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A Robust Finite Difference Model Reduction for the Uniform Spectral Observability of a Fully-Clamped Three-Layer Mead-Marcus-type Smart Laminate

Ahmet Özkan Özer and Md Zulfiqur Haider

***Abstract*— We consider a fully-clamped three-layer Mead-Marcus beam model, crucial for understanding the interactions between shear and bending motions. This model is exactly observable with a single boundary observer. We propose a semi-discrete Finite Difference approximation, where characterizing low and high-frequency eigenvalues presents significant challenges, particularly in proving the uniform gap condition, which is sufficient for achieving spectral observability. To address this, we employ the discrete multipliers method and Gershgorin's Circle Theorem, highlighting the importance of numerical filtering. Our results, assuming a small shear modulus in the core layer, are promising due to the strong coupling between shear dynamics and overall bending. This complexity contrasts with a single-layer clamped Euler-Bernoulli beam. Notably, when the model simplifies to a classical Euler-Bernoulli beam, our findings extend existing results to multi-layer beams.**

***Index Terms*— Unifrom Spectral Observability, Finite Differences, Gershgorin's Circle Theorem, Multi-layer beams, Mead-Marcus beam**

I. Introduction

Multi-layer sandwich beams, like those found in ultrasonic transducers [21] with elastic/piezoelectric components, hold increasing promise in diverse industrial applications such as aeronautics, civil engineering, defense, biomedicine, and space structures [3], [22]. A three-layer sandwich beam comprises perfectly bonded alternating piezoelectric/elastic layers sandwiching compliant viscoelastic layers [4], [9]. While most sandwich beam theories average stresses and elastic moduli through the depth, Rao & Nakra's discrete-layer theory treats each layer separately, incorporating Euler-Bernoulli and Mindlin-Timoshenko assumptions [9].

High-frequency multi-layer ultrasonic transducers with larger bandwidths offer excellent imaging performance in the biomedical field [21]. However, achieving perfect acoustic impedance matching between the piezo-element and the target medium across the frequency spectrum remains challenging. Researchers in ultrasonic imaging strive for optimal performance—high frequency, large bandwidth, and high sensitivity—yet the models in use often rely on oversimplified spring-mass systems, as noted in [20].

Throughout this paper, we denote time derivatives with dots and primes for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively. We focus on the system of partial differential equations for the multi-layer Mead-Marcus-type beam model from [2]

$$\begin{cases} \ddot{z} + z'''' - B\psi' = 0, \\ -C\psi'' + P\psi = -Bz''', \ (x,t) \in (0,L) \times \mathbb{R}^+, \\ (z, \psi, z')(x,t)|_{x=0,L} = 0, \quad t \in \mathbb{R}^+, \\ (z, \dot{z})(x,0) = (z_0, z_1)(x), \ x \in [0,L]. \end{cases} \tag{1}$$

Here, $\psi(x,t)$ and $z(x,t)$ represent the longitudinal and transverse vibrations of the centerlines of each layer. The physical constants $B, C, P > 0$ are defined in [2], [14].

A. State-Space Formulation and Exact Observability

The second equation of (1) can be used to solve for the shear angle ψ. Define the differential operator $\boldsymbol{D_x^2} = \frac{\partial^2}{\partial x^2}$ on the domain $\mathrm{Dom}(\boldsymbol{D_x^2}) = \{z \in H^2(0,L) \mid z(0) = z(L) = 0\}$. This operator is densely defined, self-adjoint, positive-definite, and unbounded. Since C and P are positive, the operator $(-C\boldsymbol{D_x^2} + P)^{-1}$ exists and is bounded on $L^2(0,L)$.

Defining the operator $\boldsymbol{J} = \frac{1}{C}[-I + P(-C\boldsymbol{D_x^2} + P\boldsymbol{I})^{-1}]$, it is shown that $\boldsymbol{J}$ is continuous, self-adjoint, and non-positive on $L^2(0,L)$. Moreover, $\boldsymbol{J} = (-C\boldsymbol{D_x^2} + P)^{-1}\boldsymbol{D_x^2}$. [14, Lemma 1]. By eliminating ψ in the second equation of (1), a simplified form of the PDE is obtained as

$$\begin{aligned} &\ddot{z} + z'''' + B^2(\boldsymbol{J}D_x^2 z')' = 0, (x,t) \in (0,L) \times \mathbb{R}^+ \\ &(z, z')(x,t)|_{x=0,L} = 0, \quad t \in \mathbb{R}^+ \\ &(z, \dot{z})(x,0) = (z_0, z_1)(x), \ x \in [0,L]. \end{aligned} \tag{2}$$

Now, define the energy of (1) on the Hilbert space $\mathcal{H} = H_0^2(0,L) \times L^2(0,L)$, so that the energy norm $\|\cdot\|_e$ on $\mathcal{H}$ is defined by

$$E(t) = \frac{1}{2}\int_0^L \left[|\dot{z}|^2 + |z''|^2 - (B^2 J z')z'\right] dx \tag{3}$$

and the associated inner product is $\langle (u,v),(f,g)\rangle = \int_0^L \left[v\bar{g} + u''\bar{f}'' - Ju'\bar{f}'\right] dx$. Define the operator $\mathcal{A}$ on $\mathrm{Dom}(\mathcal{A}) = H_0^2(0,L) \times L^2(0,L)$ as the following

$$\mathcal{A}(u,v) = \left(v, -\left[u'''' + B^2(Ju')'\right]\right) = (v, -\mathrm{A}u) \tag{4}$$

where $\mathrm{A} := (\boldsymbol{D_x^4} + B^2\boldsymbol{D_x}J\boldsymbol{D_x})$. Letting $\vec{y}(t) = (z(t,x), \dot{z}(t,x))$ the (2) can be rewritten as

$$\dot{\vec{y}}(t,x) = \mathcal{A}\vec{y}(t,x), \quad \vec{y}(0) = \vec{y}_0 = (z^0(x), z^1(x)). \tag{5}$$

Theorem 1 (Exact Observability)**.** *[14] The operator $\mathcal{A}$ defined by* (4) *is the infinitesimal generator of a unitary C_0-group on $\mathcal{H}$. Hence, for all $y_0 \in \mathcal{H}$ there exist a unique*

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[1]Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101, USA. Emails: ozkan.ozer@wku.edu, mdzulfiqur.haider458@topper.wku.edu

solution y to (5) *in* $C[\mathbb{R},\mathcal{H}]$. *Moreover, for all* $T>0$, *there exists a constant* $C(T)>0$ *such that*

$$\int_0^T |z''(L,t)|^2 dt \geq C(T)E(0). \qquad (6)$$

B. Literature on Model Reductions for Coupled PDEs

The literature on finite-dimensional model reductions for coupled PDEs emphasizes the importance of considering all vibration modes when designing sensors. Neglecting high-frequency residual modes, known as the "spill-over" effect, can hinder the system's observability [11]. Moreover, applying Finite Differences or Finite Elements blindly may introduce spurious, non-observable high-frequency modes [18]. To counter this, methods like "direct Fourier filtering", known for its efficiency, have been explored for Euler-Bernoulli and Rayleigh beams with hinged boundary conditions [12], [13].

Recent Finite Differences-based reductions, as analyzed in [16], [17] and extended to multi-layer systems in [1], incorporate direct Fourier filtering to achieve uniform observability in the natural energy space. A significant limitation of this approach is the necessity of having explicit knowledge of the entire spectrum of eigenvalues. While the spectrum for hinged boundary conditions can be analytically established, the clamped boundary case requires more intricate analysis.

C. Our Contributions

In this paper, we reduce the PDE model (1) using Standard Finite Differences and find that the reduced model lacks uniform spectral observability. Spectral observability refers to the ability to monitor and reconstruct the entire system state via its eigen-space components, particularly eigenfunctions corresponding to the system's eigenvalues. Uniform spectral observability in numerical approximations implies that this property remains consistent as the mesh size decreases.

To address the lack of uniform spectral observability, we apply direct Fourier filtering to remove problematic high-frequency components and use discrete multipliers to restore uniform spectral observability. This establishes the spectral observability of (1) with a single boundary observer. Our work pioneers a robust reduction of the Mead-Marcus three-layer beam model using Standard Finite Differences, which reduces to the Euler-Bernoulli beam model when $B=C=P\equiv 0$ in (1). Thus, our results extend to both single-layer and three-layer beam models. We rigorously establish exact observability for the PDE model (1) through discrete multipliers, extending prior findings [5], [6], [14] and providing deeper insights into the observability of the discretized model.

II. Semi-discretization in the Space Variable

Let $N\in\mathbb{N}$. Define the uniform mesh size $h:=\frac{L}{N+1}$. Consider the discretization $x_i=i\cdot h$ for $i=-1,0,\ldots,N+2$ of the interval $[0,L]$.

Define the central difference approximations of the operator $\boldsymbol{D_x}$ and $-\boldsymbol{D_x^2}$ at each x_i by the symmetric positive-definite $N\times N$ matrices $\boldsymbol{D_{1c,h}} := \frac{1}{2h} tridiag(-1,0,1)$ and $\boldsymbol{A_h} := \frac{1}{h^2} tridiag(-1,2,-1)$. The eigenpairs $(\lambda_k(h),\vec{\phi}_k(h))$ of the matrix $\boldsymbol{A_h}$ are explicitly known, e.g., see [12].

Next, define the central difference approximation of the operator $\boldsymbol{D_x^4}$ at each x_i by the symmetric positive-definite $N\times N$ matrix

$$\boldsymbol{\mathcal{B}_h} := \frac{1}{h^4}\begin{bmatrix} 7 & -4 & 1 & 0 & 0 & 0 & \ldots & 0\\ -4 & 6 & -4 & 1 & 0 & 0 & \ldots & 0\\ 1 & -4 & 6 & -4 & 1 & 0 & \ldots & 0\\ 0 & 1 & -4 & 6 & -4 & 1 & \ldots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ldots & \ldots & 1 & -4 & 6 & -4 & 1\\ 0 & \ldots & \ldots & 0 & -1 & -4 & 6 & -4\\ 0 & \ldots & \ldots & 0 & 0 & 1 & -4 & 7 \end{bmatrix}.$$

Let $z_i(t)$ be the approximations of the solutions $z(x,t)$ at each node $\{x_i\}_{i=0}^{N+1}$. Next, by defining the vectors $\vec{z} := [z_1,z_2,...,z_N]^T$ and $\vec{\psi} := [\psi_1,\psi_2,...,\psi_N]^T$, we propose space semi-discretization of (1) as follows

$$\begin{cases} \ddot{\vec{z}} + \tilde{\boldsymbol{\mathcal{A}}}_{\boldsymbol{h}}\vec{z} = 0,\\ (z_0,\psi_0,z_{N+1},\psi_{N+1})(t)=0,\\ (z_1,z_{-1},z_{N+2}-z_N)(t)=0,\ t\in\mathbb{R}^+,\\ (z,\psi,z_t,\psi_t)_i(0)\\ \quad =(z^0,\psi^0,z^1,\psi^1)(x_i),\ i=0,1,\ldots,N+1, \end{cases} \qquad (7)$$

where

$$\begin{cases} \tilde{\boldsymbol{\mathcal{A}}}_{\boldsymbol{h}} := (\boldsymbol{\mathcal{B}_h} + B^2\boldsymbol{D_{1c,h}}J_h\boldsymbol{D_{1c,h}})\\ J_h := -(C\boldsymbol{A_h}+PI)^{-1}\boldsymbol{A_h}. \end{cases} \qquad (8)$$

Define the matrix $\boldsymbol{\mathcal{A}_h}$ by $\boldsymbol{\mathcal{A}_h}(\vec{u},\vec{v}) := \left(\vec{v}, -\tilde{\boldsymbol{\mathcal{A}}}_{\boldsymbol{h}}\vec{u}\right)$. With $\vec{y}_h=(\vec{z},\dot{\vec{z}})$, (7) can be rewritten as $\dot{\vec{y}}_h = \boldsymbol{\mathcal{A}_h}\vec{y}_h$.

Note that by assuming $B=\psi\equiv 0$ in (1), the discretized model (7) simplifies to a single-layer beam, which corresponds to the semi-discretized Euler-Bernoulli beam as presented in [5]-[6]. The resulting discretized model is

$$\begin{cases} \ddot{\vec{z}} + \boldsymbol{\mathcal{B}_h}\vec{z} = 0,\\ z_0(t)=z_{N+1}(t)=0, \qquad z_1(t)=z_{-1}(t),\\ z_{N+2}(t)=z_N(t), \qquad t\in\mathbb{R}^+,\\ (\vec{z},\dot{\vec{z}})_i = (z^0,z^1)(x_i), \quad i=0,1,\ldots,N+1. \end{cases} \qquad (9)$$

The energy of the solutions of (9) is defined by

$$\begin{aligned} E_{h,EB}(t) &:= \frac{h}{2}\left\{\left\langle \dot{\vec{z}},\dot{\vec{z}}\right\rangle + \langle \boldsymbol{A_h}\vec{z},\boldsymbol{A_h}\vec{z}\rangle\right\}\\ &\quad + h\left|\tfrac{z_1}{h^2}\right|^2 + h\left|\tfrac{z_N}{h^2}\right|^2. \end{aligned} \qquad (10)$$

For the rest of the paper and the ease of the calculations, define

$$\begin{aligned} &\tfrac{1}{h^4}\Delta_x^4 z_i = \delta_x^4 z_i := \tfrac{z_{i+2}-4z_{i+2}+6z_i-4z_{i-1}+z_{i-2}}{h^4},\\ &h^3\delta_x^3 z_i = \Delta_x^3 z_i := (z_{i+2}-3z_{i+1}+3z_i-z_{i-2}),\\ &h^2\delta_x^2 z_i = \Delta_x^2 z_i := z_{i+1}-2z_i+z_{i-1},\\ &h\delta_x z_i = \Delta_x z_i = z_{i+1}-z_i. \end{aligned} \qquad (11)$$

Moreover, $(\boldsymbol{D_{1c,h}}\vec{z})_i := \frac{z_{i+1}-z_{i-1}}{2h}$, and therefore,

$$\begin{cases} \delta_x^4 z_i := \frac{1}{h^4}(\Delta_x^3 z_i - \Delta_x^3 z_{i-1}),\\ \delta_x^3 z_i := \frac{1}{h^3}(\Delta_x^2 z_{i+1} - \Delta_x^2 z_i),\\ (\boldsymbol{D_{1c}}\vec{z})_i := z_{i+1}-z_{i-1} = \Delta_x z_i + \Delta_x z_{i-1}. \end{cases} \qquad (12)$$

Next, define the inner product $\langle \vec{u}, \vec{v} \rangle := \sum_{j=1}^{N} u_i v_i$, on $\mathbb{R}^N$.

Lemma 2. *The matrix $\boldsymbol{J_h}$ in* (8) *can be expressed as $\boldsymbol{J_h} = C^{-1}[-I+P(C\boldsymbol{A_h}+PI)^{-1}]$. Furthermore, $\boldsymbol{J_h}$ is self-adjoint and non-positive on l^2.*

Proof. Starting with $\boldsymbol{J_h}\vec{u} = \vec{v}$, we have $-C^{-1}\vec{u} + C^{-1}P[(C\boldsymbol{A_h} + PI)]^{-1}\vec{u} = \vec{v}$. This implies $\vec{u} = C^2P^{-1}\boldsymbol{A_h}\vec{v} + C\vec{v} + P^{-1}C\boldsymbol{A_h}\vec{u} + \vec{u}$. Rearranging gives $-(C\boldsymbol{A_h} + PI)^{-1}\boldsymbol{A_h}\vec{u} = \vec{v}$, i.e., $\boldsymbol{J_h}\vec{u} = \vec{v}$, confirming that $\boldsymbol{J_h}$ has the form given in (8), thus proving the first claim.

For self-adjointness, let $\vec{u}, \vec{v} \in \mathbb{R}^n$ so that

$$\begin{aligned}
\langle \boldsymbol{J_h}\vec{u}, \vec{v} \rangle &= \left\langle -C^{-1}\vec{u} + C^{-1}P(C\boldsymbol{A_h} + PI)^{-1}\vec{u}, \vec{v} \right\rangle \\
&= \left\langle -C^{-1}\vec{u} + (C\boldsymbol{A_h} + PI)^{-1}PC^{-1}\vec{u}, \vec{v} \right\rangle \\
&= \left\langle [-C^{-1}I + PC^{-1}(C\boldsymbol{A_h} + PI)^{-1}]\vec{u}, \vec{v} \right\rangle = \langle \vec{u}, \boldsymbol{J_h}\vec{v} \rangle,
\end{aligned}$$

utilizing the self-adjointness of $\boldsymbol{A_h}$. To prove $\boldsymbol{J_h}$ is non-positive, let $\vec{r} := (C\boldsymbol{A_h} + PI)^{-1}\vec{s}$ so that $\vec{s} = C\boldsymbol{A_h}\vec{r} + P\vec{r}$,

$$\begin{aligned}
\langle \boldsymbol{J_h}\vec{s}, \vec{s} \rangle &= \left\langle -C^{-1}\vec{s} + C^{-1}P[(C\boldsymbol{A_h} + PI)]^{-1}\vec{s}, \vec{s} \right\rangle \\
&= \langle -\boldsymbol{A_h}\vec{r}, C\boldsymbol{A_h}\vec{r} + P\vec{r} \rangle,
\end{aligned}$$

and since $\boldsymbol{A_h}$ is positive definite, the proof follows. □

Defining $\boldsymbol{\mathcal{L}} := \boldsymbol{J_h}\boldsymbol{D_{1c,h}}$, the energy of the solutions of semi-discretized model reduction (7) can be defined by

$$E_{h,MM}(t) := E_{h,EB}(t) - \tfrac{B^2h}{2} \langle \boldsymbol{\mathcal{L}}\vec{z}, \boldsymbol{D_{1c,h}}\vec{z} \rangle. \quad (13)$$

Remark 1. By Lemma 2, and defining

$$\vec{m} := (C\boldsymbol{A}_h + P\boldsymbol{I})^{-1}\boldsymbol{D_{1c,h}}\vec{z}, \quad (14)$$

an alternative form of the energy for the solutions of the semi-discretized model reduction (7) can be expressed as

$$\begin{aligned}
E_{h,MM}(t) := E_{h,EB}(t) + \tfrac{B^2h}{2} \{ C \langle \boldsymbol{A_h}\vec{m}, \boldsymbol{A_h}\vec{m} \rangle \\
+ P \langle \boldsymbol{A_h}\vec{m}, \vec{m} \rangle \},
\end{aligned} \quad (15)$$

noting the boundary conditions $m_0 = m_{N+1} = 0$ hold.

Lemma 3. *The energy* (13) *along the solutions of* (7) *is conservative, i.e. $\dot{E}_{h,MM}(t) = 0$. So that, $E_{h,MM}(t) \equiv E_{h,MM}(0)$, for any time $t > 0$.*

Proof. By taking the time derivative of $E_{h,MM}(t)$ along the solutions of (7)

$$\begin{aligned}
\dot{E}_{h,MM}(t) &= h \sum_{i=1}^{N} \dot{z}_i \ddot{z}_i + \tfrac{h}{2} \tfrac{d}{dt} \sum_{i=1}^{N} \left| \delta_x^2 z_i \right|^2 \\
&+ h \tfrac{d}{dt} \left| \tfrac{z_1}{h^2} \right|^2 + h \tfrac{d}{dt} \left| \tfrac{z_N}{h^2} \right|^2 - \tfrac{B^2h}{2} \\
&+ \left[\left\langle \boldsymbol{\mathcal{L}}\dot{\vec{z}}, \boldsymbol{D_{1c,h}}\vec{z} \right\rangle + \left\langle \boldsymbol{\mathcal{L}}\vec{z}, \boldsymbol{D_{1c,h}}\dot{\vec{z}} \right\rangle \right].
\end{aligned} \quad (16)$$

Finally, since the following equality is valid utilizing $(7)_3 - (7)_5$

$$\langle \boldsymbol{\mathcal{B}_h}\vec{z}, \dot{\vec{z}} \rangle = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{N} \left| \delta_x^2 z_i \right|^2 + \frac{d}{dt} \left\{ \left| \frac{z_1}{h^2} \right|^2 + \left| \frac{z_N}{h^2} \right|^2 \right\},$$

and since $\boldsymbol{D_{1c,h}}$ is anti-symmetric, the conclusion follows from this together with Lemma 2 and (7). □

Next, we establish the equivalence of $E_{h,MM}(t)$ and $E_{h,EB}(t)$. To do so, we require the following lemma.

Lemma 4. *The solutions of* (9) *satisfy the inequality*

$$\tfrac{h}{2} \sum_{i=1}^{N} \left((\boldsymbol{D_{1c,h}}\vec{z})_i \right)^2 \leq \tfrac{h}{2} \langle \boldsymbol{A_h}\vec{z}, \vec{z} \rangle \leq L^2 E_{h,EB}(t). \quad (17)$$

Proof. By the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{N} \left((\boldsymbol{D_{1c,h}}\vec{z})_i \right)^2 \leq \sum_{i=0}^{N} (\delta_x z_i)^2. \quad (18)$$

Since $\sum_{i=0}^{N} (\delta_x z_i)^2 = \langle \boldsymbol{A_h}\vec{z}, \vec{z} \rangle$. By Hölder's inequality for sums, the following inequality follows

$$\sum_{i=0}^{N} (\delta_x z_i)^2 \leq \left\{ \sum_{i=1}^{N} \left(\delta_x^2 z_i \right)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{N} (z_i)^2 \right\}^{\frac{1}{2}}. \quad (19)$$

Next, the following discussion is needed to have the discrete version of Poincare's inequality to estimate the term $\sum_{i=1}^{N} |z_i|^2$ above. Since $z_0 = 0$, $z_i = h \sum_{j=0}^{i-1} \delta_x z_j$, for $i = 1, ..., N+1$. Moreover, with $z_1 = 0$,it is straightforward to show that $\delta_x z_i = h \sum_{j=1}^{i} \delta_x^2 z_j$.

By the Hölder's inequality for sums,

$$\begin{aligned}
(z_i)^2 = \left(h \sum_{j=0}^{i-1} \delta_x z_j \right)^2 &\leq \left(h \sum_{j=0}^{i-1} 1 \right) \left(h \sum_{j=0}^{i-1} (\delta_x z_j)^2 \right) \\
&\leq Lh \sum_{j=0}^{N} (\delta_x z_j)^2,
\end{aligned}$$

and analogously,

$$(\delta_x z_j)^2 = \left(h \sum_{j=1}^{i} \delta_x^2 z_j \right)^2 \leq Lh \sum_{j=1}^{N} \left(\delta_x^2 z_j \right)^2,$$

where we used $Nh < L$.

Now, using the above inequalities, it is straightforward to show that

$$\sum_{i=1}^{N} (z_i)^2 \leq L^2 \sum_{i=1}^{N} (\delta_x z_i)^2 \leq L^4 \sum_{j=1}^{N} \left(\delta_x^2 z_j \right)^2. \quad (20)$$

Finally, by (18) and (19) and the inequality above we get

$$\sum_{i=1}^{N} \left((\boldsymbol{D_{1c,h}}\vec{z})_i \right)^2 \leq L^2 \sum_{i=1}^{N} \left(\delta_x^2 z_i \right)^2.$$

The conclusion (17) follows immediately from this and (10). □

Theorem 5. *Let $F_1 := 1 + \frac{B^2L^2}{C} > 0$. The discrete energy $E_{h,MM}(t)$ and $E_{h,EB}(t)$ are equivalent, i.e.,*

$$E_{h,EB}(t) \leq E_{h,MM}(t) \leq F_1 E_{h,EB}(t). \quad (21)$$

Proof. The first inequality is obvious since $-\boldsymbol{J_h}$ is positive. We prove the second inequality by the following $\tilde{\boldsymbol{J}}_{\boldsymbol{h}} = -\boldsymbol{J_h}$ is positive and

$$\begin{aligned}
E_{h,MM}(t) &= E_{h,EB}(t) - \tfrac{B^2h}{2} \langle \boldsymbol{\mathcal{L}}\vec{z}, \boldsymbol{D_{1c,h}}\vec{z} \rangle \\
&\leq E_{h,EB}(t) + \tfrac{B^2h}{2} ||\tilde{\boldsymbol{J}}_{\boldsymbol{h}}\boldsymbol{D_{1c,h}}\vec{z}|| ||\boldsymbol{D_{1c,h}}\vec{z}|| \\
&\leq E_{h,EB}(t) + \tfrac{B^2h}{2} ||\tilde{\boldsymbol{J}}_{\boldsymbol{h}}|| \sum_{i=1}^{N} \left| (\boldsymbol{D_{1c,h}}\vec{z})_i \right|^2.
\end{aligned} \quad (22)$$

Now, if $\vec{u} = (u_1, u_2, \ldots, u_N)^T$ is the eigenvectors of $\tilde{\boldsymbol{J}}_{\boldsymbol{h}}$ corresponding to the eigenvalues λ then the eigenvalue problem, $\tilde{\boldsymbol{J}}_{\boldsymbol{h}}\vec{u} = \lambda\vec{u}$ leads to $(C\boldsymbol{A}_{\boldsymbol{h}} + PI)^{-1}\boldsymbol{A}_{\boldsymbol{h}})\vec{u} = \lambda\vec{u}$. Therefore, $\boldsymbol{A}_{\boldsymbol{h}}\vec{u} = \lambda[C\boldsymbol{A}_{\boldsymbol{h}} + P]\vec{u}$. Rearranging the terms yields $\lambda = \frac{\lambda(\boldsymbol{A}_{\boldsymbol{h}})}{C\lambda(\boldsymbol{A}_{\boldsymbol{h}})+P}$.

Therefore, $||\tilde{\boldsymbol{J}}_{\boldsymbol{h}}\vec{u}|| \leq |\lambda|||\vec{u}|| \leq \left|\frac{\lambda(\boldsymbol{A}_{\boldsymbol{h}})}{C\lambda(\boldsymbol{A}_{\boldsymbol{h}})+P}\right| ||\vec{u}|| \leq \frac{1}{C}||\vec{u}||$. This implies that $\|\tilde{\boldsymbol{J}}_{\boldsymbol{h}}\| < \frac{1}{C}$. Considering this with (22) and Lemma 4 leads to (21). Therefore, the conclusion (21) follows immediately. $\square$

III. Spectral Observability Uniformly as $h \to 0$

This section aims to prove the discrete counterpart of (24) for the eigen-space. Let $\vec{w} = (w_1, w_2, \ldots, w_N)^T$ be the normalized eigenvectors, $\langle \vec{w}, \vec{w}\rangle = 1$, corresponding to the eigenvalues $\tilde{\lambda}$ for the eigenvalue problem (7). Consider the following eigenvalue problem

$$\begin{cases} \tilde{\boldsymbol{\mathcal{A}}}_h\vec{w} := (\boldsymbol{\mathcal{B}}_{\boldsymbol{h}} + B^2\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\boldsymbol{\mathcal{L}})\vec{w} = \tilde{\lambda}\vec{w}, \\ w_0 = w_{N+1} = 0, \quad w_1(t) = w_{-1}(t), \\ w_{N+2}(t) = w_N(t), \qquad t \in \mathbb{R}^+. \end{cases} \tag{23}$$

To prove our main result, the following lemmas are in order.

Lemma 6. *The following identity holds for* (23)*,*

$$\begin{aligned} &-\tfrac{1}{2h^3}\left\{4(w_1)^2 + 4(w_N)^2 + w_1w_2 + w_Nw_{N-1})\right\} \\ &+\tfrac{L}{2}\left(\delta_x^2 w_{N+1}\right)^2 = \tfrac{B^2Ph}{2}\langle \boldsymbol{A}_h\vec{m}, \vec{m}\rangle \\ &+2h\sum_{i=1}^N\left(\delta_x^2 w_i\right)^2 + \tfrac{B^2P^2h}{2C}\sum_{i=0}^N (m_i)^2 \\ &-\tfrac{B^2P^2h^3}{4C}\langle \boldsymbol{A}_h\vec{m}, \vec{m}\rangle + \tfrac{B^2Ch^3}{4}\sum_{i=1}^N\left(\delta_x^3 m_i\right)^2 \\ &-\tfrac{\tilde{\lambda}h^3}{2}\langle \boldsymbol{A}_h\vec{w}, \vec{w}\rangle + \tfrac{B^2Ph^2}{C}\langle i\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}, \boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{m}\rangle. \end{aligned} \tag{24}$$

Proof. Multiply equation $(23)_1$ by the multiplier $2ih^2(\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w})_i$ and sum over i from 1 to N.

$$\begin{aligned} 2h^2&\left\langle(\boldsymbol{\mathcal{B}}_{\boldsymbol{h}} + B^2\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\boldsymbol{\mathcal{L}})\vec{w}, i\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}\right\rangle \\ &= 2h^2\tilde{\lambda}\langle \vec{w}, i\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}\rangle. \end{aligned} \tag{25}$$

where the right-hand side can be written as

$$2h^2\tilde{\lambda}\langle \vec{w}, i\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}\rangle = -\tilde{\lambda}h\sum_{i=0}^N w_iw_{i+1} \tag{26}$$

with a quick index shifting. Due to the following identity,

$$-\sum_{i=0}^N w_iw_{i+1} = \frac{1}{2}\sum_{i=0}^N (w_{i+1} - w_i)^2 - \langle \vec{w}, \vec{w}\rangle, \tag{27}$$

$$2h^2\tilde{\lambda}\langle \vec{w}, i\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}\rangle = \frac{\tilde{\lambda}h^3}{2}\sum_{i=0}^N (\delta_x w_i)^2 - \tilde{\lambda}h\langle \vec{w}, \vec{w}\rangle. \tag{28}$$

Since $\langle \boldsymbol{\mathcal{B}}_{\boldsymbol{h}}\vec{w}, \vec{w}\rangle = \sum_{i=0}^N\left|\delta_x^2 w_i\right|^2 - 2\left\{\left|\frac{w_1}{h^2}\right|^2 - \left|\frac{z_N}{h^2}\right|^2\right\}$, by (23) and (14), the following is immediate

$$\begin{aligned} \tilde{\lambda}h\langle \vec{w}, \vec{w}\rangle &= h\langle \boldsymbol{\mathcal{B}}_{\boldsymbol{h}}\vec{w}, \vec{w}\rangle - \left\langle B^2\boldsymbol{\mathcal{L}}\vec{w}, \boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}\right\rangle \\ &= h\sum_{i=0}^N\left(\delta_x^2 w_i\right)^2 - \tfrac{2}{h^3}((w_1)^2 - (w_N)^2) \\ &\quad +B^2Ch\langle \boldsymbol{A}_{\boldsymbol{h}}\vec{m}, \boldsymbol{A}_{\boldsymbol{h}}\vec{m}\rangle + B^2Ph\sum_{i=0}^N(\delta_x m_i)^2 \end{aligned} \tag{29}$$

Call the left-hand side of (25) as $\frac{1}{h}(S_1 + S_2)$ where

$$S_1 := \quad 2h^2\langle \boldsymbol{\mathcal{B}}_{\boldsymbol{h}}\vec{w}, i\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}\rangle, \tag{30}$$

$$S_2 := \quad 2h^2\left\langle B^2\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\boldsymbol{\mathcal{L}}\vec{w}, i\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\vec{w}\right\rangle. \tag{31}$$

First, we estimate $\frac{S_1}{h}$. By $(12)_1$ and $(12)_3$

$$\begin{aligned} \tfrac{S_1}{h} &= \tfrac{1}{h^4}\sum_{i=1}^N i\left(\Delta_x^3 w_i - \Delta_x^3 w_{i-1}\right)\left(\Delta_x w_i + \Delta_x w_{i-1}\right) \\ &= -\tfrac{1}{h^4}\sum_{i=0}^N \Delta_x^3 w_i\Delta_x w_i + (N+1)\,\Delta_x^3 w_N\Delta_x w_N \\ &\quad +\tfrac{1}{h^4}\sum_{i=1}^N i\Delta_x^3 w_i\Delta_x w_{i-1} - \tfrac{1}{h^4}\sum_{i=1}^N i\Delta_x^3 w_{i-1}\Delta_x w_i \\ &\quad +\tfrac{1}{h^4}\sum_{i=1}^N i\Delta_x^3 w_i\Delta_x w_i - \tfrac{1}{h^4}\sum_{i=1}^N i\Delta_x^3 w_i\Delta_x w_i \\ &\quad +\tfrac{1}{h^4}\sum_{i=1}^N i\Delta_x^3 w_{i-1}\Delta_x w_{i-1} - \tfrac{1}{h^4}\sum_{i=1}^N i\Delta_x^3 w_{i-1}\Delta_x w_{i-1}, \end{aligned}$$

where the last two terms are added and subtracted.

Since $\Delta_x^3 w_i = \Delta_x^2 w_{i+1} - \Delta_x^2 w_i$,

$$\begin{aligned} \frac{S_1}{h} &= -\tfrac{2}{h^4}\sum_{i=0}^N \Delta_x^2 w_{i+1}\Delta_x w_i + \tfrac{2}{h^4}\sum_{i=0}^N \Delta_x^2 w_i\Delta_x w_i \\ &\quad +\tfrac{2(N+1)}{h^4}\Delta_x^3 w_N\Delta_x w_N - \tfrac{1}{h^4}\sum_{i=0}^N i\Delta_x^2 w_{i+1}\Delta_x^2 w_i \\ &\quad +\tfrac{1}{h^4}\sum_{i=0}^N (i+1)\,\Delta_x^2 w_{i+1}\Delta_x^2 w_i - \tfrac{(N+1)}{h^4}\Delta_x^2 w_{N+1}\Delta_x^2 w_N \\ &= -\tfrac{2}{h^4}\sum_{i=1}^{N+1}\Delta_x^2 w_i\Delta_x w_{i-1} + \tfrac{2}{h^4}\sum_{i=0}^N \Delta_x^2 w_i\Delta_x w_i \\ &\quad +\tfrac{2(N+1)}{h^4}\Delta_x^3 w_N\Delta_x w_N + \tfrac{1}{h^4}\sum_{i=0}^N \Delta_x^2 w_{i+1}\Delta_x^2 w_i \\ &\quad -\tfrac{(N+1)}{h^4}\Delta_x^2 w_{N+1}\Delta_x^2 w_N. \end{aligned}$$

Now use $\Delta_x^2 w_i = \Delta_x w_i - \Delta_x w_{i-1}$ to get

$$\begin{aligned} S_1 \quad &= 2h\sum_{i=0}^N\left(\delta_x^2 w_i\right)^2 + h\sum_{i=0}^N \delta_x^2 w_{i+1}\delta_x^2 w_i \\ &\quad -\tfrac{4}{h^3}\left[(w_1)^2 - (w_N)^2\right] - \tfrac{4L}{h^4}(w_N)^2. \end{aligned}$$

By adopting (27) again, this leads to

$$\begin{aligned} S_1 &= 3h\sum_{i=0}^N\left(\delta_x^2 w_i\right)^2 - \tfrac{h^3}{2}\sum_{i=0}^N\left(\delta_x^3 w_i\right)^2 \\ &\quad -4L\left(\tfrac{w_N}{h^2}\right)^2 - \tfrac{6}{h^3}((w_1)^2 - (w_N)^2). \end{aligned} \tag{32}$$

To estimate the term $\frac{h^3}{2}\sum_{i=0}^N\left(\delta_x^3 w_i\right)^2$ in (32), the following equality is needed first

$$\begin{aligned} &\tfrac{h^3}{2}\sum_{i=1}^N \delta_x^4 w_i\delta_x^2 w_i = -\tfrac{h^3}{2}\sum_{i=0}^N\left(\delta_x^3 w_i\right)^2 \\ &-\tfrac{1}{h^3}\left(w_1w_2 - 4(w_1)^2 - 4(w_N)^2 + w_Nw_{N-1}\right). \end{aligned} \tag{33}$$

Now, define the operator $\mathbf{T}$ by $(\mathbf{T}w)_i := \left(B^2\boldsymbol{D}_{\mathbf{1c},\boldsymbol{h}}\boldsymbol{\mathcal{L}}w\right)_i$ so that $\langle \mathbf{T}\vec{w}, \boldsymbol{A}_{\boldsymbol{h}}\vec{w}\rangle = 0$. By this, (23), and (33),

$$\begin{aligned} &-\tfrac{h^3}{2}\sum_{i=0}^N\left(\delta_x^3 w_i\right)^2 = -\tfrac{\tilde{\lambda}h^3}{2}\sum_{i=1}^N\left(\delta_x w_i\right)^2 \\ &\quad +\tfrac{1}{h^3}\left(w_1w_2 - 4(w_1)^2 - 4(w_N)^2 + w_Nw_{N-1}\right). \end{aligned}$$

Finally, by substituting this in (32),

$$
\begin{aligned}
S_1 =& -\tfrac{4L}{h^4}\left(w_N\right)^2 + 3h\sum_{i=0}^{N}\left(\delta_x^2 w_i\right)^2 \\
& -\tfrac{\tilde{\lambda}h^3}{2}\sum_{i=0}^{N}\left(\delta_x w_i\right)^2 - \tfrac{1}{h^3}\Big(10\left(w_1\right)^2 - w_1 w_2 \\
& -2\left(w_N\right)^2 - w_N w_{N-1}\Big).
\end{aligned}
\tag{34}
$$

Next, we estimate S_2. By Lemma 2 and (14), i.e. $\boldsymbol{D_{1c,h}}\vec{w} = C\boldsymbol{A_h}\vec{m} + P\vec{m}$, we have

$$S_2 = -2B^2h^2\left\langle \boldsymbol{\mathcal{L}}\vec{w}, i\boldsymbol{D_{1c,h}}(C\boldsymbol{A_h}\vec{m} + P\vec{m})\right\rangle.$$

Now, by the definition of $\boldsymbol{J_h}$ in Lemma 2 and adopting similar calculations from (26)-(27), this yields

$$
\begin{aligned}
S_2 =& \tfrac{B^2h^3C}{2}\sum_{i=1}^{N}\left(\delta_x^3 m_i\right)^2 - B^2Ch\sum_{i=1}^{N}\left(\delta_x^2 m_i\right)^2 \\
& +B^2Ph\sum_{i=0}^{N}\left(\delta_x m_i\right)^2 - \tfrac{B^2PL}{h^2}\left(m_N\right)^2.
\end{aligned}
\tag{35}
$$

Then, using (29)- (31), (34), and (35), (25) reduces to

$$
\begin{aligned}
& 4h\sum_{i=0}^{N}\left(\delta_x^2 w_i\right)^2 - \tilde{\lambda}h^3\sum_{i=0}^{N}\left(\delta_x w_i\right)^2 \\
& +B^2Ph\sum_{i=0}^{N}\left(\delta_x m_i\right)^2 + \tfrac{B^2Ch^3}{2}\sum_{i=1}^{N}\left(\delta_x^3 m_i\right)^2 \\
& = \tfrac{4L}{h^4}\left(w_N\right)^2 + \tfrac{B^2PL}{h^2}\left(m_N\right)^2 - B^2Ph\sum_{i=0}^{N}\left(\delta_x m_i\right)^2 \\
& -\tfrac{1}{h^3}(12\left(w_1\right)^2 - 4\left(w_N\right)^2 - w_1w_2 - w_Nw_{N-1}).
\end{aligned}
\tag{36}
$$

Next, multiply (14) by $2ih^2(\boldsymbol{D_{1c,h}}\vec{m})_i$ and take the sum from 1 to N to get

$$
\begin{aligned}
\tfrac{B^2PL}{h^2}\left(m_N\right)^2 =& B^2Ph\sum_{i=0}^{N}\left(\delta_x m_i\right)^2 + \tfrac{B^2P^2h^3}{2C}\sum_{i=0}^{N}\left(\delta_x m_i\right)^2 \\
& -\tfrac{B^2P^2h}{C}\sum_{i=0}^{N}(m_i)^2 - \tfrac{2B^2Ph^2}{C}\left\langle i\boldsymbol{D_{1c,h}}\vec{w}, \boldsymbol{D_{1c,h}}\vec{m}\right\rangle.
\end{aligned}
$$

Now, substitute this in (36) to obtain (24). □

Lemma 7. *For* (23)*,* $\tilde{\lambda}h = E_{h,MM}(t)$*.*

Proof. Since $\langle\vec{w},\vec{w}\rangle = 1$, multiply (23) by hw_i, sum over i from 1 to N, and use (14) and Lemma 2 to conclude. □

Remark 2. The identity $\tilde{\lambda}h = E_{h,EB}(t)$ is true for the eigen-solutions of $\boldsymbol{\mathcal{B}_h}\tilde{\lambda} = \tilde{\lambda}\vec{w}$ of (9).

Lemma 8. *Let* $\sigma > 0$*, be the numerical filtering parameter, then for each* $\lambda(\boldsymbol{A_h})h^2 \in (0, 4-\sigma)$*, the following inequalities hold true for the solutions of* (23)

$$
\begin{cases}
\tfrac{(2w_1-w_2)^2}{4h^3} + \tfrac{(2w_N-w_{N-1})^2}{4h^3} \le E_{h,MM}(t), \\
\langle \boldsymbol{A}_h\vec{w}, \vec{w}\rangle \le \tfrac{4-\sigma}{h^2}.
\end{cases}
\tag{37}
$$

Proof. First note that from Lemma 2, $\boldsymbol{J}_h$ is non-positive by a quick calculation, it can be shown that

$$\langle \boldsymbol{A}_h\vec{w}, \boldsymbol{A}_h\vec{w}\rangle \le \left\langle \tilde{\mathcal{A}}_h\vec{w}, \vec{w}\right\rangle = \tilde{\lambda}\left\langle\vec{w},\vec{w}\right\rangle. \tag{38}$$

Since the eigenvectors of (23) are normalized, we have $\tilde{\lambda} \ge ||A_h\vec{w}||_0^2 = \sum_{i=1}^{N}\delta_x^2 w_i\cdot\delta_x^2 w_i$. Multiply both side by $\frac{h}{2}$ to obtain $\frac{\tilde{\lambda}h}{2} \ge \frac{(2w_1-w_2)^2}{4h^3} + \frac{(2w_N-w_{N-1})^2}{4h^3}$. Next, use Lemma-7, and apply the boundary conditions in (23) to obtain $(37)_1$. Since $\langle\boldsymbol{A}_h\vec{w},\vec{w}\rangle \le ||\boldsymbol{A}_h||$, by Gershgorin's Circle Theorem [10], for any Strictly Diagonally Dominant (SDD) matrix, $|\lambda(\boldsymbol{A_h})| \le ||\boldsymbol{A_h}||_\infty$. Thus, $\lambda(\boldsymbol{A}_h)h^2 \le 4-\sigma$. Finally, the conclusion $(37)_2$ follows immediately. □

We can now state the main result.

Theorem 9 (Spectral Observability)**.** *Let* B *be small such that* $B<1$*,* $2C > Ph^2$*, and* $4B^2L^4P << C(2C-Ph^2)$*, ensuring* $\frac{4B^2L^4P}{C(2C-Ph^2)} > 0$ *is small. Choose the numerical filtering parameter* $\sigma \in \left(\frac{B^2L^2}{B^2L^2+C} + \frac{4B^2L^4P}{C(2C-Ph^2)}, 4\right)$*. For* $0 < \xi_1 < \underbrace{\frac{\sigma}{2} - \frac{2B^2L^2}{B^2L^2+C} - \frac{2B^2L^4P}{C(2C-Ph^2)}}_{>0}$*,* $0 < \xi_2 \le \underbrace{\frac{2C-Ph^2}{2L}}_{>0}$*, since* $\tilde{\lambda}(\boldsymbol{A}_h) \in (0, 4-\sigma)$*, there exists a constant* $\mathcal{R}(h) = \frac{\frac{\sigma}{2} - \frac{B^2L^2}{B^2L^2+C} - \frac{B^2L^3P}{C\xi_2} - \xi_1}{2L + \frac{h}{2\xi_1}} > 0$ *(as* $h\to 0$*) such that the following estimate holds*

$$\left|\frac{w_N}{h^2}\right|^2 \ge \mathcal{R}(h)E_{h,MM}(0). \tag{39}$$

Proof. First, apply the generalized Young's inequality with $\xi_1 > 0$ to majorize the left-hand side of (24) in Lemma 6, and use Remark 2 to obtain

$$
\begin{aligned}
& \tfrac{L}{2}\left(\delta_x^2 w_{N+1}\right)^2 + \tfrac{1}{2h^3}\Big\{\tfrac{(w_1)^2}{2\xi_1} + \tfrac{(2w_1-w_2)^2\xi_1}{2} + \tfrac{(w_N)^2}{2\xi_1} \\
& + \tfrac{\xi_1(2w_N-w_{N-1})^2}{2}\Big\} - \tfrac{1}{h^3}\left((w_1)^2 + (w_N)^2\right) \\
& -\tfrac{B^2Ph^2}{C}\langle i\boldsymbol{D_{1c,h}}\vec{w}, \boldsymbol{D_{1c,h}}\vec{m}\rangle \ge 2E_{h,EB}(t) \\
& -\tfrac{\tilde{\lambda}h^3}{2}\langle\boldsymbol{A}_h\vec{w},\vec{w}\rangle + \left(\tfrac{B^2Ph}{2} - \tfrac{B^2P^2h^3}{4C}\right)\langle\boldsymbol{A}_h\vec{m},\vec{m}\rangle.
\end{aligned}
$$

By the symmetry of the eigenvalue problem (23)), $(w_1)^2 = (w_N)^2$. Now, use Lemmas 7, 8 to obtain

$$
\begin{aligned}
& -\tfrac{B^2Ph^2}{C}\langle i\boldsymbol{D_{1c,h}}\vec{w}, \boldsymbol{D_{1c,h}}\vec{m}\rangle \\
& \quad + \left(\tfrac{L}{2} + \tfrac{h}{8\xi_1} - \tfrac{h}{2}\right)\left(\delta_x^2 w_{N+1}\right)^2 \ge 2E_{h,EB}(t) \\
& \quad - \left(\xi_1 + \tfrac{4-\sigma}{2}\right)E_{h,MM}(t) \\
& \quad + \left(\tfrac{B^2Ph}{2} - \tfrac{B^2P^2h^3}{4C}\right)\langle\boldsymbol{A}_h\vec{m},\vec{m}\rangle.
\end{aligned}
\tag{40}
$$

Since $Nh < L$, apply generalized Young's inequality with $\xi_2 > 0$ and Lemma 4 to obtain

$$
\begin{aligned}
& -\tfrac{B^2Ph^2}{C}\langle i\boldsymbol{D_{1c,h}}\vec{w}, \boldsymbol{D_{1c,h}}\vec{m}\rangle \\
& \le \tfrac{B^2PhL}{2C\xi_2}\sum_{i=1}^{N}\left|(\boldsymbol{D_{1c,h}}\vec{w})_i\right|^2 + \tfrac{B^2PhL\xi_2}{2C}\sum_{i=1}^{N}\left|(\boldsymbol{D_{1c,h}}\vec{m})_i\right|^2 \\
& \le \tfrac{B^2PhL^3}{C\xi_2}E_{h,MM}(t) + \tfrac{B^2PhL\xi_2}{2C}\langle\boldsymbol{A_h}\vec{m},\vec{m}\rangle.
\end{aligned}
$$

Substituting this in (40) and using Theorem 5 gives

$$
\begin{aligned}
& \left(\tfrac{L}{2} + \tfrac{h}{8\xi_1}\right)\left(\delta_x^2 w_{N+1}\right)^2 \\
& \ge \left(\tfrac{\sigma}{2} + \tfrac{2C}{B^2L^2+C} - \tfrac{B^2PL^3}{C\xi_2} - \xi_1 - 2\right)E_{h,MM}(t) \\
& + \left(\tfrac{B^2Ph}{2} - \tfrac{B^2P^2h^3}{4C} - \tfrac{B^2PhL\xi_2}{2C}\right)\langle\boldsymbol{A_h}\vec{m},\vec{m}\rangle.
\end{aligned}
$$

Since $\left|\delta_x^2 w_{N+1}\right|^2 = 4\left|\frac{w_N}{h^2}\right|^2$, use Lemma 3, and choose ξ_1, ξ_2 as in the theorem to obtain (39). □

A. Numerical Experiments

To validate our theoretical findings, we consider a composite structure consisting of PZT-silicone, rubber, and aluminum layers, arranged from bottom to top, with dimensions $L = 1, \mathrm{m}$, $h_1 = h_3 = 0.1\mathrm{m}$, and $h_2 = 0.03\mathrm{m}$. The filtering parameter is set within $\sigma \in (0.1, 4)$, and the corresponding parameters are calculated as $B \approx 0.09$, $C \approx 0.7$, and $P \approx 5.4 \times 10^{-9}$. For more details, see [2].

Table I shows the effect of increasing N with a fixed filtering parameter $\sigma = 2.75$. As N increases, more eigenvalues are retained, improving numerical accuracy. For $N = 20$ (40 eigenvalues), 16 are retained, and 24 are filtered out. As N grows, more eigenvalues are retained.

Filtering par. $\sigma < 4$	# of total e-values=$2N$	# of retained e-values	# of filtered e-values
2.75	40	16	24
2.75	80	30	50
2.75	160	60	100

TABLE I: Increasing N with a fixed σ increases the number of retained eigenvalues.

Table II highlights the relationship between the desired number of retained eigenvalues and the corresponding filtering parameter σ. To retain 20 eigenvalues as N increases, σ must be reduced. For example, with 40 total eigenvalues, $\sigma = 3.97$ is needed, while for 160 eigenvalues, σ must be reduced to 0.6.

Desired # of e-values to be retained	Total # of e-values=$2N$	# of filtered e-values	$\sigma < 4$
20	40	20	3.97
20	80	60	1.97
20	160	140	0.6

TABLE II: Adjusting σ to retain 20 eigenvalues as N increases.

IV. Conclusions & Future Work

Key conclusions include the alignment of the smallness assumption for B in Theorem 9 with findings in [2], highlighting the importance of large shear in the middle layer for improved damping. The filtering parameter σ is critical for uniform spectral observability. As $\sigma \to 4$, only low-frequency eigenvalues are retained; as $\sigma \to 0$, insufficient filtering compromises observability as $h \to 0$. Therefore, appropriate filtering is essential to eliminate spurious high-frequency eigenvalues. Increasing the number of nodes N and applying suitable filtering is recommended to retain high-frequency modes without sacrificing accuracy.

Proving uniform spectral observability paves the way for our next goal of establishing uniform exact boundary observability of the reduced model of (1) as $h \to 0$. Using the discrete multiplier approach and Haraux's theorem, we aim to achieve uniform observability as $h \to 0$ for arbitrarily small observation times [7], inspired by [1].

The methodology developed in this paper is also being applied to a more complex three-layer beam model in [15], which includes additional wave equations [8].

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