

$$
\|\check{\rho}_0\|_{H^1} + |\tilde{l}_0| < \delta. \tag{54}
$$

Let $\check{\rho} \in C^0\big([0,T), H^1([0,l^*];\mathbb{R}^2)\big)$ and $\tilde{l} \in C^0([0,T],\mathbb{R})$ be the maximal solutions of the coupled PDE-ODE system (11)-(13). Following (51)-(54), we have

$$
\|\check{\rho}\|_{H^1} + |\tilde{l}| \le \varepsilon_1, \quad \forall t \in [0,T), \tag{55}
$$

$$
\|\check{\rho}\|_\sigma + |\tilde{l}|_\sigma \le \varepsilon_2, \quad \forall t \in [0,T). \tag{56}
$$

By combining (55) and Lemma 1, it gives that $T = +\infty$. Following (46), (47) and (56), we have

$$
\|\check{\rho}(x,t)\|_{H^1}^2 + |\tilde{l}(t)|^2 \le \beta_2 W(t)
$$

$$\leq \beta_2 e^{-\alpha_2 t} W(0)$$
$$\leq \beta_2^2 e^{-\alpha_2 t} \left(\|\check{\rho}_0\|_{H^1}^2 + |\tilde{l}_0|^2 \right). \quad (57)$$

The proof of Theorem 1 is thus completed. ■

IV. Numerical simulations

We consider the freeway traffic in the presence of an autonomous vehicle platoon. The traffic flow parameters are given by $v_f = 90$ km/h, $\rho_m = 120$ veh./km and $\alpha = 0.8$. The initial upstream and downstream endpoints of the platoon are set as $x_u^0 = 0$ m and $x_d^0 = 413.8$ m respectively, i.e., the initial platoon length $l_0 = 413.8$ m. The initial density of the traffic flow within $[x_u^0, x_d^0]$ is

$$\rho(0, x) = 58 + 2\cos(4\pi x). \quad (58)$$

The control objective is to stabilize the mixed traffic flow into the desired steady state with $\rho^* = 75$ veh./km and $l^* = 300$ m. Set $k_\rho = 0.6$ and $k_l = 0.55$ in (4). Taking $\mu = 0.01$ and solving matrix inequality in Theorem 1 by means of the Linear Matrix Inequality toolbox in MATLAB, we obtain $p_1 = 10.3$, $p_2 = -4.5$ and $p_3 = 16.2$.

To obtain the numerical solutions, the two-step variant of Lax-Wendroff method is introduced to discretize the mixed traffic flow system (2). The trajectory of the mixed traffic flow moving forward along the road is shown in Fig. 1 (a), from which, we can observe that the platoon length converges to a steady value. The time evolution of the platoon length is shown in Fig. 1 (b), showing that the platoon length converges to the desired value l^*.

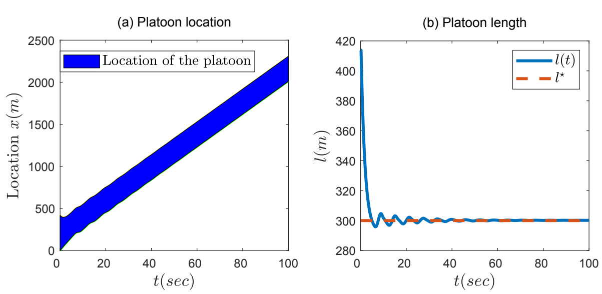

Fig. 1. Trajectories of the platoon location and the platoon length $l(t)$.

The spatiotemporal density evolution of the mixed traffic flow is presented in Fig. 2 (a), the top view of which is shown in Fig. 2 (b). The red line in Fig. 2 (b) represents the time evolution of the platoon length. From Fig. 2, we can clearly observe both $\rho(t, x)$ and $l(t)$ converge to the desired steady state values of ρ^* and l^*.

V. Conclusions

In this paper, we have studied the boundary stabilization problem for a mixed traffic flow system, which is composed of the traditional human-driven traffic flow and a platoon of autonomous vehicles. The first-order LWR model has been used to describe the mixed traffic flow, which is with a bilateral moving spatial domain governed by the platoon. Then a downstream boundary controller has been designed based on the information of upstream density and platoon length, for stabilizing the mixed traffic flow into the desired uniform density and the desired platoon length. By transforming the system into a coupled PDE-ODE system with fixed spatial domain, we have employed the Lyapunov function method to prove the well-posedness and local exponential stability of the system, where sufficient conditions are established for ensuring the local exponential stability of the system. Finally, numerical simulations are presented to verify the effectiveness of the proposed controller.

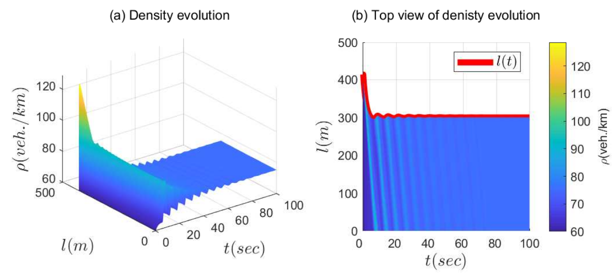

Fig. 2. The spatiotemporal evolution of the traffic density ρ.

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