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Communication-efficient Allocation of Multiple Indivisible Resources in a Federated Multi-agent System

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***Abstract*— A federated multi-agent system is a multi-agent system wherein agents collaborate with a central server to optimize system goals without sharing their private information. We develop a communication-efficient solution to resource allocation problems for a population of agents coupled through multiple indivisible shared resources in a federated multi-agent system. The agents demand resources in a probabilistic way based on their local computation and preferences, and the agents receive either one unit of a resource or do not receive it. The agents are not required to share their cost functions or derivatives of cost functions with other agents or the central server. Optimal control of a population of such agents, subject to capacity constraints, is widely found in many application domains, such as smart energy systems, intelligent transportation systems, and edge computing, to name a few. We present convergence results using multi-time scale stochastic approximation techniques and an example of electric vehicle charging point allocation illustrating the efficacy of the developed solution.**

Keywords: Distributed optimization, optimal control, multi-resource allocation, indivisible resources, smart city, electric vehicle charging.

I. Introduction

We define a *federated multi-agent system* as a multi-agent system wherein agents collaborate with a central server to optimize system goals without sharing their private information [1]. This paper considers a federated multi-agent system consisting of several agents and a central server. The agents collaborate with the central server to solve a multi-resource allocation problem and aim to achieve social welfare for the system. In several applications in smart energy systems, intelligent transportation systems, edge computing, etcetera, a population of agents such as distributed energy resources (DERs), electric vehicles (EVs), virtual machines (VMs), or IoT devices should be controlled to access limited shared resources. These agents are not required to share private information with other agents in the system. It is challenging to solve such allocation problems that minimize the cost to the system, satisfy constraints, and provide a quality of service guarantee to each agent in the federated multi-agent system. For example, managing the distributed energy resources and loads on smart grids to achieve social optimum cost [2], or controlling the electric vehicle charging [3], [4], in these examples, resource utilization should be maximized, and also a certain quality of service should be guaranteed to each agent. These problems are formulated as optimal control problems to control the agents' population in the system [5]–[7].

This paper considers the problem of controlling a population of agents coupled through multiple indivisible shared resources in a federated multi-agent system; each agent demands the indivisible resources in a stochastic way based on their local computation, and the agents do not share their cost function or derivative of the cost function with other agents or the central server in the system. This work is a novel extension of [8] in which the optimal control of a population of agents is considered for a single indivisible resource. Controlling a population of agents, which demand resources in a *stochastic way*, is widely found in many application areas, as stated above. In these applications, the probabilistic intent of agents can be modeled to optimize system-level goals. For example, when a population of electric vehicles demands different types of charging points, such as level 1 (slow charger) or level 2 (faster charger) charging points, to minimize voltage fluctuation or minimize the cost to the network, their objective functions depend on the consumption of both types of charging points. Generally speaking, in these scenarios, agents are coupled through the allocation of multiple indivisible resources. Note that *indivisible resources* are either allocated one unit or not allocated. The developed solution is also suitable for client selection in federated learning, wherein clients collaborate with a server to train a global model without sharing their on-device data [9], [10]. The server selects a subset of clients and sends them a global model; the clients train the model on their device data and send the learned parameters to the server. The server then aggregates the parameters and adjusts the weights to update the global model. This process is repeated until the training loss converges or time exceeds a set limit. The server selects the subset of clients at a time step to participate in the training process without considering clients' preferences [9]. Our solution can be used to incorporate the choices and preferences of clients to participate in the training process, and on average, the number of times clients participate will reach optimal value. Interested readers may refer to our recent work on differentially private client selection in federated settings at [11].

Our main contribution to this paper is to develop a stochastic solution for a federated multi-agent system in which several agents are coupled with multiple indivisible shared resources. Each agent demands the shared resources in a probabilistic way based on its private cost function, derivative of the cost function, etc. This approach is a novel extension of the single indivisible resource allocation solution proposed

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in [8]. We show almost sure convergence of the average allocations based on the ideas from multi-time scale stochastic approximation techniques [12]. To check the efficacy of our algorithm, we present an application to control a population of electric vehicles that share a limited number of level 1 and level 2 charging points. It illustrates that the agents receive the optimal charging points in long-term averages and maximize the social welfare of the system. The agents do not need to communicate among themselves to achieve social optimum cost; however, the central server keeps track of the aggregate utilization of resources and broadcasts price (feedback) signals at each time step. Using the value of the price signal, agents calculate their probabilistic intent to demand resources at the next time step. Assuming each price signal is of μ bits floating point, then for a system with m shared resources, the communication overhead will be μm bits per time step. Furthermore, the upper bound on the communication complexity is $\mathcal{O}(m)$-bits per time step, which is independent of the number of agents in the system.

II. Preliminaries

Suppose there are n agents collaborating with a central server in a federated multi-agent system; the agents are coupled through m indivisible shared resources. Each agent has a cost function that depends on the allocation of these shared indivisible resources. Let the capacity of resources be $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$, respectively.

Let $X_{ji}(k) \in \{0, 1\}$ denote an independent Bernoulli random variable representing the instantaneous allocation of resource j of agent i at time step k. Furthermore, let $y_{ji}(k) \in [0, 1]$ denote the average allocation of resource j of agent i at time step k. We define $y_{ji}(k)$ as follows,

$$y_{ji}(k) \triangleq \frac{1}{k+1} \sum_{\ell=0}^{k} X_{ji}(\ell), \tag{1}$$

for $i = 1, 2, \dots, n$, and $j = 1, 2, \dots, m$. Let $[\mathbf{y}]_i \in [0, 1]^m$ denote $(y_{1i}, y_{2i}, \dots, y_{mi})$ and $\mathbf{y}_j \in [0, 1]^n$ denote $(y_{j1}, y_{j2}, \dots, y_{jn})$. Additionally, let $\mathbf{y} \in ([0, 1]^n)^m$ denote $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m)$. Note that we use bold letters to denote vectors. Moreover, agent i has a cost function f_i that associates a cost to the amount of allocated shared resources to that agent. We make the following assumption for the cost function.

Assumption 2.1: The cost function $f_i : [0, 1]^m \to \mathbb{R}_+$ is twice continuously differentiable, strictly convex, and increasing in all variables, for $i = 1, 2, \dots, n$.

As we work with the average allocation of indivisible resources, we formulate the multi-resource allocation problem as the following optimization problem:

$$\begin{aligned} \min_{y_{11}, \dots, y_{mn}} \quad & \sum_{i=1}^{n} f_i(y_{1i}, \dots, y_{mi}), \\ \text{subject to} \quad & \sum_{i=1}^{n} y_{1i} = \mathcal{C}_1; \dots; \sum_{i=1}^{n} y_{mi} = \mathcal{C}_m, \\ & y_{1i} \in [0, 1]; \dots; y_{mi} \in [0, 1], \quad i = 1, 2, \dots, n. \end{aligned} \tag{2}$$

Let $\mathbf{y}^* = (y^*_{11}, \dots, y^*_{mn}) \in ((0, 1]^n)^m$ denote the solution to (2). Let $\mathbb{N}$ denote the set of natural numbers, and let $k \in \mathbb{N}$ denote the time steps. Our objective is to develop an iterative stochastic algorithm wherein agents share their states with a central server; however, they keep their cost function or partial derivatives of the cost function private. The algorithm determines the instantaneous allocations $X_{ji}(k)$ and ensures that the long-term average allocations, as defined in (1) converge to optimal allocations, that is,

$$\lim_{k \to \infty} y_{11}(k) = y^*_{11}; \dots; \lim_{k \to \infty} y_{mn}(k) = y^*_{mn}.$$

Hence, agents achieve the minimum overall cost over long-term averages. By compactness of the constraint set, optimal solutions exist. The assumption that the cost function f_i is strictly convex and increasing leads to a unique optimal solution. After finding the Lagrangian of the optimization problem (2) and following a similar analysis as [6], [13], [14], we find that the partial derivatives of the cost functions of all agents competing for a particular resource reach consensus at the optimal point. That is, for $i, u \in \{1, 2, \dots, n\}$ and $j = 1, 2, \dots, m$, the following holds:

$$\begin{aligned} & \frac{\partial}{\partial y_{ji}} f_i(y_{1i}, \dots, y_{mi}) \Big|_{y_{ji} = y^*_{ji}} \\ & \quad = \frac{\partial}{\partial y_{ju}} f_u(y_{1u}, \dots, y_{mu}) \Big|_{y_{ju} = y^*_{ju}}. \end{aligned} \tag{3}$$

We use the consensus of derivatives to show that the proposed algorithm reaches optimal values asymptotically. The consensus of derivatives of cost functions is also used in [6], [15], [16] to show the convergence of allocations to optimal values.

III. Allocating multiple indivisible resources

This section presents multi-indivisible resource allocation algorithm that generalizes the single indivisible resource algorithm of [8]; however, we chose a different update rule for the feedback (public) signal, and the agents are coupled through multiple shared resources.

Each agent in the federated multi-agent system runs the algorithm to demand resources. Let $\tau_j \in (0, 1)$ be the gain parameter, and let $\Theta_j(k)$ denote a feedback signal updated and broadcast by the central server; we also call it the *price or public signal*. The central server updates $\Theta_j(k)$ according to (5) at each time step k and broadcasts it to all agents in the federated multi-agent system, for all j. When an agent joins the system at time step k, it receives the parameter $\Theta_j(k)$ for resource j, for all j. Each agent's algorithm updates its resource demand at a time step—either by demanding one unit of the resource or not demanding it. The price signal $\Theta_j(k)$ depends on its value at the previous time step, τ_j, capacity constraint $\mathcal{C}_j$, and the total utilization of resource j at the previous time step, for all j and k. After receiving this signal, agent i's algorithm responds in a probabilistic way. It calculates its probability $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$ using its average allocation $[\mathbf{y}]_i(k)$ of resource j and the derivative of its cost function, for all j and k, as described in (6). Agent i finds out the outcome of Bernoulli trial for resource j at time step k, outcome 1 occurs with probability $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$ and outcome 0 with probability $1 - \sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$; based on the value 0 or 1, the algorithm decides whether to demand one unit of the resource j or not. If the value is 1, then the algorithm demands one unit of the resource; otherwise, it does not demand the resource, as stated below.

$$X_{ji}(k+1) = \begin{cases} 1 & \text{with probability } \sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)); \\ 0 & \text{with probability } 1 - \sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)). \end{cases} \tag{4}$$

Analogously, it is done for all the resources in the federated multi-agent system. This process repeats over time. Following this, the average allocations converge to optimal allocations. The proposed *multi-indivisible resource allocation* algorithm for the central server is presented in Algorithm 1, and the algorithm for each agent is presented in Algorithm 2.

Algorithm 1: Algorithm of the central server.

Input: $\mathcal{C}_1, \ldots, \mathcal{C}_m$, $\tau_1, \ldots, \tau_m$, $X_{11}(k), \ldots, X_{mn}(k)$, for $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$.
Output: $\Theta_1(k+1), \Theta_2(k+1), \ldots, \Theta_m(k+1)$, for $k \in \mathbb{N}$.
Initialization: $\Theta_1(0)$, $\Theta_2(0), \ldots, \Theta_m(0)$ with real values,
foreach $k \in \mathbb{N}$ **do**
  foreach $j \in \{1, 2, \ldots, m\}$ **do**
    calculate $\Theta_j(k+1)$ according to (5) and broadcast it in the federated multi-agent system;
  end
end

We choose a *gain parameter* τ_j, a small positive real number in $(0, 1)$. The public signal $\Theta_j(k+1)$ depends on the utilization of resources at a time step; for resource j and time step k, it is defined as follows:

$$\Theta_j(k+1) \triangleq \Theta_j(k) - \frac{\tau_j}{(k+1)^{\frac{2}{3}}} \left(\sum_{i=1}^{n} X_{ji}(k+1) - \mathcal{C}_j \right), \tag{5}$$

$j = 1, 2, \ldots, m$ and $k \in \mathbb{N}$. After receiving the price signal $\Theta_j(k)$ from the central server at time step k, agent i responds with probability $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$ to demand resource j at next time step, defined as:

$$\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)) \triangleq \Theta_j(k) \frac{y_{ji}(k)}{\frac{\partial}{\partial y_{ji}} f_i([\mathbf{y}]_i(k))\Big|_{y_{ji}=y_{ji}(k)}}. \tag{6}$$

Notice that $\Theta_j(k)$ is used to bound the probability $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)) \in (0, 1)$, for all i, j and k. Furthermore, the agents update probabilities $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$ at time step k before demanding the resources at the next time step;

Algorithm 2: Multi resource allocation algorithm of agent i.

Input: $\Theta_1(k), \Theta_2(k), \ldots, \Theta_m(k)$, for $k \in \mathbb{N}$.
Output: $X_{1i}(k+1), X_{2i}(k+1), \ldots, X_{mi}(k+1)$, for $k \in \mathbb{N}$.
Initialization: $X_{ji}(0) \leftarrow 1$ and $y_{ji}(0) \leftarrow X_{ji}(0)$, for $j \in \{1, 2, \ldots, m\}$.
foreach $k \in \mathbb{N}$ **do**
  foreach $j \in \{1, 2, \ldots, m\}$ **do**
    $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)) \leftarrow \Theta_j(k) \frac{y_{ji}(k)}{\frac{\partial}{\partial y_{ji}} f_i([\mathbf{y}]_i(k))\Big|_{y_{ji}=y_{ji}(k)}}$;
    generate Bernoulli independent random variable $b_{ji}(k)$ with the parameter $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$;
    if $b_{ji}(k) = 1$ **then**
      $X_{ji}(k+1) \leftarrow 1$;
    else
      $X_{ji}(k+1) \leftarrow 0$;
    end
  end
end

the occurrence of an instantaneous allocation $X_{ji}(k+1)$ will be independent of the occurrence of the instantaneous allocations $X_{\nu i}(k+1)$ of other resources in the federated multi-agent system, for resources $j, \nu \in \{1, 2, \ldots, m\}$, $j \neq \nu$.

Following the algorithm, the long-term average allocations converge to optimal allocations. Let $\boldsymbol{X}_j(k) = (X_{j1}(k), \ldots, X_{jn}(k)) \in \{0, 1\}^n$ and $\mathbf{y}_j(k) \in [0, 1]^n$ denote the vectors with entries $X_{ji}(k), y_{ji}(k)$, respectively, and $\boldsymbol{\sigma}_j(\Theta_j(k), \mathbf{y}(k))$ denotes the vector with entries $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k = 0, 1, 2, \ldots$

For $j = 1, 2, \ldots, m$, we reformulate the average allocation $\mathbf{y}_j(k)$ as:

$$\mathbf{y}_j(k+1) = \frac{k}{k+1} \mathbf{y}_j(k) + \frac{1}{k+1} \boldsymbol{X}_j(k+1). \tag{7}$$

Which may be reformulated as follows:

$$\begin{aligned} &\mathbf{y}_j(k+1) \\ &= \mathbf{y}_j(k) + \frac{1}{k+1} \left[(\boldsymbol{\sigma}_j(\Theta_j(k), \mathbf{y}(k)) - \mathbf{y}_j(k)) \right. \\ &\qquad \left. + \, (\boldsymbol{X}_j(k+1) - \boldsymbol{\sigma}_j(\Theta_j(k), \mathbf{y}(k))) \right]. \end{aligned} \tag{8}$$

Let $(\boldsymbol{X}_j(k+1) - \boldsymbol{\sigma}_j(\Theta_j(k), \mathbf{y}(k)))$ be denoted by $\mathbf{M}_j(k+1)$, and the step-size $\frac{1}{k+1}$ be denoted by $a(k)$, for $k \in \mathbb{N}$. Also, let $(\boldsymbol{\sigma}_j(\Theta_j(k), \mathbf{y}(k)) - \mathbf{y}_j(k))$ be denoted by $\omega_j(\mathbf{y}_j(k))$. After replacing these values in (8), we obtain

$$\mathbf{y}_j(k+1) = \mathbf{y}_j(k) + a(k) \left[\omega_j(\mathbf{y}_j(k)) + \mathbf{M}_j(k+1) \right]. \tag{9}$$

Here, for a fixed j, $\{\mathbf{M}_j(k)\}$ is a *martingale difference sequence* with respect to the σ-algebra.

A. Convergence of average allocations

In this subsection, we show the convergence of the price signal and the average allocation using the stochastic approximation results for multiple timescales [12, Chapter 6]—wherein two positive decreasing step sizes are considered; one step size converges to 0 at a faster rate than the other step size. Examples of such step sizes are, for $k \in \mathbb{N}$, $a(k) = \frac{1}{k}$ and $b(k) = \frac{1}{k^{3/4}}$; notice that $a(k) \to 0$ at a faster rate than $b(k)$. We now state the following result on the convergence for multi-time scales with decreasing step sizes from [12, Chapter 6].

Assumption 3.1: Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n_+$, and let the maps $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. For a fixed constant $\mathbf{x}(0)$ and $\mathbf{z}(0)$, let $\mathbf{x}(k)$ and $\mathbf{z}(k)$ be formulated as follows

$$\mathbf{x}(k+1) = \mathbf{x}(k) + a(k)\big[\omega(\mathbf{x}(k), \mathbf{z}(k)) + \mathbf{M}_1(k+1)\big], \tag{10}$$

$$\mathbf{z}(k+1) = \mathbf{z}(k) + b(k)\big[h(\mathbf{x}(k), \mathbf{z}(k)) + \mathbf{M}_2(k+1)\big]. \tag{11}$$

We have the following assumptions.

(i) The maps ω and h are Lipschitz continuous.

(ii) Step sizes $\{a(k)\}_{k\in\mathbb{N}}$ and $\{b(k)\}_{k\in\mathbb{N}}$ are such that the following are satisfied:

$$\lim_{k\to\infty} a(k) = 0, \lim_{k\to\infty} b(k) = 0,$$
$$\sum_{\ell=0}^{\infty} a(\ell) = \infty, \sum_{\ell=0}^{\infty} b(\ell) = \infty,$$
$$\sum_{\ell=0}^{\infty} a(\ell)^2 + \sum_{\ell=0}^{\infty} b(\ell)^2 < \infty, \text{ and}$$
$$\lim_{k\to\infty} \frac{a(k)}{b(k)} = 0.$$

Thus, $\mathbf{z}(k)$ is a fast transient, and $\mathbf{x}(k)$ is a slow component; $\mathbf{x}(k)$ is quasi-static and it is almost a constant for a large k.

(iii) $\{\mathbf{M}_1(k)\}_{k\in\mathbb{N}}$ and $\{\mathbf{M}_2(k)\}_{k\in\mathbb{N}}$ are martingale difference sequences. Let $\mathcal{F}_k$ be a σ-algebra generated by the events up to time step k, then we have

$$\mathbb{E}\left(\mathbf{M}_1(k+1) \mid \mathcal{F}_k, \text{ for } k \in \mathbb{N}\right) = 0, \text{ and,}$$
$$\mathbb{E}\left(\mathbf{M}_2(k+1) \mid \mathcal{F}_k, \text{ for } k \in \mathbb{N}\right) = 0.$$

(iv) $\sup_k \left(\|\mathbf{x}(k)\|_1 + \|\mathbf{z}(k)\|_1\right) < \infty$ almost surely.

Theorem 3.2 (Convergence with multi-time-scale step sizes): [12, Chapter 6] Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n_+$, and let the maps $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. For a fixed constant $\mathbf{x}(0)$ and $\mathbf{z}(0)$, let $\mathbf{x}(k)$ be formulated as in (10) and $\mathbf{z}(k)$ be formulated as in (11), and Assumptions 3.1 (i) to (iv) are satisfied, then $(\mathbf{x}(k), \mathbf{z}(k))$ converges almost surely.

We now proceed to prove the convergence results for the multi-resource allocation case. The following Hoeffding's result will be useful to show the upper bound of $\Theta_j(k)$.

Theorem 3.3 (Hoeffding's inequality [17]): Let X_i be the independent Bernoulli random variables, and let $\mathbb{E}(X_i)$ be their expectations, then for any $\epsilon > 0$, we have

$$\mathbb{P}\left(\Big|\sum_{i=1}^{n} X_i - \mathbb{E}(\sum_{i=1}^{n} X_i)\Big| \geq \epsilon\right) \leq 2\exp\left(-2\epsilon^2/n\right). \tag{12}$$

We state the following upper bound result for the feedback signal.

Lemma 3.4: For a fixed j, let $\Theta_j(0) = \Theta_{j0} > 0$, and let $\Theta_j(k)$ be as in (5) and $\tau_j > 0$. For any $\epsilon > 0$, we have $\mathbb{P}\left(|(\Theta_j(k+1) - \mathbb{E}(\Theta_j(k+1)|\Theta_j(k))| \geq \epsilon\right) \leq 2\exp\left(\frac{-2((k+1)^{2/3}\epsilon)^2}{\tau_j^2 n}\right)$.

Proof: From (5), we have

$$\Theta_j(k+1) = \Theta_j(k) - \frac{\tau_j}{(k+1)^{2/3}}\left(\sum_{i=1}^{n} X_{ji}(k+1) - \mathcal{C}_j\right).$$

We obtain

$$\begin{aligned}&\mathbb{E}(\Theta_j(k+1)|\Theta_j(k))\\ &= \Theta_j(k) - \frac{\tau_j}{(k+1)^{2/3}}\mathbb{E}\left(\sum_{i=1}^{n} X_{ji}(k+1)\right) + \frac{\tau_j}{(k+1)^{2/3}}\mathcal{C}_j.\end{aligned}$$

Thus,

$$\begin{aligned}&\Theta_j(k+1) - \mathbb{E}\Big(\Theta_j(k+1)|\Theta_j(k)\Big)\\ &= \frac{\tau_j}{(k+1)^{\frac{2}{3}}}\mathbb{E}\Big(\sum_{i=1}^{n} X_{ji}(k+1)\Big) - \frac{\tau_j}{(k+1)^{\frac{2}{3}}}\sum_{i=1}^{n} X_{ji}(k+1).\end{aligned}$$

From Hoeffding's inequality (see Theorem 3.3), we obtain

$$\begin{aligned}&\mathbb{P}\left(|\Theta_j(k+1) - \mathbb{E}(\Theta_j(k+1)|\Theta_j(k))| \geq \epsilon\right)\\ &= \mathbb{P}\left(\left|-\frac{\tau_j}{(k+1)^{\frac{2}{3}}}\sum_{i=1}^{n} X_{ji}(k+1)\right.\right.\\ &\qquad \left.\left. + \frac{\tau_j}{(k+1)^{\frac{2}{3}}}\mathbb{E}\left(\sum_{i=1}^{n} X_{ji}(k+1)\right)\right| \geq \epsilon\right)\\ &= \mathbb{P}\left(\left|-\sum_{i=1}^{n} X_{ji}(k+1)\right.\right.\\ &\qquad \left.\left. + \mathbb{E}\left(\sum_{i=1}^{n} X_{ji}(k+1)\right)\right| \geq \frac{(k+1)^{\frac{2}{3}}}{\tau_j}\epsilon\right)\\ &\leq 2\exp\left(\frac{-2((k+1)^{\frac{2}{3}}\epsilon)^2}{\tau_j^2 n}\right).\end{aligned}$$

■

We state the following result on bounds.

Lemma 3.5: For a fixed j, let $\Theta_j(0) = \Theta_{j0} > 0$, and let $\Theta_j(k)$ be as in (5), we have $\sup_k \left(\|\mathbf{y}_j(k)\|_1 + \|\Theta_j(k)\|_1\right) < \infty$ almost surely.

Proof: As $\mathbf{y}_j(k) \in [0,1]^n$, and from Lemma 3.4, we obtain the result. ■

We now show that, for $j = 1, 2, \ldots, m$, the sequence $\{(\mathbf{y}_j(k), \Theta_j(k))\}_{k\in\mathbb{N}}$ converges almost surely.

Theorem 3.6: For any j, and fixed constants $\mathbf{y}_j(0)$ and $\Theta_j(0)$, the sequence $\{(\mathbf{y}_j(k), \Theta_j(k))\}_{k\in\mathbb{N}}$ converges, almost surely.

Proof:

Recall that the instantaneous allocation $X_{ji}(k+1)$ is updated as in (4) and the probability $\sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k))$ is calculated as in (6), for all i, j, and k. Moreover, the average allocation $\mathbf{y}_j(k+1)$ is formulated as in (7). Let us define the map $\omega_j : \mathbb{R} \times (\mathbb{R}^n)^m \to \mathbb{R}^n$, and let $\big(\boldsymbol{\sigma}_j(\Theta_j(k), \mathbf{y}(k)) - \mathbf{y}_j(k)\big)$ be denoted by $\omega_j(\Theta_j(k), \mathbf{y}(k))$. Also, let $\big(\boldsymbol{X}_j(k+1) - \boldsymbol{\sigma}_j(\Theta_j(k), \mathbf{y}(k))\big)$ be denoted by $\mathbf{M}_{j,1}(k+1)$. Analogous to (9), we obtain:

$$\mathbf{y}_j(k+1) = \mathbf{y}_j(k) + a(k)\left[\omega_j(\Theta_j(k), \mathbf{y}(k)) + \mathbf{M}_{j,1}(k+1)\right], \tag{13}$$

where $a(k) = \frac{1}{k+1}$. Recall the definition of $\Theta_j(k)$ presented in (5), we choose a small constant $\tau_j \in (0,1)$. We reformulate (5) as

$$\begin{aligned}
&\Theta_j(k+1) \\
&= \Theta_j(k) + \frac{\tau_j}{(k+1)^{2/3}} \left(\left(\mathcal{C}_j - \sum_{i=1}^{n} \sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)) \right) \right. \\
&\left. + \left(\sum_{i=1}^{n} \sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)) - \sum_{i=1}^{n} X_{ji}(k+1) \right) \right).
\end{aligned} \tag{14}$$

For $j = 1, 2, \dots, m$, let the map $h_j : \mathbb{R} \times (\mathbb{R}^n)^m \to \mathbb{R}$, and let $(\mathcal{C}_j - \sum_{i=1}^{n} \sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)))$ be denoted by $h_j(\Theta_j(k), \mathbf{y}(k))$.

Let $(\sum_{i=1}^{n} \sigma_{ji}(\Theta_j(k), [\mathbf{y}]_i(k)) - \sum_{i=1}^{n} X_{ji}(k+1))$ be denoted by $M_{j,2}(k+1)$. Let the step-size $\frac{\tau_j}{(k+1)^{2/3}}$ be denoted by $b_j(k)$. Then from (14), we obtain:

$$\begin{aligned}
&\Theta_j(k+1) \\
&= \Theta_j(k) + b_j(k)\Big(h_j(\Theta_j(k), \mathbf{y}(k)) + M_{j,2}(k+1)\Big).
\end{aligned} \tag{15}$$

Assumption 3.1 (i) is satisfied, as the maps ω_j and h_j are Lipschitz continuous, for $j = 1, 2, \dots, m$. We have step-size $a(k) = \frac{1}{k+1}$, and for $\tau_j \in (0,1)$, we have the step-size $b_j(k) = \frac{\tau_j}{(k+1)^{2/3}}$; thus, we obtain:

$$\begin{aligned}
&\lim_{k\to\infty} a(k) = 0, \lim_{k\to\infty} b_j(k) = 0, \\
&\sum_{\ell=0}^{\infty} a(\ell) = \infty, \sum_{\ell=0}^{\infty} b_j(\ell) = \infty, \\
&\sum_{\ell=0}^{\infty} a(\ell)^2 + \sum_{\ell=0}^{\infty} b_j(\ell)^2 < \infty, \text{ and} \\
&\frac{a(k)}{b_j(k)} \to 0, \text{ when } k \to \infty.
\end{aligned}$$

These satisfy Assumption 3.1 (ii).

Recall that $\mathcal{F}_k$ denotes a σ-algebra generated by the events up to time step k, then we obtain the following expectation:

$$\mathbb{E}\left(\mathbf{M}_{j,1}(k+1) \mid \mathcal{F}_k, \text{ for } k \in \mathbb{N}\right) = 0,$$

and

$$\mathbb{E}\left(M_{j,2}(k+1) \mid \mathcal{F}_k, \text{ for } k \in \mathbb{N}\right) = 0.$$

Thus, $\{\mathbf{M}_{j,1}(k)\}_{k\in\mathbb{N}}$ and $\{M_{j,2}(k)\}_{k\in\mathbb{N}}$ are martingale difference sequences that satisfy Assumption 3.1 (iii).

From Lemma 3.5, for a fixed j, we have $\sup_k \left(\|\mathbf{y}_j(k)\|_1 + \|\Theta_j(k)\|_1\right) < \infty$ almost surely that satisfies Assumption 3.1 (iv). Thus for all j, from Theorem 3.2, we conclude that $(\Theta_j(k), \mathbf{y}_j(k))$ converges, almost surely. ■

Note that the proof of convergence in [18] is based on a constant price signal Θ_j; however, the current paper shows the convergence with a varying price signal $\Theta_j(k)$. Furthermore, price signal $\Theta_j(k)$ has a decreasing step size for all j. We now make the following remark about the communication complexity of the model.

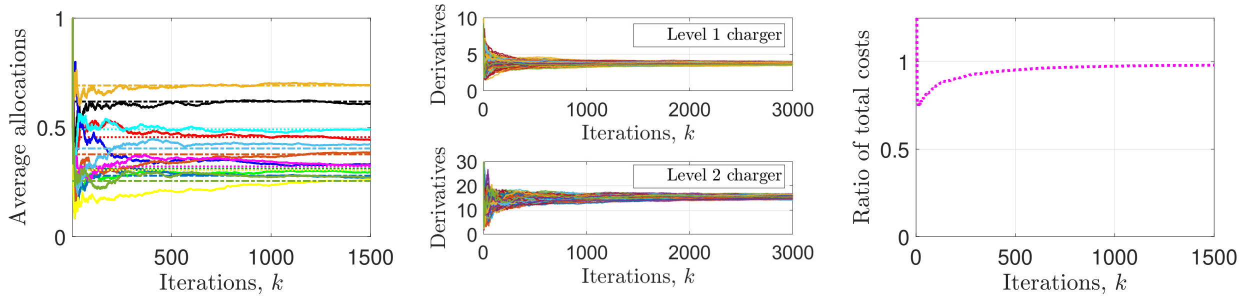
Remark 3.7: Although the agents do not need to communicate with other agents in the federated multi-agent system, the central server broadcasts the price signals $\Theta_j(k)$ at each time step; because of the broadcast, the federated multi-agent system incurs very little communication overhead. Suppose that $\Theta_j(k)$ takes μ bits floating-point values. If the system has m indivisible resources, the communication overhead will be μm bits per time step. The upper bound on the communication complexity will be $\mathcal{O}(m)$-bits per time unit, which is independent of the number of agents in the system.

IV. Application to electric vehicle charging

This section presents a hypothetical scenario to regulate the number of electric vehicles (EVs) that share a limited number of level 1 and level 2 charging points. We illustrate through numerical results that EVs receive the optimal charging points in long-term average allocations. To compare the results, we solved the optimization problem (2) by the CVX solver.

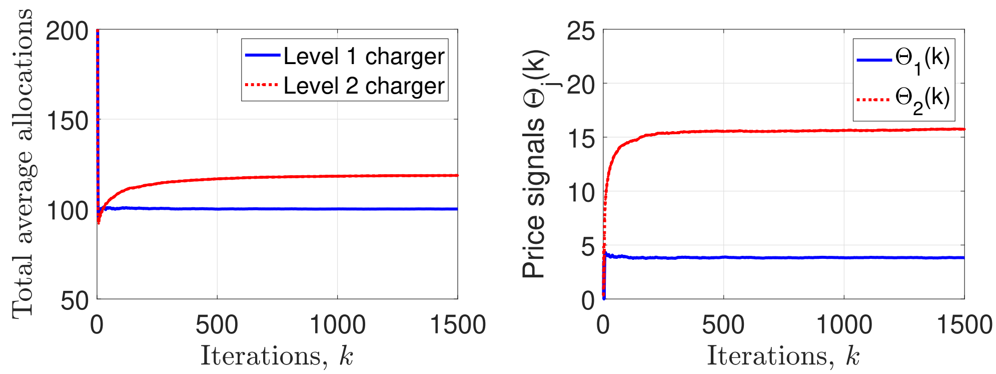
Let us assume that the government sets several public Electric Vehicle (EV) charging stations near workplaces in Fredericton City to promote daytime charging. Daytime charging changes the power demand peaks better than home charging; furthermore, reducing the load on fossil fuel generators, as the solar panels could produce power during the daytime [19], [20]. Let us consider that *electric vehicle supply equipments* (EVSEs) are installed that support a combination of slow (level 1) and fast (level 2) chargers to further load balancing on the power grid or to reduce the cost to the system.

Consider that $n = 250$ electric vehicles in the city are cooperating to access level 1 and level 2 charging points. At the start of the day, each EV owner calculates its probabilistic intent to demand the charging points for the day, and it notifies the managing agency about their outcome. Moreover, each EV has a private cost function, which associates a cost that depends on the average allocations of level 1 and level 2 chargers, i'th EV has cost function f_i, for $i = 1, 2, \dots, n$. We assume that a central server owned by the government agency keeps track of the aggregate utilization of chargers in a day, and it broadcasts the price signals $\Theta_1(k)$ and $\Theta_2(k)$ in the network for days $k \in \mathbb{N}$. The cost functions may be classified into different classes based on several factors: the type of vehicle, its battery capacity, onboard charger capacity, and a few others. We consider that a vehicle belongs to one of the classes. Let a_i, b_i, c_i, and d_i be uniformly distributed random



(a) (b) (c)

Fig. 1: (a) The evolution of average allocations of randomly selected charging points, the dotted lines denote the optimal values obtained by the CVX solver, (b) the evolution of the derivatives of cost functions f_i of all the EVs in the network with respect to level 1 and level 2 chargers, and (c) the evolution of the ratio of total costs by our solution and the total optimal cost by the CVX solver.



(a) (b)

Fig. 2: (a) The evolution of the sum of average allocations of charging points, capacities of level 1 and level 2 chargers are $\mathcal{C}_1 = 100$ and $\mathcal{C}_2 = 120$, respectively, (b) the evolution of price signals $\Theta_1(k)$ and $\Theta_2(k)$.

variables, where $a_i \in (1, 1.5)$, $b_i \in (1, 2)$, $c_i \in (3, 4.5)$, and $d_i \in (8.5, 17)$, for $i \in \{1, 2, \ldots, n\}$. The cost functions are listed as follows:

$$f_i(x_i, y_i) = \begin{cases} (i) a_i x_i + b_i y_i + c_i (x_i)^2 + d_i (y_i)^2, \\ (ii) a_i x_i + b_i y_i + \frac{2}{3} c_i (x_i)^2 + \frac{1}{4} c_i (x_i)^4 \\ \qquad + d_i (y_i)^4. \end{cases} \tag{16}$$

We categorize EVs into groups: EVs 1 to 125 belong to class 1, and EVs 126 to 250 belong to class 2. The cost functions of class 1 are listed in (16)(i), and the cost functions of class 2 are listed in (16)(ii). We consider that there are 100 level 1 chargers, $\mathcal{C}_1 = 100$, and there are 120 level 2 chargers, $\mathcal{C}_2 = 120$. The initial values assigned to a few parameters are: $\Theta_1(0) = 0.328$, $\Theta_2(0) = 0.35$, $\tau_1 = 0.05$, and $\tau_2 = 0.06$. We use the proposed Algorithm 1 and Algorithm 2 to allocate charging points to the electric vehicles in the network. Recall that an EV owner requests the charging point from the city agency in a probabilistic way (say, using a web application) based on its private cost function f_i and its previous average allocation of level 1 and level 2 charging points. The EV users do not share their cost functions or the partial derivatives of the cost functions with other EV users or with the government agency. Note a limitation of this application: following the proposed algorithm, in some cases, an EV user can receive access to both level 1 and level 2 charging points for a single EV, which may not be desired in real-world applications. Through experimental results, we observe that the EVs asymptotically receive close to optimal allocations of level 1 and level 2 charging points as shown in Figure 1(a), and they minimize the overall cost to the network. The average allocations are close to the dotted lines; the dotted lines are plotted with the optimal values obtained by the CVX solver. For further verification, the partial derivatives of the cost functions with respect to a charger type for all EVs are plotted in Figure 1(b); it illustrates that they make a consensus, satisfying the KKT conditions for optimality, as described in (3). Figure 1(c) illustrates the evolution of the ratio of total costs by our solution and the optimal total cost obtained by the CVX solver. We observe that the ratio of total costs $\frac{\sum_{i=1}^n f_i(x_i(k), y_i(k))}{\sum_{i=1}^n f_i(x_i^*, y_i^*)}$ converge close to 1.

Figure 2(a) illustrates the sum of the average allocations of charging points $\sum_{i=1}^n x_i(k)$ and $\sum_{i=1}^n y_i(k)$ over time. We observe that the sum of the average allocations converges to the respective capacity, satisfying the capacity constraints. Finally, Figure 2(b) illustrates the convergence of price signals $\Theta_1(k)$ and $\Theta_2(k)$, defined in (5).

V. Related work

In this section, we briefly present the related literature. Norkin and co-authors in [21] proposed an optimal allocation approach for indivisible resources based on branch and bound technique. The authors in [22] developed a mechanism to compute allocations of indivisible resources. A game theoretic mechanism for optimal allocation of indivisible resources is developed in [23]; they consider the agents having private valuations for resources.

In other directions, fair allocation of mixed resources, divisible and indivisible, is studied in [24]. Moreover, group fairness for indivisible resources is studied in [25]. A federated multi-agent system for actor-critic reinforcement learning is proposed in [26]. For details on the fair allocation of indivisible resources, interested readers can refer to the recent survey at [27], readers can also refer to [13, Chapter 1.4] and more recent work on distributed optimization and federated optimization.

VI. Conclusion

We proposed a new distributed stochastic algorithm to solve multi-indivisible resource allocation problems in a federated multi-agent system. The solution does not require communication between agents. However, a little communication is required with a central server that keeps track of the utilization of resources and broadcasts price signals in the network. We presented the theoretical results on the convergence of average allocations of resources. The ideas from multi-time scale stochastic approximation techniques inspire our solution approach. We present an application to control a population of electric vehicles for a limited set of level 1 and level 2 charging points. Experiments show that the long-term average allocations converge close to optimal values.

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