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Event-Triggered Global Robust Exact Output Regulation for a Class of Nonlinear Uncertain Systems

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Abstract— **Recently, the event-triggered global robust practical output regulation problem of a class of nonlinear uncertain systems has been solved with the exact output regulation problem remaining open. In this paper, we investigate the exact output regulation problem by dynamic event-triggered output feedback control. To solve the problem, we first convert the event-triggered global robust output regulation problem into the event-triggered global robust stabilization problem of an augmented system based on the internal model principle. Then, we develop the dynamic event-triggered mechanism and the dynamic output feedback control law. By Lyapunov analysis, we show that the event-triggered global robust stabilization problem can be solved, thus leading to the solution of the event-triggered global robust output regulation problem, and meanwhile, the Zeno behavior can be strictly excluded.**

I. INTRODUCTION

In the past few decades, with its advantages in reducing unnecessary communication costs and saving computation resources, event-triggered control has attracted a lot of attention, see [1]–[4] and references therein. The applications of event-triggered control can be found in many practical scenarios, see [5]–[8].

Recently, the robust output regulation problem by event-triggered control has attracted many researchers' attention, see [9]–[20]. For the robust output regulation problem, both the system uncertainty and the external disturbance are considered. The problem is first studied for linear systems in [9]–[15]. In particular, in [9], by proposing an event-triggered output feedback control law and a static event-triggered mechanism, practical output regulation for a class of linear uncertain minimum-phase systems is achieved, that is, for the system uncertainties in an arbitrarily large compact set, the steady-state tracking error of the closed-loop system can be made to a small neighborhood of the origin. Later, the robust output regulation of nonlinear systems by event-triggered control is studied in [17]–[20]. Particularly, the event-triggered global robust output regulation problem of nonlinear systems in normal form with unity relative degree is studied in [17]. Based on the internal model approach, an output-based event-triggered control law together with a static event-triggered mechanism is developed and practical output regulation for nonlinear systems is achieved. Subsequently, the event-triggered global robust practical output regulation problem of nonlinear systems in normal form with any relative degree is addressed in [18]. Resorting to the distributed internal model approach, the event-triggered control methods are employed to solve the event-triggered cooperative practical output regulation problem of multi-agent nonlinear systems in [19]. Lately, based on the high gain feedback control method, it is shown in [20] that the same event-triggered cooperative practical output regulation problem in [19] can be solved by a merely static distributed feedback control law which does not need to involve the internal model dynamics. It is worth mentioning that, in the previous works, the robust output regulation problem or the cooperative robust output regulation problem can only be solved practically by event-triggered control.

In this paper, we further address the event-triggered global robust output regulation problem of a class of nonlinear uncertain systems. First, we introduce the internal model to tackle system uncertainties and disturbances and convert the event-triggered global output regulation problem into an event-triggered global robust stabilization problem of an augmented system consisting of the nonlinear uncertain system and the internal model. Next, we delicately design the dynamic variable in the event-triggered mechanism by constraining its upper bound by a tracking-error-related function. Resorting to the changing supply function technique and Lyapunov analysis approach, we construct the dynamic event-triggered mechanism and the Lyapunov function for the augmented closed-loop system. Then, by the contradiction method, we provide a rigorous proof to show that exact robust output regulation can be achieved while prohibiting the Zeno behavior. Compared with the existing results on event-triggered robust output regulation of nonlinear systems in [17]–[20], we solve the event-triggered global robust output regulation problem exactly, that is, the regulation error tends to zero asymptotically and the Zeno behavior can be explicitly excluded. Moreover, the newly developed dynamic event-triggered mechanism approach has the potential to be extended to event-triggered global robust output regulation problems of more complex nonlinear uncertain systems. It is interesting to extend the current work to some other nonlinear control problems [21]–[24].

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Notation. $\mathbb{R}$, $\mathbb{N}$ denote the sets of real numbers and natural numbers, respectively. For any column vectors $x_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, $\text{col}(x_1, \ldots, x_m) = [x_1^T, \ldots, x_m^T]^T$. $||x||$ denotes the Euclidean norm of vector x and $||A||$ denotes the Euclidean norm of matrix A. I_n denotes the n-dimensional identity matrix. The C^1 function denotes the continuously differentiable function.

II. Problem Formulation

Consider a class of nonlinear uncertain systems in output feedback form described by

$$\begin{aligned} \dot{z}(t) &= f(z(t), y(t), v(t), w) \\ \dot{y}(t) &= g(z(t), y(t), v(t), w) + b(w)u(t) \\ e(t) &= y(t) - y_0(t) \end{aligned} \tag{1}$$

where $z(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ denote the state, $e(t) \in \mathbb{R}$ denotes the tracking error, $u(t) \in \mathbb{R}$ denotes the control input, $w \in \mathbb{R}^{n_w}$ denotes the uncertain constant parameter vector and can take arbitrary values in any compact subset $\mathbb{W} \subset \mathbb{R}^{n_w}$, and $v(t)$ denotes the exogenous signal and is generated by a linear system as follows:

$$\begin{aligned} \dot{v}(t) &= Sv(t) \\ y_0(t) &= q(v(t), w) \end{aligned} \tag{2}$$

where $v(t) \in \mathbb{R}^{n_v}$ and $y_0(t) \in \mathbb{R}$. The system (2) is called the exosystem and is used to formulate both reference input signals and disturbance signals in system (1). The functions $f(\cdot)$, $g(\cdot)$ and $q(\cdot)$ are assumed to be sufficiently smooth in their arguments satisfying $f(0,0,0,w) = 0$, $g(0,0,0,w) = 0$ and $q(0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$. The function $b(w)$ is assumed to be a continuous function and satisfies $b(w) > 0$ for all $w \in \mathbb{R}^{n_w}$. Then, for any compact subset $\Upsilon \subset \mathbb{R}^{n_w}$, there exist some known positive numbers b_m and b_M such that $b_m \leq b(w) \leq b_M$ for all $w \in \Upsilon$.

The general form of our output-based event-triggered feedback control laws is given as follows:

$$\begin{aligned} u(t) &= \kappa(e(t_k), \eta(t_k)) \\ \dot{\eta}(t) &= \varpi(\eta(t), u(t)), t \in [t_k, t_{k+1}), k \in \mathbb{N} \end{aligned} \tag{3}$$

where $\kappa(\cdot)$ and $\varpi(\cdot)$ are some globally defined functions to be specified later. The time instants t_k denote triggering time instants with $t_0 = 0$ and $k \in \mathbb{N}$, and are generated by an event-triggered mechanism as follows:

$$\begin{aligned} t_{k+1} &= \inf\{t > t_k | h(\tilde{e}(t), \tilde{\eta}(t), e(t)) \geq \gamma(t)\} \\ \dot{\gamma}(t) &= \rho(\gamma(t), \tilde{e}(t), \tilde{\eta}(t), e(t)) \end{aligned} \tag{4}$$

where $h(\cdot)$ and $\rho(\cdot)$ are some functions to be designed, and

$$\begin{aligned} \tilde{e}(t) &= e(t_k) - e(t) \\ \tilde{\eta}(t) &= \eta(t_k) - \eta(t) \end{aligned} \tag{5}$$

for any $t \in [t_k, t_{k+1})$ with $k \in \mathbb{N}$. Since the internal variable $\gamma(t)$ is generated by a dynamic system, (4) is called a dynamic event-triggered mechanism.

Remark 2.1: Denote $x_c(t)$ as the solution of the closed-loop system composed of (1) and (3) with the triggering mechanism (4). Suppose that the solution $x_c(t)$ is right maximally defined for all $t \in [0, T_M)$ with $0 < T_M \leq \infty$. Let $\{t_k\}_{k \in \mathbb{S}}$ with $\mathbb{S} \subset \mathbb{N}$ denote the time sequence generated by the event-triggered mechanism (4). Then, as in [25], one of the following three cases may occur:

1) $\mathbb{S} = \mathbb{N}$ and $\lim_{k\to\infty} t_k < \infty$.
2) $\mathbb{S} = \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$.
3) $\mathbb{S}$ is a finite set and is denoted by $\mathbb{S} = \{0, 1, \cdots, k^*\}$ with $k^* \in \mathbb{N}$. In this case, $t_{k^*} < T_M$.

In the three cases, the solution $x_c(t)$ is defined for all $t \in \bigcup_{k\in\mathbb{S}} [t_k, t_{k+1})$. Case 1) means that the Zeno behavior occurs, which is undesirable in practical implementation. In this paper, to make the system well behaved for all $t \in [0, +\infty)$, we wish to design the control law (3) and the event-triggered mechanism (4) such that $T_M = +\infty$ and the Zeno behavior is excluded.

Now, our problem is described as follows.

Problem 2.1: Given the plant (1), the exosystem (2), any compact subsets $\mathbb{V} \subset \mathbb{R}^{n_v}$ and $\mathbb{W} \subset \mathbb{R}^{n_w}$ with $0 \in \mathbb{V}$ and $0 \in \mathbb{W}$, design an output-based event-triggered feedback control law of the form (3) and a dynamic event-triggered mechanism of the form (4) such that, for any initial states $z(0), y(0)$, and $\eta(0)$, and for any $v(t) \in \mathbb{V}$ and $w \in \mathbb{W}$, the closed-loop system has the following two properties:

Property 1: the trajectory of the closed-loop system exists and is bounded for all $t \geq 0$;

Property 2: $\lim_{t\to\infty} e(t) = 0$.

Remark 2.2: Problem 2.1 is called the event-triggered global robust output regulation problem. As mentioned in [17], compared with the classical output regulation problem in [26], a piecewise continuous control law instead of a continuous control law has to be designed to guarantee the two properties in Problem 2.1, which makes Problem 2.1 more challenging. In addition, as in [17], the control law (3) can be directly implemented on the digital platform.

III. Internal Model and Problem Conversion

In this section, we construct an internal model and show that the event-triggered global robust output regulation problem can be converted into an event-triggered global stabilization problem of an augmented system.

First, we need to introduce some standard assumptions.

Assumption 3.1: The exosystem (2) is neutrally stable, i.e., all eigenvalues of S are semi-simple with zero real parts.

Assumption 3.2: There exists a globally defined smooth function $\mathbf{z} : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$ to $\mathbb{R}^n$ with $\mathbf{z}(0, w) = 0$ such that

$$\frac{\partial \mathbf{z}(v, w)}{\partial v} Sv = f(\mathbf{z}(v, w), q(v, w), v, w) \tag{6}$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$.

Assumption 3.3: The function $\mathbf{u}(v, w)$ is a polynomial in v with coefficients depending on w.

Remark 3.1: Assumption 3.1 is used to guarantee the boundedness of the signal $v(t)$. Under Assumption 3.1, for any $v(0) \in \mathbb{V}_0$ for a compact subset $\mathbb{V}_0 \subset \mathbb{R}^{n_v}$, there exists a compact subset $\mathbb{V} \subset \mathbb{R}^{n_v}$ such that $v(t) \in \mathbb{V}$ for all time.

Remark 3.2: Assumption 3.2 is a necessary condition for the solvability of the output regulation problem [27], [28].

Under Assumption 3.2, let

$$\begin{aligned}\mathbf{y}(v,w) =& q(v,w)\\ \mathbf{u}(v,w) =& b^{-1}(w)\Big(\frac{\partial q(v,w)}{\partial v}Sv - g(\mathbf{z}(v,w),q(v,w),v,w)\Big).\end{aligned}$$

Then, functions $\mathbf{z}(v,w), \mathbf{y}(v,w)$ and $\mathbf{u}(v,w)$ characterize the steady-state states and the steady-state input for the states z, y and the control input u, respectively, which are also the invariant manifolds for the system (1) [29].

Remark 3.3: Assumption 3.3 is made for facilitation of the internal model design and is used in relevant results, see [17], [26]. It can also be relaxed by the result in [30].

Under Assumptions 3.1-3.3, define the following system

$$\begin{aligned}\frac{\partial \vartheta(v,w)}{\partial v}Sv = \Phi\vartheta(v,w)\\ \mathbf{u}(v,w) = \Gamma\vartheta(v,w)\end{aligned} \tag{7}$$

where $\vartheta(v,w) = \mathrm{col}(\mathbf{u}(v,w), \dot{\mathbf{u}}(v,w), \ldots, \mathbf{u}^{(s-1)}(v,w))$, $\Phi = \left[\begin{array}{c|c} 0_{(s-1)\times 1} & I_{s-1} \\ \hline a_1 & a_2 \ \cdots \ a_s\end{array}\right]$ and $\Gamma = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$ for some constants $a_1, a_2, \ldots, a_s$ guaranteeing that the polynomial $P(\lambda) = \lambda^s - a_1 - a_2\lambda - \cdots - a_s\lambda^{s-1}$ has distinct roots with zero real parts. By Definition 6.2 and Proposition 6.12 of [29], system (7) is called the steady-state generator for system (1) with output u and can reproduce the steady-state input $\mathbf{u}(v,w)$ [29].

Then, we can construct the internal model. Note that the matrix pair (Φ,Γ) is observable. Then, for any controllable matrix pair (M,N) with $M \in \mathbb{R}^{s\times s}$ being Hurwitz and $N \in \mathbb{R}^{s\times 1}$, the Sylvester equation $T\Phi - MT = N\Gamma$ has a unique nonsingular solution T [31]. Thus, by Proposition 6.21 of [29], we can define the following system:

$$\dot{\eta} = M\eta + Nu \tag{8}$$

which is an internal model of the system (1).

Next, we show that the event-triggered global robust output regulation problem can be converted into an event-triggered global robust stabilization problem of the augmented system consisting of (1) and (8). To do this, we perform the following coordinate and input transformation on the augmented system:

$$\begin{aligned}\bar{z} = z - \mathbf{z}(v,w),\ \bar{\eta} = \eta - T\vartheta(v,w) - Nb^{-1}e\\ e = y - q(v,w),\ \bar{u} = u - \Gamma T^{-1}\eta(t_k)\end{aligned} \tag{9}$$

where $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Then,

$$\begin{aligned}\dot{\bar{z}} =& \bar{f}(\bar{z},e,\mu)\\ \dot{\bar{\eta}} =& M\bar{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z},e,\mu)\\ \dot{e} =& \bar{g}(\bar{z},e,\mu) + b\Gamma T^{-1}\bar{\eta} + \Gamma T^{-1}Ne + b\bar{u} + b\Gamma T^{-1}\tilde{\eta}\end{aligned} \tag{10}$$

where $\mu = \mathrm{col}(v,w)$ and

$$\begin{aligned}\bar{f}(\bar{z},e,\mu) =& f(\bar{z}+\mathbf{z}, e+q, v, w) - f(\mathbf{z},q,v,w)\\ \bar{g}(\bar{z},e,\mu) =& g(\bar{z}+\mathbf{z}, e+q, v, w) - g(\mathbf{z},q,v,w).\end{aligned} \tag{11}$$

It can be verified that for all $\mu \in \mathbb{R}^{n_v\times n_w}$, functions $\bar{f}(\cdot)$ and $\bar{g}(\cdot)$ vanish at their origin, respectively.

Consider a control law of the following form:

$$\bar{u}(t) = \varsigma(e(t_k)), t \in [t_k, t_{k+1}), k \in \mathbb{S} \tag{12}$$

where $\varsigma(\cdot)$ is a smooth function vanishing at the origin. Let $\bar{x}_c(t) = \mathrm{col}(\bar{z}(t), \bar{\eta}(t), e(t), \gamma(t))$ denote the solution of the closed-loop system composed of (4), (10) and (12) and $\Omega = \mathbb{V}\times\mathbb{W}$. Now, we are ready to show the problem conversion by the following proposition.

Proposition 3.1: Under Assumptions 3.1-3.3, if for any compact subset Ω, there exists a control law of the form (12), such that for any initial condition $\bar{x}_c(0)$ and for all $\mu \in \Omega$, $\bar{x}_c(t)$ exists and is bounded for all $t \in [0,\infty)$, and $\lim_{t\to\infty}\bar{x}_c(t) = 0$ asymptotically, then Problem 2.1 can be solved by the dynamic output feedback control law:

$$\begin{aligned}u(t) =& \varsigma(e(t_k)) + \Gamma T^{-1}\eta(t_k)\\ \dot{\eta}(t) =& M\eta(t) + Nu(t), t \in [t_k, t_{k+1}), k \in \mathbb{N}\end{aligned} \tag{13}$$

together with the dynamic event-triggered mechanism of the form (4).

Proof: Since $\bar{x}_c(t) = \mathrm{col}(\bar{z}(t), \bar{\eta}(t), e(t), \gamma(t))$, then, $\lim_{t\to\infty}\bar{x}_c(t) = 0$ asymptotically implies $\lim_{t\to\infty} e(t) = 0$ asymptotically. That is, Property 2 in Problem 2.1 is satisfied.

Next, denote $x_c(t) = \mathrm{col}(z(t), \eta(t), y(t), \gamma(t))$. Under Assumptions 3.1, since $\mathbf{z}(v,w), b(w), \vartheta(v,w)$ and $q(v,w)$ are all smooth functions of their arguments, $\mathbf{z}(v,w), b(w), \vartheta(v,w)$ and $q(v,w)$ are all bounded for any compact set Ω and for all $t \in [0,\infty)$. By (9), $x_c(t)$ satisfies

$$\begin{aligned}x_c(t) =& \bar{x}_c(t) + \mathrm{col}(\mathbf{z}(v(t),w), T\vartheta(v(t),w)\\ &+ Nb^{-1}(w)e(t), q(v(t),w), 0).\end{aligned} \tag{14}$$

Since $\lim_{t\to\infty}\bar{x}_c(t) = 0$, we have that $x_c(t)$ exists and is bounded for all $t \in [0,\infty)$, which means Property 1 in Problem 2.1 holds. The proof is thus completed. □

IV. Main Result

In this section, we solve the event-triggered global robust stabilization problem of the augmented system (10).

First, design the following output feedback control law:

$$\bar{u}(t) = \zeta(t_k), t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{15}$$

where

$$\zeta(t) = -\xi(e(t))e(t) \tag{16}$$

and $\xi(\cdot)$ is a sufficiently smooth positive function to be specified later. Define

$$\tilde{\zeta}(t) = \zeta(t_k) - \zeta(t). \tag{17}$$

We develop the dynamic event-triggered mechanism as follows:

$$\begin{aligned}t_{k+1} =& \inf\Big\{t > t_k \Big| \theta\big(\tilde{\zeta}(t) + \Gamma T^{-1}\tilde{\eta}(t)\big)^2\\ &- \theta\sigma|e(t)\zeta(t)| \geq \gamma(t)\Big\}\\ \dot{\gamma}(t) =& -\beta\gamma(t) - \alpha\big(\tilde{\zeta}(t) + \Gamma T^{-1}\tilde{\eta}(t)\big)^2\\ &+ \alpha\sigma|e(t)\zeta(t)|,\ \gamma(0) > 0\end{aligned} \tag{18}$$

where θ is any positive number, β, α and σ are some positive constants to be specified later. Then, the closed-loop system composed of (10), (15) and (18) can be written as:

$$\begin{aligned}\dot{\bar{z}} =& \bar{f}(\bar{z},e,\mu)\\ \dot{\bar{\eta}} =& M\bar{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z},e,\mu)\\ \dot{e} =& \bar{g}(\bar{z},e,\mu) + b\Gamma T^{-1}\bar{\eta} + \Gamma T^{-1}Ne + b\zeta + b\tilde{\zeta} + b\Gamma T^{-1}\tilde{\eta}\\ \dot{\gamma} =& -\beta\gamma - \alpha\big(\tilde{\zeta} + \Gamma T^{-1}\tilde{\eta}\big)^2 + \alpha\sigma|e\zeta|,\ \gamma(0) > 0.\end{aligned} \tag{19}$$

To solve the stabilization problem of system (10), we need to introduce one more assumption.

Assumption 4.1: For any compact subset $\Omega \subseteq \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$, there exists a C^1 function $V_1(\bar{z})$ such that, for any $\mu \in \Omega$, any $\bar{z}$ and any e

$$\begin{aligned}\underline{\alpha}_1(\|\bar{z}\|) \leq& V_1(\bar{z}) \leq \overline{\alpha}_1(\|\bar{z}\|)\\ \frac{\partial V_1(\bar{z})}{\partial \bar{z}}\bar{f}(\bar{z},e,\mu) \leq& -\alpha_1(\|\bar{z}\|) + \varpi_1(e)\end{aligned} \tag{20}$$

where $\underline{\alpha}_1(\cdot)$ and $\overline{\alpha}_1(\cdot)$ are some class $\mathcal{K}_\infty$ functions, $\varpi_1(\cdot)$ is a known smooth positive definite function, and $\alpha_1(\cdot)$ is a known class $\mathcal{K}_\infty$ function satisfying $\lim_{s\to 0^+}\sup\big(s^2/\alpha_1(s)\big) < \infty$.

Remark 4.1: Assumption 4.1 is a standard assumption and is used in relevant results, see [17], [26], [27]. Assumption 4.1 implies that for any $\mu \in \Omega$, the subsystem $\dot{\bar{z}} = \bar{f}(\bar{z},e,\mu)$ is input-to-state stable (ISS) with respect to state $\bar{z}$ and continuous-time input e.

Lemma 4.1: Under Assumptions 3.1-3.3 and 4.1, let $\theta > 0$, $0 < \sigma < \frac{b_m}{b_M^2}$, $0 < \alpha < b_M^2$, and $\beta \geq \frac{b_M^2-\alpha}{\theta} + \iota$, where ι is any positive real number. Then, there exists a smooth positive function $\xi(\cdot)$, a C^1 function $U(\bar{z},\bar{\eta},e,\gamma)$ that satisfies

$$\underline{\beta}(\|(\bar{z},\bar{\eta},e,\gamma)\|) \leq U(\bar{z},\bar{\eta},e,\gamma) \leq \overline{\beta}(\|(\bar{z},\bar{\eta},e,\gamma)\|) \tag{21}$$

for some class $\mathcal{K}_\infty$ functions $\underline{\beta}(\cdot)$ and $\overline{\beta}(\cdot)$, such that, for all $\mu \in \Omega$, along the trajectory of (19),

$$\dot{U}(\bar{z},\bar{\eta},e,\gamma) \leq -\|\bar{z}\|^2 - \|\bar{\eta}\|^2 - e^2 - \iota\gamma. \tag{22}$$

Proof: 1) Consider the $\bar{z}$ subsystem in (19). Under Assumption 4.1, by the changing supply function technique in [32], given any smooth function $\Delta(\bar{z}) > 0$, there exists a C^1 function $\bar{V}_1(\bar{z})$ such that, for any $\bar{z} \in \mathbb{R}^n$, $e \in \mathbb{R}$ and $\mu \in \Omega$,

$$\begin{aligned}\underline{\varrho}_1(\|\bar{z}\|) \leq& \bar{V}_1(\bar{z}) \leq \overline{\varrho}_1(\|\bar{z}\|)\\ \frac{\partial \bar{V}_1(\bar{z})}{\partial \bar{z}}\bar{f}(\bar{z},e,\mu) \leq& -\Delta(\bar{z})\|\bar{z}\|^2 + \pi(e)e^2\end{aligned} \tag{23}$$

where $\underline{\varrho}_1(\cdot)$ and $\overline{\varrho}_1(\cdot)$ are some class $\mathcal{K}_\infty$ functions, and $\pi(\cdot)$ is a known smooth positive function.

2) Consider the $\bar{\eta}$ subsystem in (19). Since $\bar{g}(\bar{z},e,\mu)$ is a smooth function and $\bar{g}(0,0,\mu) = 0$ for all $\mu \in \Omega$, by Lemma 7.8 in [29], there exist smooth positive functions $\varphi(\cdot)$ and $\chi(\cdot)$ such that for all $\bar{z} \in \mathbb{R}^n$, $e \in \mathbb{R}$ and $\mu \in \Omega$,

$$|\bar{g}(\bar{z},e,\mu)|^2 \leq \varphi(\bar{z})\|\bar{z}\|^2 + \chi(e)e^2. \tag{24}$$

Let $\bar{V}_2(\bar{\eta}) = \ell\bar{\eta}^T P_1\bar{\eta}$, where $\ell > 0$ is a positive real number to be defined and P_1 is the positive definite solution to the Lyapunov equation $M^\top P_1 + P_1 M = -I$. Then, for all $\mu \in \Omega$, along the trajectory of the $\bar{\eta}$ subsystem, as $b_m \leq b(w) \leq b_M$, it follows from (24) that

$$\begin{aligned}\dot{\bar{V}}_2(\bar{\eta}) =& -\ell\|\bar{\eta}\|^2 + 2\ell\bar{\eta}^T P_1 MNb^{-1}e\\ &- 2\ell\bar{\eta}^T P_1 Nb^{-1}\bar{g}(\bar{z},e,\mu)\\ \leq& -(\ell - \frac{1}{2})\|\bar{\eta}\|^2 + \frac{4\ell^2}{{b_m}^2}\|P_1N\|^2\varphi(\bar{z})\|\bar{z}\|^2\\ &+ \Big(\frac{4\ell^2}{{b_m}^2}\|P_1MN\|^2 + \frac{4\ell^2}{{b_m}^2}\|P_1N\|^2\chi(e)\Big)e^2.\end{aligned} \tag{25}$$

Let $V_2(\bar{z},\bar{\eta}) = \bar{V}_1(\bar{z}) + \bar{V}_2(\bar{\eta})$. Then, it follows from (23) and (25) that

$$\begin{aligned}&\dot{V}_2(\bar{z},\bar{\eta})\\ \leq& -\Big(\Delta(\bar{z}) - \frac{4\ell^2}{{b_m}^2}\|P_1N\|^2\varphi(\bar{z})\Big)\|\bar{z}\|^2 - (\ell - \frac{1}{2})\|\bar{\eta}\|^2\\ &+ \Big(\pi(e) + \frac{4\ell^2}{{b_m}^2}\|P_1MN\|^2 + \frac{4\ell^2}{{b_m}^2}\|P_1N\|^2\chi(e)\Big)e^2.\end{aligned} \tag{26}$$

3) Consider the (e,γ) subsystem in (19). Let $\tilde{g}(\bar{z},\bar{\eta},e,\mu) = \bar{g}(\bar{z},e,\mu) + b\Gamma T^{-1}\bar{\eta} + \Gamma T^{-1}Ne$. Then, it follows from (24) that

$$\begin{aligned}&e\tilde{g}(\bar{z},\bar{\eta},e,\mu)\\ =& e\big(\bar{g}(\bar{z},e,\mu) + b\Gamma T^{-1}\bar{\eta} + \Gamma T^{-1}Ne\big)\\ \leq& \varphi(\bar{z})\|\bar{z}\|^2 + \frac{1}{4}\|\bar{\eta}\|^2\\ &+ \Big(\frac{1}{4} + \chi(e) + b_M^2\|\Gamma T^{-1}\|^2 + \|\Gamma T^{-1}N\|\Big)e^2.\end{aligned} \tag{27}$$

Under the dynamic event-triggered mechanism (18), it follows that

$$\big(\tilde{\zeta} + \Gamma T^{-1}\tilde{\eta}\big)^2 - \sigma|e\zeta| \leq \frac{1}{\theta}\gamma \tag{28}$$

which implies

$$\dot{\gamma} \geq -(\beta + \frac{\alpha}{\theta})\gamma. \tag{29}$$

Thus, by Comparison Lemma in [33] and noting $\gamma(0) > 0$, it is obtained that

$$\gamma(t) \geq \gamma(0)e^{-(\beta+\frac{\alpha}{\theta})t} > 0, t \geq 0. \tag{30}$$

Let $V_3(e,\gamma) = \frac{1}{2}e^2 + \gamma$. For all $t_k \leq t < t_{k+1}, k \in \mathbb{S}$, and for all $\mu \in \Omega$, by (16), we obtain that the derivative of $V_3(e,\gamma)$ along the trajectory of the (e,γ) subsystem satisfies

$$\begin{aligned}&\dot{V}_3(e,\gamma)\\ =& e\Big(\tilde{g}(\bar{z},\bar{\eta},e,\mu) + b\zeta + b\tilde{\zeta} + b\Gamma T^{-1}\tilde{\eta}\Big) + \dot{\gamma}\\ \leq& e\tilde{g}(\bar{z},\bar{\eta},e,\mu) - b_m\xi(e)e^2 + b_M|e||\tilde{\zeta} + \Gamma T^{-1}\tilde{\eta}|\\ &- \beta\gamma - \alpha\big(\tilde{\zeta} + \Gamma T^{-1}\tilde{\eta}\big)^2 + \alpha\sigma|e\zeta|\\ \leq& e\tilde{g}(\bar{z},\bar{\eta},e,\mu) - \Big((b_m - b_M^2\sigma)\xi(e) - \frac{1}{4}\Big)e^2\\ &- \beta\gamma + (b_M^2 - \alpha)\big((\tilde{\zeta} + \Gamma T^{-1}\tilde{\eta})^2 - \sigma|e\zeta|\big).\end{aligned} \tag{31}$$

By (27) and (28), it follows that

$$\begin{aligned}
&\dot{V}_3(e,\gamma)\\
&\leq \varphi(\bar{z})||\bar{z}||^2 + \frac{1}{4}||\bar{\eta}||^2\\
&\quad - \Big((b_m - b_M^2\sigma)\xi(e) - \frac{1}{2} - \chi(e) - b_M^2||\Gamma T^{-1}||^2\\
&\quad - ||\Gamma T^{-1}N||\Big)e^2 - \Big(\beta - \frac{b_M^2 - \alpha}{\theta}\Big)\gamma.
\end{aligned} \tag{32}$$

4) Consider the overall closed-loop system (19). Denote $U(\bar{z},\bar{\eta},e,\gamma) = V_2(\bar{z},\bar{\eta}) + V_3(e,\gamma)$. Then, by (26) and (32), for all $\mu \in \Omega$, along the trajectory of (19),

$$\begin{aligned}
&\dot{U}(\bar{z},\bar{\eta},e,\gamma)\\
&\leq -\big(\Delta(\bar{z}) - k_1\varphi(\bar{z})\big)||\bar{z}||^2 - (\ell - \frac{3}{4})||\bar{\eta}||^2\\
&\quad - \Big((b_m - b_M^2\sigma)\xi(e) - \pi(e) - k_1\chi(e) - k_2\Big)e^2\\
&\quad - (\beta - \frac{b_M^2 - \alpha}{\theta})\gamma
\end{aligned} \tag{33}$$

where $k_1 = 1 + \frac{4\ell^2}{b_m{}^2}||P_1N||^2$, $k_2 = \frac{1}{2} + \frac{4\ell^2}{b_m{}^2}||P_1MN||^2 + b_M^2||\Gamma T^{-1}||^2 + ||\Gamma T^{-1}N||$. Choose $\ell \geq \frac{7}{4}$, $\Delta(\bar{z}) \geq k_1\varphi(\bar{z}) + 1$ and $\xi(e) \geq \frac{1}{b_m - b_M^2\sigma}(\pi(e) + k_1\chi(e) + k_2 + 1)$. Then,

$$\dot{U}(\bar{z},\bar{\eta},e,\gamma) \leq -||\bar{z}||^2 - ||\bar{\eta}||^2 - e^2 - \iota\gamma. \tag{34}$$

The proof is thus complete. □

Based on Proposition 3.1 and Lemma 4.1, we can obtain the solution to the event-triggered global robust output regulation problem by the following theorem.

Theorem 4.1: Under Assumptions 3.1-3.3 and 4.1, the event-triggered global robust output regulation problem for the system consisting of (1) and (2) can be solved by the following dynamic output feedback control law:

$$\begin{aligned}
u(t) =& -\xi(e(t_k))e(t_k) + \Gamma T^{-1}\eta(t_k)\\
\dot{\eta}(t) =& M\eta(t) + Nu(t), t \in [t_k, t_{k+1}), k \in \mathbb{N}
\end{aligned} \tag{35}$$

together with the dynamic event-triggered mechanism (18), where $\xi(\cdot)$, θ, β, α and σ are defined in Lemma 4.1. Moreover, the Zeno behavior can be strictly excluded.

Proof: Due to space limit, we give a sketch of the proof.

1) By Lemma 4.1, we have that $\bar{z}(t), \bar{\eta}(t), e(t)$ and $\gamma(t)$ are all bounded over $[0, T_M)$.

2) Show that there is no Zeno behavior by contradiction. Suppose that there exists Zeno behavior, i.e., $\lim_{k\to\infty} t_k = T_0$, $k \in \mathbb{S}$, for some positive constant T_0. Then, by the definition of limits of sequences, for some positive constant ϵ_0, there exists an integer $\bar{k}(\epsilon_0)$ such that, for all $k > \bar{k}(\epsilon_0)$,

$$T_0 - \epsilon_0 < t_k \leq T_0. \tag{36}$$

Let $\hat{k} \geq \bar{k}(\epsilon_0)$ and $\hat{k} \in \mathbb{S}$. We can show that $t_{\hat{k}+1} - t_{\hat{k}} \geq \epsilon_0$ which contradicts (36). Therefore, Zeno behavior is excluded. Consider Case 3) in Remark 2.1. In this case, the closed-loop system reduces to a continuous-time system for $t > t_{k^*}$. Then, by Lemma 4.1, for any initial states $\bar{x}_c(0)$, and any $\mu \in \Omega$, the solution $\bar{x}_c(t)$ exists for all $t \in [0,\infty)$.

3) By Lemma 4.1, we have that $\lim_{t\to\infty}\bar{x}_c(t) = 0$. The proof is thus complete by invoking Proposition 3.1. □

V. Example

Consider a class of uncertain Lorenz systems with the following dynamics [17], [26]:

$$\begin{aligned}
\dot{z}_1 =& (-4 + w_1)z_1 - (-4 + w_1)y\\
\dot{z}_1 =& (-5 + w_2)z_2 + z_1y\\
\dot{y} =& (2 + w_3)z_1 - y - z_1z_2 + (3 + w_4)u\\
e =& y - v_1
\end{aligned} \tag{37}$$

where $w = [w_1, w_2, w_3, w_4]^\top$ represents the uncertainty of the system. It is assumed that $w \in \mathbb{W} = \{w | w \in \mathbb{R}^4, |w_i| \leq 1, i = 1,2,3,4\}$. The exosystem takes the following form:

$$\dot{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} v \tag{38}$$

and $v \in \mathbb{V} = \{v | v \in \mathbb{R}^2, |v_i| \leq 1, i = 1,2\}$. Then, Assumption 3.1 is satisfied.

Like in [26], we can verify that Assumptions 3.2 and 3.3 are satisfied. We can further verify that $\frac{d^4\mathbf{u}(v,w)}{dt^4} + 10\frac{d^2\mathbf{u}(v,w)}{dt^2} + 9\mathbf{u}(v,w) = 0$. By Remark 3.3, we have that $\Phi = \left[\begin{array}{c|c} 0_{3\times1} & I_3 \\ \hline -9 & 0 \;\; -10 \;\; 0 \end{array}\right]$, $\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top$. The controllable pair (M,N) is chosen as follows: $M = \left[\begin{array}{c|c} 0_{3\times1} & I_3 \\ \hline -6 & -17 \;\; -17 \;\; -7 \end{array}\right]$, $N = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top$. Then, by solving the Sylvester equation $T\Phi - MT = N\Gamma$, we have that $\Gamma T^{-1} = [-3, 17, 7, 7]$. According to [26], it can be verified that Assumption 4.1 is also satisfied. Therefore, by Theorem 4.1, choosing $\iota = 0.2$, we can design the dynamic output feedback control law of the form (35) with $\xi(e(t_k)) = 6(e^6(t_k) + 1)$, and the dynamic event-triggered mechanism of the form (18) with $\theta = 600, \sigma = 1 \times 10^{-4}, \alpha = 6$ and $\beta = 0.5$. The simulation is performed with $w = [0.5, -0.6, 0.7, -0.3]^\top$, $z_1(0) = -1.61$, $z_2(0) = 0.45$, $y(0) = -1.68, v(0) = v(0) = [-0.94, 0.10]^\top$, $\eta(0) = [0.49, -0.31, 0.13, 0.20]^\top$ and $\gamma(0) = 50$.

The simulation results are shown in Figs. 1-3. The event-triggered condition and the inter-event time under the dynamic event-triggered mechanism (18) are depicted in Fig. 1 and Fig. 2, respectively. It can be observed from Fig. 3 that the tracking error approaches zero asymptotically, which means that the global robust output regulation problem is solved exactly.

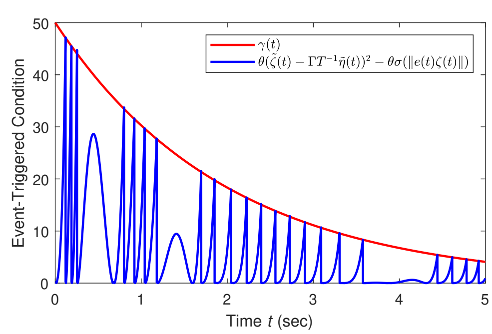

Fig. 1. Event-triggered condition.

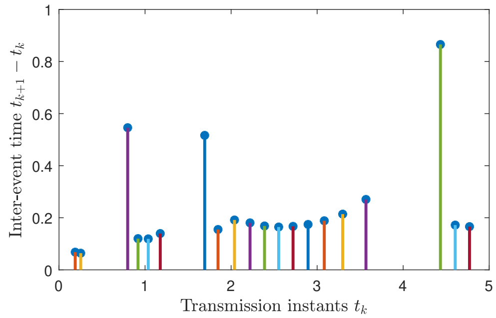

Fig. 2. Inter-event time of the dynamic event-triggered mechanism.

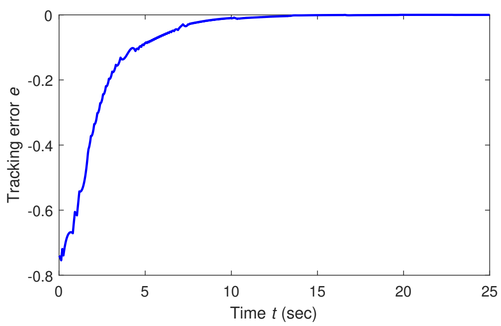

Fig. 3. Tracking error e.

VI. Conclusion

In this paper, the event-triggered global robust output regulation problem of a class of nonlinear uncertain systems has been addressed. Based on the internal model principle, a dynamic output-based event-triggered feedback control law together with a dynamic event-triggered mechanism has been developed. It has been shown that the global robust output regulation problem can be solved exactly, and the Zeno behavior can be explicitly excluded.

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