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Displacement-Based Formation Control with Measurement Noises

Weiqiang Chen[1,2], Liangming Chen[2], Jie Mei[1], *Member, IEEE* and Hongwei Zhang[1], *Senior Member, IEEE*

***Abstract*— Multi-agent formations have many practical applications. Measurement noises are inevitable in multi-agent formations, in which, however, the existing results mainly focus on special types of noises, and the analytical discussion on the effect of general noises is challenging and remains open. This motivates us to study the effect of stochastic measurement noises on displacement-based multi-agent formations, which are described by a general form of stochastic processes with finite second-order moments. First, for the case of unbiased measurement noises, a sufficient and necessary condition is derived for the existence of solutions in the stochastic dynamics of multi-agent formations. Then, several statistical features and convergence of formation errors are analyzed. In particular, for the case of unbiased measurement noises described by zero-mean wide-sense stationary processes, an upper bound on the mean square convergence of formation errors is obtained. Finally, we demonstrate the effectiveness of our theoretical results through a simulation example.**

I. INTRODUCTION

Recently, formation control has been extensively studied in the research community of multi-agent systems (MASs) due to its wide applications [1]. The goal of formation control is to coordinate the movement of multiple agents, such as robots and drones [2], to achieve a desired relative spatial configuration or prescribed geometric shape for agents' positions [3]. Various strategies have been proposed to solve the formation control problem, such as leader-follower approach [4], virtual structure approach [5], and behavior-based approach [6], in which as one of behavior-based control approaches, consensus-based displacement formation control has been widely studied due to its high flexibility and robustness [7]. In addition, several types of constraints have been utilized to describe the geometric shape of multi-agent formations, including displacement constraints [8], distance constraints [9], bearing constraints [10] and angle constraints [11], [12]. It is noteworthy that a desired formation described by displacement constraints can not only ensure fixed orientation and scale of the formation, but also easily guarantee global convergence of the formation. Therefore, due to the above two aspects, it is valuable to further investigate the formation control problem by using the displacement-based approach.

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[1]Weiqiang Chen, Jie Mei and Hongwei Zhang are with the School of Mechanical Engineering and Automation, Harbin Institute of Technology, Shenzhen, Shenzhen, 518055, China. Emails: aqiang833@163.com, jmei@hit.edu.cn and hwzhang@hit.edu.cn

[2]Weiqiang Chen and Liangming Chen are with the Center for Control Science and Technology and Shenzhen Key Laboratory of Control Theory and Intelligent Systems, Southern University of Science and Technology, Shenzhen 518055, China. Email: chenlm6@sustech.edu.cn

For the existing works on formation control, most of them consider that the system states or measurements are deterministic signals [9], [12], [13]. However, in engineering practices, stochastic noises are inevitable which can be caused by a variety of factors, such as environmental disturbances, electrical noises, and mechanical vibration [14]. It is worth noting that the existence of even unbiased stochastic noises can make a stable deterministic system become unstable, see the examples in [15, Chapters 11.1 and 11.2]. Therefore, it is also important to investigate the effect of system or measurement noises on multi-agent formations. The work [16] studies the displacement-based multi-agent formation control problem, in which each agent's dynamics are considered as a stochastic differential equation. This study employs Wiener processes (normally distributed processes) as stochastic noises in the agents' dynamics, where these noises are not strictly differentiable in the mathematical sense [17]. Distributed formation control of discrete-time MASs with measurement noises is investigated in [18], in which each agent can measure inter-agent relative distances subject to zero-mean measurement noises with fixed variances. By using an adaptive estimator-based formation control law, the mean square formation errors have an upper bound in [18], in which other statistical properties of the formation errors, such as autocorrelations and autocovariances of formation errors, are not discussed. In addition, for the consensus problem of continuous-time MASs, consensus conditions under the existence of measurement noises are derived in [19]. It is worth mentioning that the results presented in [19] assume that the measurement noises in the inter-agent displacements are independent Gaussian white noises. In practice, the measurement noises may not be described by one special type of stochastic noises, but a more general form whose probability distribution is time-varying. In such case, due to the time-varying property of the noises' distribution, it is challenging to directly derive the relevant statistical properties and convergence of the formations, which has not been adequately investigated in the existing literature. Thus, the displacement-based formation control problem in the presence of general stochastic noises is worthy of further investigation.

Motivated by the aforementioned discussions, this paper aims to investigate multi-agent formation control problem utilizing inter-agent displacement measurements that are subjected to stochastic noises represented by a general form of stochastic processes with finite second-order moments. Specifically, we consider a class of noises, namely unbiased (zero-mean) noises. To achieve the desired multi-agent formation, the consensus-based strategy is employed to design a distributed control law by using inter-agent displacement measurements. The main contributions of this paper are summarized as two aspects. Firstly, a sufficient and necessary condition is derived for the existence of a non-trivial solution in the stochastic dynamics of multi-agent formations with unbiased measurement noises, in which the stochastic noises are described by zero-mean stochastic processes with finite second-order moments. Secondly, the statistical features including the mean values, autocorrelations and autocovariances of the formation errors are derived, and the convergence of the formation errors is further analyzed.

The rest of this paper are organized as follows. Section II introduces some basic mathematical knowledge and problem formulation. Section III discusses the case of displacement-based formation control in the presence of unbiased measurement noises. A numerical simulation is provided in Section IV.

II. PRELIMINARIES

Notations: Let $\mathbb{E}\{\cdot\}$ denote the mathematical expectation of a stochastic vector or matrix. Let $\mathbf{I}_n$ be the identity matrix with dimension n, and $\mathbf{0}_{m\times n}$ denote the zero matrix of m rows and n columns. Denote $\mathbf{1}_n = [1,...,1]^\top \in \mathbb{R}^n$ and $\mathbf{0}_n = [0,...,0]^\top \in \mathbb{R}^n$. Let $\sup\{\cdot\}$ denote the supremum of the set. Let $\|\cdot\|$ represent the Euclidean norm and $\mathbb{T}$ denote time domain described by $\{t : 0 \le t < \infty\}$, respectively. Let $\otimes$ denote the Kronecker product. For two symmetric matrices X and Y, $X \preceq Y$ (resp. $X \prec Y$) represents $Y - X$ is a positive semidefinite (resp. positive definite) matrix.

A. Graph theory

Consider a multi-agent system consisting of N agents, which are labeled from 1 to N. An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to describe the measurement topology among the agents, where $\mathcal{V} = \{1, 2, ..., N\}$ is the vertex set with agent i represented by vertex i, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set with the displacement measurement between i and j represented by (i, j). Denote $|\mathcal{E}|$ as the number of edges in $\mathcal{G}$. Let $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}, i \ne j\}$ represent the neighbor set of agent i. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as $a_{ij} = 1$ if $(j, i) \subseteq \mathcal{E}$, and $a_{ij} = 0$, otherwise. The indegree matrix of $\mathcal{G}$ is defined as $D = [d_{ij}] \in \mathbb{R}^{N\times N}$, where $d_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$ and $d_{ij} = 0$ for $i \ne j$. The Laplacian matrix of $\mathcal{G}$ is defined as $L \triangleq D - A$. The incidence matrix $H = [h_{ij}] \in \mathbb{R}^{N\times|\mathcal{E}|}$ is related the vertices to the edges, where the entries with arbitrary edge orientations is given by $h_{ij} = 1$ if the j-th edge sinks at vertex i, $h_{ij} = -1$ if the j-th edge leaves vertex i, and $h_{ij} = 0$, otherwise. In this paper, we consider the measurement graph as an undirected graph $\mathcal{G}$, in which one has $L = HH^\top$ [20].

***Lemma** 1 ([20]):* A connected and undirected graph $\mathcal{G}$ satisfies the following two properties

1) The matrix $L = HH^\top$ is positive semidefinite with a single zero eigenvalue.
2) The matrix $H^\top H$ is positive definite if the graph $\mathcal{G}$ has no cycles.

B. Stochastic processes

***Definition** 1 ([21]):* Consider a stochastic process $\{\nu(t) \in \mathbb{R}, t \in [a, b], 0 \le a < b < \infty\}$ which satisfies $\mathbb{E}\{\nu^2(t)\} < \infty$, and a deterministic function $h(z, t) \in \mathbb{R}$, where $z \in \mathbb{R}$ is a parameter. The stochastic process is said to be integrable in mean square sense (m.s. integrable) if the following equation holds

$$\lim_{\max\Delta t_i\to 0} \mathbb{E}\left\{\left|\psi(z) - \sum_{i=1}^{n} h(z, t_i^*)\nu(t_i^*)\Delta t_i\right|^2\right\} = 0, \quad (1)$$

where $\Delta t_i = t_{i+1} - t_i$ for $t_i^* \in [t_i, t_{i+1}]$, $\{t_i \in [a, b] : a \le t_i \le t_{i+1} \le b, 1 \le i \le n\}$ and $\psi(z)$ is a function of z. In addition, we denote the stochastic integral as $\psi(z) = \int_a^b h(z, \tau)\nu(\tau)\mathrm{d}\tau$.

***Definition** 2 ([21]):* A stochastic process $\{\nu(t) \in \mathbb{R}, t \in [a, b], 0 \le a < b < \infty\}$ is called wide-sense stationary(W.S.S), if its mean $\mathbb{E}\{\nu(t)\}$ is constant, and its autocorrelation $\mathbb{E}\{\nu(t_1)\nu(t_2)\}$ only depends on $\Delta t = t_2 - t_1$, where $t_1, t_2 \in [a, b]$.

***Lemma** 2:* [21, Theorem 9A-3] The stochastic process $\{\nu(t) \in \mathbb{R}, t \in [a, b], 0 \le a < b < \infty\}$ is m.s. integrable, if and only if

$$\int_a^b \int_a^b \mathbb{E}\{\nu(t_1)\nu(t_2)\}\, \mathrm{d}t_1\mathrm{d}t_2 < \infty. \quad (2)$$

C. Problem formulation

For an N-agent formation, we consider that its measurement graph $\mathcal{G}$ is connected and undirected without cycles. The agents' dynamics are described by a single integrator model

$$\dot{p}_i(t) = u_i(t), \qquad i = 1, ..., N \quad (3)$$

where $p_i(t) \in \mathbb{R}^2$ is the position of agent i in a 2-D plane, and $u_i(t) \in \mathbb{R}^2$ is the control input of agent i.

Assume that agent i can only measure inter-agent displacements subject to measurement noises, i.e.,

$$y_{ji}(t) \triangleq p_{ji}(t) + \xi_{ji}(t), \quad (4)$$

where $y_{ji}(t) \in \mathbb{R}^2$ is the measured displacement of agent j with respect to (w.r.t.) agent i, $p_{ji}(t) = p_i(t) - p_j(t)$ denotes the real displacement of agent j w.r.t. agent i, and $\xi_{ji}(t) = [\xi^x_{ji}(t), \xi^y_{ji}(t)]^\top \in \mathbb{R}^2$ is the measurement noise associated with the edge (j, i). In this paper, we consider that the measurement noises in x and y directions are mutually independent, and the second-order moments of $\xi^x_{ji}(t)$ and $\xi^y_{ji}(t)$ exist for any $(j, i) \in \mathcal{E}$ and $t \in \mathbb{T}$, i.e., $\mathbb{E}\{(\xi^x_{ji}(t))^2\} < \infty$ and $\mathbb{E}\{(\xi^y_{ji}(t))^2\} < \infty$.

According to the displacement-based formation control algorithm [22, Eq. (4.23)], the displacement-based formation controller under the measured noisy displacements (4) can be described as

$$u_i(t) = -\alpha \sum_{j\in\mathcal{N}_i} (y_{ji}(t) - \delta_{ji}), \quad (5)$$

where $\alpha \in \mathbb{R}^+$ is a control gain, and $\delta_{ji} \in \mathbb{R}^2$ is the desired constant displacement of agent i w.r.t. agent j. Based on

[23, Theorem 3], if the undirected graph $\mathcal{G}$ is connected, there exists $\delta_i \in \mathbb{R}^2$ and $\delta_j \in \mathbb{R}^2$ such that $\delta_{ji} = \delta_i - \delta_j$ for all $(i,j) \in \mathcal{E}$. Define the formation error of agent i as $\tilde{p}_i(t) \triangleq p_i(t) - \delta_i$. Then the formation errors' dynamics can be written as

$$\dot{\tilde{p}}_i(t) = \alpha \sum_{j \in \mathcal{N}_i} (\tilde{p}_j(t) - \tilde{p}_i(t) + \xi_{ji}(t)). \tag{6}$$

The aim of this paper is to investigate how the multi-agent formation governed by (6) evolves under two types of stochastic measurement noises, namely unbiased noise with $\mathbb{E}\{\xi_{ji}(t)\} = \mathbf{0}_2$ for any $t \in \mathbb{T}$ and biased noise with $\mathbb{E}\{\xi_{ji}(t_1)\} \neq \mathbf{0}_2$, $\exists t_1 \in \mathbb{T}$. We will discuss these two cases respectively in the following two sections. Particularly, the statistical features and convergence of the formation errors will be our interests.

III. Formation Control with Unbiased Measurement Noises

First, we denote edge $(i,j) \in \mathcal{E}$ as the k-th edge in the measurement graph $\mathcal{G}$, where $k = 1, 2, ..., |\mathcal{E}|$. Then, the displacement measurement $y_k(t)$ can be rewritten as

$$y_k(t) = p_k(t) + \xi_k(t), \qquad k = 1, 2, ..., |\mathcal{E}|. \tag{7}$$

The statistical features of $\xi_k(t)$ are selected as

$$\begin{aligned} &\mu_{\xi_k}(t) \triangleq \mathbb{E}\{\xi_k(t)\}, \qquad t \in \mathbb{T} \\ &R_{\xi_k}(t_1,t_2) \triangleq \mathbb{E}\left\{\xi_k(t_1)\xi_k^\top(t_2)\right\}, \qquad t_1, t_2 \in \mathbb{T} \\ &Q_{\xi_k}(t_1,t_2) \triangleq \mathbb{E}\left\{(\xi_k(t_1) - \mu_{\xi_k}(t_1))(\xi_k(t_2) - \mu_{\xi_k}(t_2))^\top\right\}, \end{aligned} \tag{8}$$

where $\mu_{\xi_k}(t) \in \mathbb{R}^2$ is the mean value of $\xi_k(t)$, and $R_{\xi_k}(t_1,t_2) \in \mathbb{R}^{2\times 2}$ and $Q_{\xi_k}(t_1,t_2) \in \mathbb{R}^{2\times 2}$ are the autocorrelation and autocovariance matrices of $\xi_k(t)$ between time instants t_1 and t_2, respectively.

A. Statistical properties of formation error dynamics (6)

We consider that the measurement noise $\xi_k(t)$ satisfies the following assumption.

***Assumption** 1:* For any $k = 1, 2, ..., |\mathcal{E}|$, each element of the measurement noise $\{\xi_k(t), t \in \mathbb{T}\}$ with finite second-order moment satisfies the following conditions

1) $\mu_{\xi_k}(t) \equiv \mathbf{0}_2, \qquad \forall t \in \mathbb{T}$,
2) $R_{\xi_k}(t_1,t_2) \neq \mathbf{0}_{2\times 2}, \qquad \exists t_1, t_2 \in \mathbb{T}$.
3) $\mathbb{E}\left\{\xi_m(t)\xi_n^\top(t)\right\} \equiv \mathbf{0}_{2\times 2}, \quad m \neq n, \quad m, n = 1, 2, ..., |\mathcal{E}|$.

Since $\xi_k^x(t)$ and $\xi_k^y(t)$ are independent, we have $R_{\xi_k}(t_1,t_2) = \begin{bmatrix} R_{\xi_k}^x(t_1,t_2) & 0 \\ 0 & R_{\xi_k}^y(t_1,t_2) \end{bmatrix}$, and $Q_{\xi_k}(t_1,t_2) = \begin{bmatrix} Q_{\xi_k}^x(t_1,t_2) & 0 \\ 0 & Q_{\xi_k}^y(t_1,t_2) \end{bmatrix}$. When $t = t_1 = t_2$, $Q_{\xi_k}^x(t,t)$ and $Q_{\xi_k}^y(t,t)$ are variances of the measurement noises in x and y directions, respectively. In addition, if $\mu_{\xi_k}(t) \equiv \mathbf{0}_2$, we have

$$Q_{\xi_k}(t_1,t_2) = R_{\xi_k}(t_1,t_2). \tag{9}$$

***Remark** 1:* It is worth mentioning that Assumption 1 is more general than those assumptions in most existing works where the stochastic noises are independent or uncorrelated between two time instants, such as Gaussian white noises. □

Define $\tilde{p}(t) \triangleq [\tilde{p}_1(t)^\top, ..., \tilde{p}_N(t)^\top]^\top$, $\xi(t) \triangleq [\xi_1(t)^\top, ..., \xi_{|\mathcal{E}|}(t)^\top]^\top$, $\mathcal{L} \triangleq L \otimes \mathbf{I}_2$, and $\mathcal{H} \triangleq H \otimes \mathbf{I}_2$. We can obtain the overall formation error dynamics

$$\dot{\tilde{p}} = -\alpha\mathcal{L}\tilde{p}(t) - \alpha\mathcal{H}\xi(t). \tag{10}$$

According to Assumption 1, the measurement noise $\xi(t)$ satisfies the following statistical properties

$$\begin{aligned} &\mu_\xi(t) \triangleq \mathbb{E}\{\xi(t)\} = \mathbf{0}_{2|\mathcal{E}|}, \\ &R_\xi(t_1,t_2) \triangleq \mathbb{E}\left\{\xi(t_1)\xi^\top(t_2)\right\} \\ &\qquad = \begin{bmatrix} R_{\xi_1}(t_1,t_2) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_{\xi_{|\mathcal{E}|}}(t_1,t_2) \end{bmatrix}. \end{aligned} \tag{11}$$

Now we present the following conclusion.

***Lemma** 3:* Consider that the multi-agent system (3) is controlled by (5), Assumption 1 holds, and the graph $\mathcal{G}$ is connected and undirected. The overall formation error dynamics (10) exists a non-trivial solution in mean square sense if and only if all the 2-by-2 block diagonal submatrices of the matrix $\lim_{t\to\infty} \int_0^t \int_0^t e^{-\alpha\mathcal{L}(t-s)}\mathcal{H}R_\xi(s,l)\mathcal{H}^\top e^{-\alpha\mathcal{L}(t-l)}\mathrm{d}s\mathrm{d}l$ exist, in which each block submatrix belongs to $\mathbb{R}^{2\times 2}$. Moreover, if there is a solution, then this non-trivial solution can be written as

$$\tilde{p}(t) = e^{-\alpha\mathcal{L}t}\tilde{p}(0) - \alpha\int_0^t e^{-\alpha\mathcal{L}(t-\tau)}\mathcal{H}\xi(\tau)\mathrm{d}\tau. \tag{12}$$

Proof: Note that if $\xi(t)$ is a deterministic signal, it is straightforward that the solution of (10) can be written as (12). However, this is not the case for the stochastic process $\{\xi(t), t \in \mathbb{T}\}$. To ensure the existence of solutions in mean square sense for (10), it must hold that the stochastic integral $\int_0^t e^{-\alpha\mathcal{L}(t-\tau)}\mathcal{H}\xi(\tau)\mathrm{d}\tau$ exists in mean square sense on $\mathbb{T}$, i.e., the stochastic function $e^{-\alpha\mathcal{L}(t-\tau)}\mathcal{H}\xi(\tau)$ is m.s. integrable on $\mathbb{T}$.

For simplicity, we define

$$\begin{aligned} &\Phi(t,\tau) \triangleq e^{-\alpha\mathcal{L}(t-\tau)}\mathcal{H}, \\ &\eta(t) \triangleq \int_0^t \Phi(t,\tau)\xi(\tau)\mathrm{d}\tau. \end{aligned} \tag{13}$$

Then the stochastic integral $\eta(t)$ can be further derived as

$$\begin{aligned} \eta(t) &= \int_0^t \begin{bmatrix} \Phi_{11}(t,\tau) & \cdots & \Phi_{1|\mathcal{E}|}(t,\tau) \\ \vdots & \ddots & \vdots \\ \Phi_{N1}(t,\tau) & \cdots & \Phi_{N|\mathcal{E}|}(t,\tau) \end{bmatrix} \begin{bmatrix} \xi_1(\tau) \\ \vdots \\ \xi_{|\mathcal{E}|}(\tau) \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} \int_0^t \left(\sum_{j=1}^{|\mathcal{E}|} \Phi_{1j}(t,\tau)\xi_j(\tau) \right) \mathrm{d}\tau \\ \vdots \\ \int_0^t \left(\sum_{j=1}^{|\mathcal{E}|} \Phi_{Nj}(t,\tau)\xi_j(\tau) \right) \mathrm{d}\tau \end{bmatrix}, \end{aligned} \tag{14}$$

where $\Phi_{ij}(t,\tau) \in \mathbb{R}^{2\times 2}$ represents the i-th row, j-th column, and 2-by-2 submatrix of $\Phi(t,\tau)$, $1 \le i \le N$ and $1 \le j \le |\mathcal{E}|$. To show that the i-th subvector $\int_0^t \left(\sum_{j=1}^{|\mathcal{E}|} \Phi_{ij}(t,\tau)\xi_j(\tau)\right) \mathrm{d}\tau$ of $\eta(t)$ exists in mean square sense for any $i = 1, ..., N$, according to Lemma 2 and Assumption 1, we can obtain that

$$\begin{aligned}&\mathbb{E}\left\{\int_0^t \int_0^t \left(\sum_{j=1}^{|\mathcal{E}|} \Phi_{ij}(t,s)\xi_j(s)\right)\left(\sum_{j=1}^{|\mathcal{E}|} \Phi_{ij}(t,l)\xi_j(l)\right)^\top \mathrm{d}s\mathrm{d}l\right\}\\ &= \int_0^t \int_0^t \left(\sum_{j=1}^{|\mathcal{E}|} \Phi_{ij}(t,s)\mathbb{E}\left\{\xi_j(s)\xi_j^\top(l)\right\}\Phi_{ij}^\top(t,l)\right)\mathrm{d}s\mathrm{d}l\\ &= \int_0^t \int_0^t \left(\sum_{j=1}^{|\mathcal{E}|} \Phi_{ij}(t,s)R_{\xi_j}(s,l)\Phi_{ij}^\top(t,l)\right)\mathrm{d}s\mathrm{d}l.\end{aligned} \tag{15}$$

Note that the i-th 2-by-2 block diagonal submatrix of $\Phi(t,s)R_\xi(s,l)\Phi^\top(t,l)$ is $\sum_{j=1}^{|\mathcal{E}|} \Phi_{ij}(t,s)R_{\xi_j}(s,l)\Phi_{ij}^\top(t,l)$. Therefore, according to Lemma 2, the subvectors of stochastic integral $\eta(t)$ exist in mean square if and only if (15) exists for any $1 \le i \le N$ when $t \to \infty$. This condition is equivalent to that all the 2-by-2 block diagonal submatrices of the matrix $\lim_{t\to\infty} \int_0^t \int_0^t \Phi(t,s)R_\xi(s,l)\Phi^\top(t,l)\mathrm{d}s\mathrm{d}l$ exist. □

Based on Lemma 3 and result in *Fubini's* Theorem [24], the mean value and autocorrelation of $\eta(t)$ can be derived as

$$\mu_\eta(t) \triangleq \mathbb{E}\{\eta(t)\} = \int_0^t \Phi(t,\tau)\mu_\xi(\tau)\mathrm{d}\tau \equiv \mathbf{0}_{2N}, \tag{16}$$

$$\begin{aligned}R_\eta(t_1,t_2) &\triangleq \mathbb{E}\left\{\eta(t_1)\eta^\top(t_2)\right\}\\ &= \int_0^{t_1}\int_0^{t_2} \Phi(t_1,s)R_\xi(s,l)\Phi^\top(t_2,l)\mathrm{d}s\mathrm{d}l.\end{aligned} \tag{17}$$

Before further discussing statistical properties of the non-trivial solution $\tilde{p}(t)$, we give the following assumption.

Assumption 2: The initial formation error $\tilde{p}(0) \in \mathbb{R}^{2N}$ is a multivariate stochastic variable, where each element of $\tilde{p}(0)$ is uncorrelated with every element of $\xi(t)$ for any $t \in \mathbb{T}$.

Based on Lemma 3, we can obtain the following theorem on statistical properties of the non-trivial solutions $\tilde{p}(t)$.

Theorem 1: Consider that the multi-agent system (3) is controlled by (5) and the graph $\mathcal{G}$ is connected and undirected. If Assumptions 1, 2 and the condition stated in Lemma 3 hold, the mean value, autocorrelation and autocovariance of $\tilde{p}(t)$ can be described as

$$\mu_{\tilde{p}}(t) \triangleq \mathbb{E}\{\tilde{p}(t)\} = e^{-\alpha\mathcal{L}t}\tilde{p}_0, \tag{18}$$

$$\begin{aligned}R_{\tilde{p}}(t_1,t_2) &\triangleq \mathbb{E}\left\{\tilde{p}(t_1)\tilde{p}^\top(t_2)\right\}\\ &= e^{-\alpha\mathcal{L}t_1}R_{\tilde{p}_0}e^{-\alpha\mathcal{L}t_2} + \alpha^2 R_\eta(t_1,t_2),\end{aligned} \tag{19}$$

$$\begin{aligned}Q_{\tilde{p}}(t_1,t_2) &\triangleq \mathbb{E}\left\{(\tilde{p}(t_1)-\mu_{\tilde{p}}(t_1))(\tilde{p}(t_2)-\mu_{\tilde{p}}(t_2))^\top\right\}\\ &= e^{-\alpha\mathcal{L}t_1}Q_{\tilde{p}_0}e^{-\alpha\mathcal{L}t_2} + \alpha^2 R_\eta(t_1,t_2),\end{aligned} \tag{20}$$

where $\tilde{p}_0 = \mathbb{E}\{\tilde{p}(0)\}$, $R_{\tilde{p}_0} = \mathbb{E}\left\{\tilde{p}(0)\tilde{p}^\top(0)\right\}$, and $Q_{\tilde{p}_0} = \mathbb{E}\left\{(\tilde{p}(0)-\tilde{p}_0)(\tilde{p}(0)-\tilde{p}_0)^\top\right\}$.

Proof: Combining (12) and (16), one has that the mean value of $\tilde{p}(t)$ can be rewritten as $\mu_{\tilde{p}}(t) = \mathbb{E}\left\{e^{-\alpha\mathcal{L}t}\tilde{p}(0)\right\} - \alpha\mu_\eta(t) = e^{-\alpha\mathcal{L}t}\tilde{p}_0$. Since each element of $\tilde{p}(0)$ and every element of $\xi(t)$ are uncorrelated, $\tilde{p}(0)$ and $\eta(t)$ are uncorrelated for any $t \in \mathbb{T}$. Then we have

$$\mathbb{E}\left\{\tilde{p}(0)\eta^\top(t)\right\} = \tilde{p}_0\mu_\eta^\top(t) \equiv \mathbf{0}_{2N\times 2N}. \tag{21}$$

Next, the autocorrelation of $\tilde{p}(t)$ between t_1 and t_2 can be derived as

$$\begin{aligned}R_{\tilde{p}}(t_1,t_2) = e^{-\alpha\mathcal{L}t_1}&\mathbb{E}\left\{\tilde{p}(0)\tilde{p}^\top(0)\right\}e^{-\alpha\mathcal{L}t_2}\\ &- \alpha e^{-\alpha\mathcal{L}t_1}\mathbb{E}\{\tilde{p}(0)\}\mu_\eta^\top(t_2)\\ &- \alpha\mu_\eta(t_1)\mathbb{E}\left\{\tilde{p}^\top(0)\right\}e^{-\alpha\mathcal{L}t_2}\\ &+ \alpha^2\mathbb{E}\left\{\eta(t_1)\eta^\top(t_2)\right\}.\end{aligned} \tag{22}$$

By substituting (16) and (18) into (22), the autocorrelation $R_{\tilde{p}}(t_1,t_2)$ can be derived as (19). Furthermore, similar to the above calculations, the autocovariance of $Q_{\tilde{p}}(t_1,t_2)$ can be obtained as (20). □

Based on the above theorem, we now further discuss the convergence of $\tilde{p}(t)$.

Corollary 1: Consider that the multi-agent system (3) is controlled by (5) and the graph $\mathcal{G}$ is connected and undirected. If Assumptions 1, 2 and the condition stated in Lemma 3 hold, one has the following conclusions

1) For all $\tilde{p}_0$, the mean values of the agents' formation errors reach consensus, i.e., $\lim_{t\to\infty}\left\|\mu_{\tilde{p}_i}(t) - \mu_{\tilde{p}_j}(t)\right\| = 0$, where $j, i \in \mathcal{V}$. Furthermore, the desired formation is achieved in mean sense, i.e., $\lim_{t\to\infty}\|\mathbb{E}\{p_{ji}(t) - \delta_{ji}\}\| = 0$.
2) If $\lim_{t\to\infty} R_\eta(t,t)$ exists, the variances of the formation errors in x and y directions converge to constant values when $t \to \infty$.

Proof: From (18) in Theorem 1, the mean value $\mu_{\tilde{p}}(t)$ can be regarded as a solution of one deterministic consensus system described by $\dot{\mu}_{\tilde{p}}(t) = -\mathcal{L}\mu_{\tilde{p}}(t)$, in which its initial state $\mu_{\tilde{p}}(0)$ equals $\tilde{p}_0$. According to the result in [25, Lemma 1], it is straightforward to obtain that $\lim_{t\to\infty}\left\|\mu_{\tilde{p}_i}(t) - \mu_{\tilde{p}_j}(t)\right\| = 0$, where $j, i \in \mathcal{V}$. Furthermore, we can get

$$\begin{aligned}&\lim_{t\to\infty}\left\|\mu_{\tilde{p}_i}(t) - \mu_{\tilde{p}_j}(t)\right\|\\ &= \lim_{t\to\infty}\|\mathbb{E}\{p_i(t)\} - \mathbb{E}\{p_j(t)\} - \delta_{ji}\|\\ &= \lim_{t\to\infty}\|\mathbb{E}\{p_{ji}(t) - \delta_{ji}\}\| = 0.\end{aligned} \tag{23}$$

Since the autocorrelation $R_\eta(t_1,t_2)$ is a positive semidefinite matrix, if $\lim_{t\to\infty} R_\eta(t,t)$ exists, we can obtain that $\lim_{t\to\infty} R_\eta(t,t) \preceq \bar{R}_\eta\mathbf{I}_{2N}$, where $0 \le \bar{R}_\eta < \infty$. From (20), it can be further derived that

$$\begin{aligned}\lim_{t\to\infty} Q_{\tilde{p}}(t,t) &\preceq \lim_{t\to\infty}\left(e^{-\alpha\mathcal{L}t}Q_{\tilde{p}_0}e^{-\alpha\mathcal{L}t}\right) + \alpha^2\bar{R}_\eta\mathbf{I}_{2N}\\ &= \frac{1}{N^2}\mathbf{1}_{2N}\mathbf{1}_{2N}^\top Q_{\tilde{p}_0}\mathbf{1}_{2N}\mathbf{1}_{2N}^\top + \alpha^2\bar{R}_\eta\mathbf{I}_{2N}.\end{aligned} \tag{24}$$

We can see that all the 2-by-2 diagonal submatrices of $Q_{\tilde{p}}(t,t)$ converge to constant matrices when $t \to \infty$. Therefore, we can conclude that the variances of formation errors in x and y directions converge to bounded constants as $t \to \infty$. □

B. A special class of unbiased measurement noises

In some situations, the statistical properties of displacement measurements remain invariant over time, and keep uncorrelated with the measuring time instants. Thus, we consider that the unbiased measurement noise $\xi(t)$ satisfies the following assumption.

Assumption 3: For any $k = 1, 2, ..., |\mathcal{E}|$, the measurement noise $\{\xi_k(t), t \in \mathbb{T}\}$ is W.S.S, and satisfies the following conditions

1) $\mu_{\xi_k}(t) \equiv \mathbf{0}_2, \qquad \forall t \in \mathbb{T}$,
2) $R_{\xi_k}(\Delta t) \triangleq \mathbb{E}\left\{\xi_k(t+\Delta t)\xi_k^\top(t)\right\} \preceq R_{\xi_k}(0)\mathbf{I}_2$, where $0 < R_{\xi_k}(0) < \infty$,
3) $\mathbb{E}\left\{\xi_m(t)\xi_n^\top(t)\right\} \equiv \mathbf{0}_{2\times 2}, \quad m \neq n, \; m, n = 1, 2, ..., |\mathcal{E}|$.

For this scenario, we present the following theorem.

Theorem 2: Consider that the multi-agent system (3) is controlled by (5) and the graph $\mathcal{G}$ is connected and undirected without cycles. If Assumption 3 holds, the overall formation error dynamics (10) exists a non-trivial solution in mean square sense, whose form is the same as (12).

Proof: Firstly, the autocorrelation $R_\eta(t,t)$ can be obtained as

$$\begin{aligned} &\int_0^t \int_0^t e^{-\alpha\mathcal{L}(t-s)}\mathcal{H}R_\xi(s,l)\mathcal{H}^\top e^{-\alpha\mathcal{L}(t-l)}\mathrm{d}s\mathrm{d}l \\ &= e^{-\alpha\mathcal{L}t}\left(\int_0^t \int_0^t e^{\alpha\mathcal{L}s}\mathcal{H}R_\xi(s,l)\mathcal{H}^\top e^{\alpha\mathcal{L}l}\mathrm{d}s\mathrm{d}l\right)e^{-\alpha\mathcal{L}t} \\ &\preceq \bar{R}_\xi(0)e^{-\alpha\mathcal{L}t}\left(\int_0^t \int_0^t e^{\alpha\mathcal{L}s}\mathcal{H}\mathcal{H}^\top e^{\alpha\mathcal{L}l}\mathrm{d}s\mathrm{d}l\right)e^{-\alpha\mathcal{L}t} \end{aligned} \tag{25}$$

where $\bar{R}_\xi(0) \triangleq \sup_{1\le k\le|\mathcal{E}|} R_{\xi_k}(0)$. According to Lemma 1, since matrix $H^\top H$ is positive definite, we can get

$$\begin{aligned} &\int_0^t \int_0^t e^{\alpha\mathcal{L}s}\mathcal{H}\mathcal{H}^\top e^{\alpha\mathcal{L}l}\mathrm{d}s\mathrm{d}l \\ &= \int_0^t \int_0^t e^{\alpha\mathcal{L}s}\mathcal{H}\mathcal{H}^\top\mathcal{H}\left(\mathcal{H}^\top\mathcal{H}\right)^{-2}\mathcal{H}^\top\mathcal{H}\mathcal{H}^\top e^{\alpha\mathcal{L}l}\mathrm{d}s\mathrm{d}l \\ &= \frac{1}{\alpha^2}\left(e^{\alpha\mathcal{L}t} - \mathbf{I}_{2N}\right)\mathcal{W}\left(e^{\alpha\mathcal{L}t} - \mathbf{I}_{2N}\right), \end{aligned} \tag{26}$$

where $\mathcal{W} \triangleq \mathcal{H}\left(\mathcal{H}^\top\mathcal{H}\right)^{-2}\mathcal{H}^\top \in \mathbb{R}^{2N\times 2N}$. Then substituting (26) into (25), when $t \to \infty$, we can obtain that

$$\begin{aligned} &\lim_{t\to\infty}\int_0^t \int_0^t e^{-\alpha\mathcal{L}(t-s)}\mathcal{H}R_\xi(s,l)\mathcal{H}^\top e^{-\alpha\mathcal{L}(t-l)}\mathrm{d}s\mathrm{d}l \\ &\preceq \frac{\bar{R}_\xi(0)}{\alpha^2}\left(\mathbf{I}_{2N} - e^{-\alpha\mathcal{L}t}\right)\mathcal{W}\left(\mathbf{I}_{2N} - e^{-\alpha\mathcal{L}t}\right) \\ &= \frac{\bar{R}_\xi(0)}{\alpha^2}\left(\mathbf{I}_{2N} - \frac{1}{N}\mathbf{1}_{2N}\mathbf{1}_{2N}^\top\right)\mathcal{W}\left(\mathbf{I}_{2N} - \frac{1}{N}\mathbf{1}_{2N}\mathbf{1}_{2N}^\top\right). \end{aligned} \tag{27}$$

If Assumption 3 holds, we have $\frac{\bar{R}_\xi(0)}{\alpha^2} < \infty$. Therefore, based on Lemma 3, the condition that the matrix $\lim_{t\to\infty} R_\eta(t,t)$ exists is a sufficient condition for the existence of $\tilde{p}(t)$ in mean square sense. □

Substituting (27) into (24), we can straightforwardly obtain a deterministic upper bound of $Q_{\tilde{p}}(t,t)$ when $t \to \infty$, which is omitted here. Furthermore, according to Corollary 1 and Theorem 2, we can conclude that the mean square formation errors exist deterministic upper bounds.

IV. Numerical simulation

In this section, we will give a numerical simulation for multi-agent formation control with measurement noises. We consider an MAS consisting of 4 agents, and denote edges $\{(1,2),(1,3),(1,4)\}$ as $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, respectively. The corresponding measurement topology $\mathcal{G}$ is shown as Fig. 1.

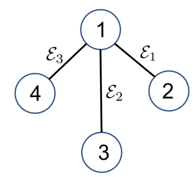
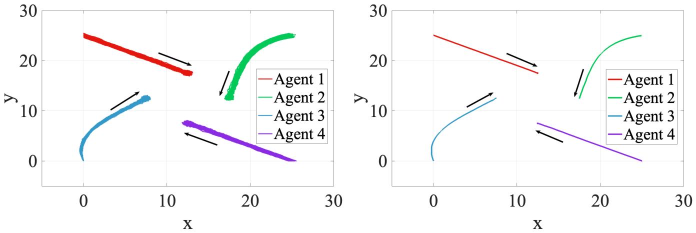


Fig. 1: Four agents' measurement topology $\mathcal{G}$

We consider the unbiased measurement noise $\xi_k(t) = \begin{bmatrix}\xi_k^x(t)\\ \xi_k^y(t)\end{bmatrix}$ as $\begin{bmatrix}A_k^x\cos(\omega_k^x t + \psi_k^x)\\ A_k^y\cos(\omega_k^y t + \psi_k^y)\end{bmatrix}$, where $A_k = [A_k^x, A_k^y]^\top \in \mathbb{R}^2$ and $\omega_k = [\omega_k^x, \omega_k^y]^\top \in \mathbb{R}^2$ are constant vectors for $1 \le k \le 3$. In addition, $\psi_k = [\psi_k^x, \psi_k^y]^\top \in \mathbb{R}^2$ is the 2-variate stochastic variable, where ψ_k^x and ψ_k^y are mutually independent, and both of them obey the uniform distribution, i.e., $\psi_k^x \sim \mathbf{U}(-\pi,\pi)$ and $\psi_k^y \sim \mathbf{U}(-\pi,\pi)$. Then, the mean value and autocorrelation of $\xi_k(t)$ can be derived as $\mu_{\xi_k}(t) = \mathbf{0}_2$ and $R_{\xi_k}(t, t+\Delta t) = \begin{bmatrix}\frac{(A_k^x)^2}{2}\cos(\omega_k^x\Delta t) & 0\\ 0 & \frac{(A_k^y)^2}{2}\cos(\omega_k^y\Delta t)\end{bmatrix}$. From the above results, the measurement noise $\xi_k(t)$ is W.S.S for any k, which satisfies Assumption 3.

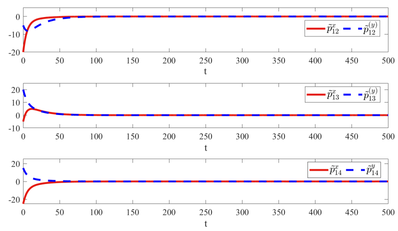
We set $A_1 = [16,16]^\top$, $A_2 = [18,18]^\top$, $A_3 = [20,20]^\top$, and $\omega_1 = \omega_2 = \omega_3 = [2.5\pi, 2\pi]^\top$. The desired formation is described by $\delta_{12} = [5,-5]^\top$, $\delta_{13} = [-5,-5]^\top$, $\delta_{14} = [0,-10]^\top$,. The initial positions are given by $p(0) = [0, 25\beta_1, 25\beta_2, 25\beta_3, 0, 0, 25\beta_4, 0]^\top$, where β_i denotes the stochastic variable, in which they are mutually independent and obeyed the Gaussian distribution for $i = 1,2,3,4$, i.e., $\beta_i \sim \mathbf{N}(1, 0.01)$. The control gain is given by $\alpha = 0.05$. This simulation is performed independently 500 times with the same parameters.

The four agents' probabilistic trajectories are shown as Fig. 2a. In addition, the corresponding expected trajectories of the agents and expected formation errors in terms of inter-agent displacements over these 500 experiments are shown as Fig. 2b and Fig. 3a, respectively. The norm values of autocovariances of $\tilde{p}_k(t)$ are shown as Fig. 3b. In addition, we can see that the variances of all agents' formation errors converge to a bounded domain, which is consistent with Theorem 2.

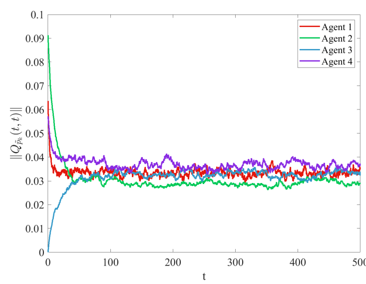


(a) Probabilistic formation (b) Expected formation

Fig. 2: Trajectories of four agents under the existence of unbiased measurement noises over 500 experiments, in which the black dotted lines represent the desired formation



(a) Mean values



(b) Autocovariances

Fig. 3: Statistical features of formation errors under the existence of unbiased measurement noises over 500 experiments

V. CONCLUSIONS

This paper has studied the multi-agent formation control problem by using displacement measurements subject to the stochastic noises. A sufficient and necessary condition has been derived for the existence of non-trivial solutions in the stochastic dynamics of multi-agent formations, in which the stochastic noises have finite second-order moments. The corresponding statistical properties and the corresponding convergence of formation errors have been analyzed in mean square sense. Specifically, for the case of the stochastic noises described by zero-mean wide-sense stationary processes, the existence of non-trivial solution in the stochastic dynamics with these noises has been analyzed, and the upper bound has been derived for the autocorrelation of these noises' stochastic integrals. In the future, formation control of MASs subjected to biased measurement noises is an interesting problem we want to study.

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