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Smooth Indirect Solution Method for State-constrained Optimal Control Problems with Nonlinear Control-affine Systems

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Abstract— **This paper presents an indirect solution method for state-constrained optimal control problems to address the long-standing issue of discontinuous control and costate under state constraints. It is known in optimal control theory that a state inequality constraint introduces discontinuities in control and costate, rendering the classical indirect solution methods ineffective. This study re-examines the necessary conditions of optimality for a class of state-constrained optimal control problems, and shows the uniqueness of the optimal control that minimizes the Hamiltonian under state constraints, which leads to a unifying form of the necessary conditions. The unified form of the necessary conditions opens the door to addressing the issue of discontinuities in control and costate by modeling them via smooth functions. This paper exploits this insight to transform the originally discontinuous problems to smooth two-point boundary value problems that can be solved by existing nonlinear root-finding algorithms. This paper also shows the formulated solution method to have an anytime algorithm-like property, and then numerically demonstrates the solution method by an optimal orbit control problem.**

I. Introduction

This paper presents a new indirect solution method for state-constrained optimal control problems to address the long-standing issue of discontinuous costate and control under state constraints. It is widely known that imposing pure state path constraints, $S(x,t) \leq 0$, in optimal control problems introduces discontinuities in control and costate [1]–[3]. Despite such difficulties, state-constrained optimal control problems arise in many fields, including aerospace, robotics, and other engineering and science applications.

In practice, many state-constrained optimal control problems are solved via direct methods. Direct methods convert continuous-time optimal control problems into finite-dimensional problems by parameterizing the problems via multiple shooting [4], numerical collocation [5], or exact time discretization [6]. The resulting parameter optimization problems are solved via nonlinear programming [7], sequential convex programming [8], model predictive control [9], etc. Many successful commercial software for solving general optimal control problems, such as GPOPS [10], fall under the category of direct methods. To incorporate inequality constraints in the direct method framework, they almost always employ the penalty function method and its variants [11], [12], which impose soft constraints by penalizing the violation of constraints and increase the penalty weight over iterations, asymptotically achieving the feasibility and optimality. These methods are usually combined with techniques in numerical optimization, such as the interior point method [7] and the augmented Lagrangian method [13].

Although direct methods provide flexibility in solving a variety of problems, they often compromise the feasibility and/or optimality due to the nature of the penalty function method [11], [12] and problem parameterization, which often involves approximation [4], [5], [9]. The computational complexity can be also an issue due to the large number (hundreds to thousands) of variables introduced by the parameterization [4], [5], [8]. In contrast, indirect methods possess an advantage of the fewer dimensionality of the problem (<10 unknowns). The fewer dimensionality is attractive particularly for complex control problems, such as those involving uncertainty [14] and multiple phases [15].

Motivated by these facts, this paper revisits the use of indirect methods. However, classical indirect methods struggle to solve state-constrained problems. Due to the discontinuities in control and costate resulting from Pontryagin's minimum principle [2], classical approaches divide the entire trajectory into sub-arcs and solve each sub-arc as separate two-point boundary value problems (TPBVPs) while satisfying the transversality conditions [1]–[3]. This approach requires *a priori* information about the sub-arc structure (e.g., number of constrained arcs), which is usually unknown in practice.

To address the issue, some recent studies have developed indirect solution methods with various approaches, such as using saturation function for problems with lower/upper bounds on state [16]; introducing slack variables and penalty function [17]; deriving a variant of the minimum principle by requiring the continuity of Lagrange multipliers associated with state constraints [18], [19]. On the other hand, like any approaches, they have their own assumptions and limitations (e.g., type of constraints considered, continuity).

This study tackles state-constrained optimal control via a different indirect method approach that explicitly addresses the discontinuities arising from the minimum principle. First, we revisit necessary conditions of optimality for constrained problems and show the uniqueness of optimal control that minimizes the constrained control Hamiltonian. This analysis leads to a unifying form of optimality necessary conditions, which enables us to address the issue of discontinuities in control and costate by modeling them via smooth functions with a sharpness parameter ρ. This approach transforms the originally discontinuous problem to a smooth TPBVP that can be solved by existing root-finding algorithms, where the sharpness parameter ρ controls the optimality of the resulting solutions. It is also shown that intermediate solutions of the continuation process respect the state constraint, which is

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often a desirable property for safety-critical systems.

II. Preliminary

A. Problem Statement

Consider a nonlinear, control-affine dynamical system:

$$\dot{x} = f(x, u, t) = f_0(x, t) + F(x, t)u \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $t \in \mathbb{R}$ are the state, control, and time, respectively; this system arises in many aerospace [20], [21] and robotic systems [22]. $F(\cdot)$ is twice continuously differentiable, which holds for many practical problems (e.g., for most robotic/aerospace problems, $F = [0_{m\times m}, I_m]^\top$).

Without loss of generality, consider the cost functional in the Lagrange form[1]:

$$J = \int_0^T L(x, u, t)\mathrm{d}t. \tag{2}$$

A feasible trajectory must satisfy terminal constraints

$$\psi(x(0), x(T), T) = 0 \tag{3}$$

where $\psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^q$, and a scalar state constraint

$$S(x, t) \leq 0. \tag{4}$$

S is assumed to be twice continuously differentiable and first order, implying that the first time derivative of S contains the control term u explicitly, i.e.,

$$S^{(1)}(x, u, t) \triangleq \dot{S} = S_x f + S_t = S_x f_0 + S_x F u + S_t \tag{5}$$

where $S_x \in \mathbb{R}^{1\times n}, S_t \in \mathbb{R}$ are the gradients of S with respect to x and t, i.e., $S_x \triangleq \nabla_x S$ and $S_t \triangleq \nabla_t S$. The first-order constraint assumption is thus equivalent to $S_x F \neq 0,\ \forall x, t$. Note that this assumption may limit the applicability of the proposed method, as $S_x F = 0$ for some practical constraints.

Thus, our original problem is formulated as in Problem 1.

Problem 1 (Original problem)**.** *Find $x^*(t), u^*(t), T^*$ that minimize the cost Eq.* (2) *subject to the dynamics Eq.* (1)*, terminal constraints Eq.* (3)*, and state constraint Eq.* (4)*.*

Although the analysis in this study is focused on a scalar state constraint S for conciseness, generalization to a vector-valued state constraint S is straightforward by requiring the following condition in addition to $S_x F \neq 0,\ \forall x, t$; for $S \in \mathbb{R}^l$ with $l > 1$, rows of $S_x F$ are independent (similar to the linear independence constraint qualification, or LICQ, which is often assumed in nonlinear programming algorithms [7]).

B. Optimality Necessary Conditions

The necessary conditions of optimality take different forms depending on whether the trajectory is on a constrained arc ($S = 0$) or on an unconstrained arc ($S < 0$) [2]. We use u_o^* to refer to the optimal control on an unconstrained arc while u_c^* to that on a constrained arc. u^* collectively denotes the optimal control regardless of constrained/unconstrained arc.

Clearly, a trajectory is characterized as follows:

$$\begin{cases} \text{unconstrained arc} & \text{if } S < 0 \\ \text{constrained arc} & \text{if } S = 0 \ \wedge \ S_o^{(1)} \geq 0 \\ \text{infeasible} & \text{if } S > 0 \end{cases} \tag{6}$$

where $S_o^{(1)}$ is a shorthand notation of $S^{(1)}(\cdot)$ under u_o^*, i.e., $S_o^{(1)} \triangleq S^{(1)}(x, u_o^*, t)$. When $S = 0$ and $S_o^{(1)} \leq 0$ at $t = t_i$, the trajectory is leaving the constrained arc, called an exit point. Likewise, $S = 0$ and $S_o^{(1)} > 0$ at an entry point.

1) Unconstrained arc: On an unconstrained arc, from Pontryagin's minimum principle, u_o^* must satisfy

$$u_o^* \in \arg\min_u H_o \tag{7}$$

H_o is the unconstrained control Hamiltonian, defined as:

$$H_o(x, u, \lambda, t) = L(x, u, t) + \lambda^\top [f_0(x, t) + F(x, t)u] \tag{8}$$

where $\lambda \in \mathbb{R}^n$ is the Lagrange multiplier associated with Eq. (1), or costate. The costate dynamics are given by

$$-\dot{\lambda}^\top = \nabla_x H_o = \nabla_x L + \lambda^\top \nabla_x f \tag{9}$$

2) Constrained arc: On a constrained arc, a feasible control must satisfy $S^{(1)}(x, u, t) \leq 0$. Among several approaches in literature [1], [2], this study takes an approach known as *indirect adjoining* to derive necessary conditions.

The indirect adjoining approach adjoins the control-dependent constraint $S^{(1)}(x, u, t) \leq 0$ in addition to Eqs. (1) and (3) to yield the control Hamiltonian [2]:

$$H_c(x, u, \lambda, \mu, t) = L + \lambda^\top f + \mu S^{(1)} \tag{10}$$

where H_c is the Hamiltonian on a constrained arc; $\mu (\geq 0)$ is a Lagrange multiplier associated with $S^{(1)} \leq 0$. Denoting the optimal multiplier by μ^*, the following must hold [2]:

$$(u_c^*, \mu^*) \in \{u, \mu \geq 0 \mid \min_{u,\mu} H_c \text{ s.t. } S^{(1)} = 0\} \tag{11}$$

The costate dynamics are given by

$$-\dot{\lambda}^\top = \nabla_x H_c(x, u, \lambda, \mu, t) = \nabla_x L + \lambda^\top \nabla_x f + \mu \nabla_x S^{(1)}$$

3) Discontinuity at corners: It is known that μ^* may be discontinuous at the time of entry to a constrained arc, called a *corner*[2]. At a corner (say $t = t_i$), λ and H obey [2], [23]:

$$\begin{aligned} \Delta\lambda_i &\triangleq \lambda(t_i^+) - \lambda(t_i^-) = -\mu(t_i^+)[S_x(t_i)]^\top, \\ \Delta H_i &\triangleq H(t_i^+) - H(t_i^-) = \mu(t_i^+) S_t(t_i) \end{aligned} \tag{12}$$

III. Uniqueness of Optimal Control and Multiplier

Although Eq. (11) states necessary conditions for (u^*, μ^*) to satisfy, it does not clarify whether such (u^*, μ^*) is unique. The uniqueness of (u^*, μ^*) is crucial for numerically solving optimal control problems via indirect methods. Numerical solutions for problems with no such uniqueness guarantee typically have to rely on direct methods (e.g., GPOPS [10]) with greater computational complexity.

[1] The other forms of cost, the Mayer and Bolza forms, can be converted to the Lagrange form. Also, a time interval $\tau \in [\tau_0, \tau_f]$ can be transformed to $t \in [0, T]$ via $t = \tau - \tau_0$ and $T = \tau_f - \tau_0$.

[2] In fact, one may also choose the discontinuity to occur at the exit corner or distribute at both of the entry and exit corners (e.g., [1]). This paper chooses to have the corner discontinuity at entry corners.

A. Optimal Control on Unconstrained Arcs

We assume that the unique solution to Eq. (7) is available in a closed form and continuously differentiable by satisfying the sufficient condition for the minimum Hamiltonian (known as the Legendre-Clebsch condition [2]):

$$\begin{aligned}\nabla_u H_o = \nabla_u L(x,u,t) + \lambda^\top F = 0 \quad \wedge \\ \nabla^2_{uu} H_o = \nabla^2_{uu} L(x,u,t) \succ 0\end{aligned} \tag{13}$$

Note that this assumption holds for many problems that are solved by indirect methods, as demonstrated in Section III-D. We generically express u_o^* that satisfies Eq. (13) as:

$$u_o^* = g(x,\lambda,t) \tag{14}$$

B. Optimal Control on Constrained Arcs

It can be shown that, with the assumption made in Section III-A, Problem 1 has the unique pair of (u_c^*, μ^*) $\forall t$ on constrained arcs. Lemma 1 formally states this fact.

Lemma 1. *Under the assumption that the unique solution to Eq.* (7) *is available in a closed form Eq.* (14)*, the solution to Eq.* (11) *is uniquely determined in the following form:*

$$u_c^* = g(x, \lambda + \mu S_x^\top, t), \tag{15}$$

$$\mu^* = h(x,\lambda,t) \geq 0, \tag{16}$$

where h is continuously differentiable.

Proof. The goal is to find the unique pair of u_c^* and $\mu*$ that solves the following Hamiltonian minimization:

$$(u_c^*, \mu^*) = \arg\min_{\mu \geq 0} \min_{u} H_c \text{ s.t. } S^{(1)} = 0 \tag{17}$$

where H_c is given by Eq. (10). Expanding H_c, we have $H_c = L + (\lambda^\top + \mu S_x)(f_0 + Fu) + \mu S_t$. The Legendre-Clebsch condition for H_c leads to

$$\nabla_u H_c = \nabla_u L(x,u,t) + (\lambda^\top + \mu S_x) F = 0 \quad \wedge \tag{18a}$$

$$\nabla^2_{uu} H_c = \nabla^2_{uu} L(x,u,t) \succ 0 \tag{18b}$$

Noting Eq. (13), it is clear that u_c^* that satisfies Eq. (18) is given by Eq. (15). Next, note that Eqs. (18a) and (18b) must be satisfied for any μ. Thus, the partial derivative of $\nabla_u H_c$ with respect to μ must be identically zero, i.e.,

$$\begin{aligned} 0 =& \nabla_\mu [\nabla_u H_c] = \nabla^2_{\mu u} L(x, u_c^*, t) + S_x F \\ =& S_x \left\{ \left[\nabla_\lambda g(x, \lambda + \mu S_x^\top, t)\right]^\top \nabla^2_{uu} L(x, u_c^*, t) + F \right\} \end{aligned}$$

which implies that $[\nabla_\lambda g(x, \lambda + \mu S_x^\top, t)]^\top \nabla^2_{uu} L(x, u_c^*, t) + F = 0$, yielding ($\because \nabla^2_{uu} L(x, u_c^*, t) \succ 0$, so invertible)

$$\nabla_\lambda g(x, \lambda + \mu S_x^\top, t) = -[\nabla^2_{uu} L(x, u_c^*, t)]^{-1} F^\top \tag{19}$$

Eq. (19) becomes handy shortly.

From Eq. (17), μ^* must satisfy $S^{(1)} = 0$ under u_c^*, i.e.,

$$S^{(1)} = S_x f_0 + S_x F g(x, \lambda + \mu^* S_x^\top, t) + S_t = 0 \tag{20}$$

Differentiating $S^{(1)}$ with respect to μ and using Eq. (19),

$$\begin{aligned} \nabla_\mu S^{(1)} =& S_x F [\nabla_\mu g(x, \lambda + \mu S_x^\top, t)] \\ =& S_x F [\nabla_\lambda g(x, \lambda + \mu S_x^\top, t)] S_x^\top \\ =& - S_x F [\nabla^2_{uu} L(x, u_c^*, t)]^{-1} F^\top S_x^\top \end{aligned} \tag{21}$$

which implies $\nabla_\mu S^{(1)} < 0$ because $[\nabla^2_{uu} L(x, u_c^*, t)]^{-1} \succ 0$ due to Eq. (18b) and $S_x F \neq 0$, $\forall x, t$ as discussed in Section II-A. Thus, $S^{(1)}$ is monotonically decreasing in μ. Note also that, when $\mu = 0$, we have that $g(x, \lambda + \mu S_x^\top, t) = g(x, \lambda, t) = u_o^*$ and hence $S^{(1)} = S_o^{(1)} \geq 0$ on a constrained arc (recall Eq. (6)). Therefore, μ^* that satisfies Eq. (20) is unique and must be non-negative, and the unique solution can be represented by a function $\mu^* = h(x, \lambda, t)$, i.e., Eq. (16). Finally, $h(x, \lambda, t)$ is continuously differentiable because $S(\cdot)$, $f_0(\cdot)$, $F(\cdot)$, and $g(\cdot)$ are twice differentiable. □

C. Unified Form of Optimality Necessary Conditions

Thus, we can express the control Hamiltonian and necessary conditions of optimality for Problem 1 in a unified form without needing to separate the discussion, as follows:

$$H(x, u, \lambda, \mu, t) = L(\cdot) + \lambda^\top f(\cdot) + \mu S^{(1)}(\cdot),, \tag{22a}$$

$$u^* = g(x, \lambda + \mu^* S_x^\top, t), \tag{22b}$$

$$\dot{\lambda} = -[\nabla_x H(x, u^*, \lambda, \mu^*, t)]^\top, \tag{22c}$$

$$\mu^* = \begin{cases} h(x, \lambda, t) & \text{if } S = 0 \ \wedge \ S_o^{(1)} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{22d}$$

where Eqs. (22a) to (22c) give the unconstrained arc solution by substituting $\mu^* = 0$; Eq. (22d) is based on Eq. (6). This is in sharp contrast to the classical discussion in literature (e.g., [2]), which divides a state-constrained optimal control problem into constrained and unconstrained arcs.

Remark 1. *Even with the unifying form given by Eq.* (22)*, Problem 1 still experiences discontinuities in μ^* (and hence in λ^* and H^*) when the trajectory enters a constrained arc.*

Section IV addresses the issue identified in Remark 1.

D. Analytical Example

Let us consider an example to demonstrate the wide applicability of the assumption made in Section III-A as well as the procedure to find u^* and μ^* on constrained arcs based on Lemma 1. This example considers $L(\cdot)$ in the form of $L(\cdot) = c(x,t) + u^\top R u + x^\top P u$, where $R \succ 0$. With this, one can find the solution to Eq. (7) by solving Eq. (13) as:

$$u_o^* = -\frac{1}{2} R^{-1} (Px + F^\top \lambda) \tag{23}$$

which is continuously differentiable ($\because$ F is continuously differentiable), hence satisfying the assumption made in Section III-A. It is also straightforward to derive u_c^* by applying Eqs. (18a) and (18b), yielding:

$$u_c^* = -\frac{1}{2} R^{-1} [Px + F^\top (\lambda + \mu^* S_x^\top)] \tag{24}$$

which takes the form given in Eq. (15). Substituting Eq. (24) into Eq. (20) and solving for μ^* yields

$$\mu^* = \frac{2 S_x f_0 + 2 S_t - S_x F R^{-1} (Px + F^\top \lambda)}{S_x F R^{-1} F^\top S_x^\top} \tag{25}$$

where $S_x F R^{-1} F^\top S_x^\top \neq 0$ since $S_x F \neq 0$ and $R \succ 0$. Eq. (25) is continuously differentiable because $S(\cdot)$, $f_0(\cdot)$, and $F(\cdot)$ are twice differentiable.

IV. Solution Method

This section presents the proposed solution method to address discontinuities in Problem 1 and analyzes its properties.

A. Smooth Approximation of Discontinuous Multiplier

Let us address the discontinuity issue stated in Remark 1. As clear from Eq. (22d), the discontinuity is triggered by the two conditions $S = 0$ and $S_o^{(1)} \geq 0$. Hence, this study models μ^* via $\tilde{\mu}^*$ with two smooth activation functions as:

$$\tilde{\mu}^* = h(x, \lambda, t)\phi_1(S)\phi_2(S_o^{(1)}), \tag{26}$$

where $\phi_1(\cdot)$ and $\phi_2(\cdot)$ aim to smoothly characterize the activation via $S = 0$ and $S_o^{(1)} \geq 0$, respectively.

There are three guiding principles in designing ϕ_1 and ϕ_2: (i) for constraint satisfaction, both ϕ_1 and ϕ_2 should take unity when $S = 0$ and $S_o^{(1)} \geq 0$; (ii) ϕ_2 should remain unity for any values of $S_o^{(1)} \geq 0$ since $S_o^{(1)}$ can be arbitrarily large on constrained arcs; (iii) for the approximation to be asymptotically precise, ϕ_1 should be close to zero when $S < 0$ while ϕ_2 should be close to zero when $S_o^{(1)} < 0$.

Based on these guiding principles, we choose a hyperbolic tangent-based function (a variant of the logistic function) and a variant of the bump function to design ϕ_1 and ϕ_2, respectively. Formally defining these activation functions,

$$\phi_1(x) = \frac{e^2+1}{2e^2}\left(1 + \tanh\frac{x+\rho_1}{\rho_1}\right), \tag{27a}$$

$$\phi_2(x) = \begin{cases} 1 & x > 0 \\ \exp\left[\frac{1}{\rho_2^2}\left(1 - \frac{1}{1-x^2}\right)\right] & x \in (-1, 0] \\ 0 & x \leq -1 \end{cases} \tag{27b}$$

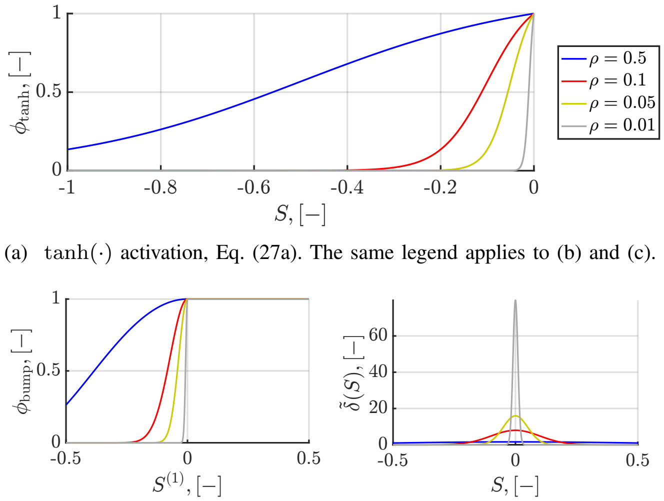
where $\rho_1 \in \mathbb{R}$ and $\rho_2 \in \mathbb{R}$ are positive scalars that control the sharpness of the approximation; smaller ρ_1 and ρ_2 lead to sharper approximations. Fig. 1 illustrates their behavior with different values of ρ, where $\phi_{\text{tanh}}, \phi_{\text{bump}}$ represent ϕ_1, ϕ_2.

Lemma 2 formally states three key properties the proposed approach possesses. The first property (1) is crucial to facilitating the numerical convergence of the solution method. This property implies the smoothness of u and $\dot{x}$, eliminating the need to divide the trajectory into sub-arcs. The second property (2) is vital when applying to safety-critical systems where we do not want to compromise the state constraint satisfaction for any ρ. The third property (3) ensures the convergence to the local optimum of the original problem.

Lemma 2. *The smooth approximation $\tilde{\mu}^*$ as defined in Eqs.* (26) *and* (27) *has the following properties:*

1) *$\tilde{\mu}^*$ is continuously differentiable over an optimal trajectory including at corners;*
2) *$\tilde{\mu}^*$ respects the inequality constraint regardless of the values of ρ_1 and ρ_2; and*
3) *$\tilde{\mu}^*$ approaches μ^* defined in Eq.* (22d) *as $\rho_1, \rho_2 \to +0$.*

Proof. (1) is shown first. It is clear from Eq. (27) that ϕ_1 and ϕ_2 are continuously differentiable in S and h, respectively, including at corners. S and h are both continuously differentiable as discussed earlier, thus proving (1).



(a) $\tanh(\cdot)$ activation, Eq. (27a). The same legend applies to (b) and (c).

(b) Bump function, Eq. (27b) (c) Costate jump approx, Eq. (31)

Fig. 1. Activation functions with various sharpness parameters ρ

For (2), it is clear from Eq. (27) that, when $S = 0 \wedge S_o^{(1)} > 0$, $\phi_1(S)$ and $\phi_2(S_o^{(1)})$ take both unity independent of the values of ρ_1 and ρ_2, and hence $\tilde{\mu}^* = h$, yielding $S^{(1)} = 0$, which ensures the constraint satisfaction on constrained arcs.

(3) is verified by showing the following facts: (a) $\lim_{\rho_1 \to +0} \phi_1(S) = 0$ if $S < 0$; (b) $\phi_1(S) = 1$ if $S = 0$ independent from ρ_1; (c) $\phi_2(S_o^{(1)}) = 1$ if $S_o^{(1)} \geq 0$ independent from ρ_2; (d) $\lim_{\rho_2 \to +0} \phi_2(S_o^{(1)}) = 0$ if $S_o^{(1)} < 0$; (e) $h = 0$ if $S_o^{(1)} = 0$. (a) and (b) are shown by noting that $\tanh(-\infty) = -1$ and that $\frac{e^2+1}{2e^2} = \frac{1}{1+\tanh 1}$, respectively. (c) is true by definition, see Eq. (27b). (d) is shown by noting that $\lim_{\rho_2 \to +0} \phi_2(S_o^{(1)}) = 0$ if $S_o^{(1)} < 0$ ($\because \exp(-\infty) = 0$). (e) is shown by noting that the unique μ^* that simultaneously satisfies $S_o^{(1)} = 0$ and $S^{(1)} = 0$ is necessarily zero, i.e., $\mu^* = h = 0$ ($\because S_o^{(1)} = S_x f_0 + S_x F g(x, \lambda, t) + S_t,\ S^{(1)} = S_x f_0 + S_x F g(x, \lambda + \mu^* S_x^\top, t) + S_t$). □

B. Smooth Approximation of Discontinuous Control

With the smooth multiplier established in Section IV-A, it is straightforward to approximate u^* via a smooth function:

$$\tilde{u}^* = g(x, \lambda + \tilde{\mu}^* S_x^\top, t), \tag{28}$$

which has the following desired properties due to Lemma 2: (1) $\tilde{u}^*$ is continuously differentiable; (2) $\tilde{u}^*$ respects the inequality constraint regardless of the values of ρ_1 and ρ_2; (3) $\tilde{u}^*$ asymptotically approaches u^* as $\rho_1, \rho_2 \to +0$.

C. Smooth Approximation of Jump in Costate & Hamiltonian

Recalling Eq. (12), the discontinuous change in costate at the i-th corner, denoted by $\Delta\lambda_i$, is given by

$$\Delta\lambda_i = -\mu^*(t_i^+)[S_x(t_i)]^\top = -h(x(t_i), \lambda(t_i), t_i)[S_x(t_i)]^\top$$

where $\mu^*(t_i^+)$ is replaced by $h(\cdot)$ due to Eq. (22d). The following formulation directly applies to ΔH_i as well.

By using the Dirac delta function, $\Delta\lambda_i$ is equivalent to

$$\Delta\lambda_i = -\int_{-\infty}^{\infty} h(x, \lambda, t)[S_x(t)]^\top \delta(t - t_i)\mathrm{d}t \tag{29}$$

where $\delta(x)$ satisfies the following conditions:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1 \tag{30}$$

Noting that $t = t_i$ corresponds to a time when $S = 0$ and that S never becomes positive under u^* with μ^*, the integral of the Dirac delta can be expressed in terms of S as:

$$\int_{-\infty}^{\infty} \delta(t - t_i)\mathrm{d}t = 2\int_{-\infty}^{0} \delta(S)\mathrm{d}S = 2\int_{-\infty}^{t_i} \delta(S(t))S^{(1)}\mathrm{d}t$$

where $\mathrm{d}S = \dot{S}\mathrm{d}t = S^{(1)}\mathrm{d}t$.

Obviously, $\delta(x)$ is discontinuous at t_i and needs a smooth approximation for integrating the costate differential equation across t_i. Thus, we smoothly approximate $\delta(x)$ as follows:

$$\tilde{\delta}(x) \triangleq \frac{1}{\rho_3\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{x^2}{\rho_3^2}\right), \tag{31}$$

which corresponds to the probability distribution function (pdf) of a zero-mean normal distribution with standard deviation ρ_3, where ρ_3 serves as another sharpness parameter.

It is straightforward to verify that $\tilde{\delta}(x)$ given by Eq. (31) has the following desirable properties: $\tilde{\delta}(x)$ asymptotically satisfies the left of Eq. (30) as $\rho_3 \to 0$; and $\tilde{\delta}(x)$ always satisfies the right of Eq. (30) regardless of the value of ρ_3, since $\tilde{\delta}(\cdot)$ represents a pdf. Fig. 1(c) illustrates the behavior of $\tilde{\delta}(S)$ with respect to various ρ_3.

Using Eq. (31), the jump condition Eq. (29) is smoothly approximated via $\widetilde{\Delta\lambda}_i$, calculated as:

$$\widetilde{\Delta\lambda}_i = -2\int_{-\infty}^{t_i} h(x, \lambda, t)[S_x(t)]^\top \tilde{\delta}(S(t))S^{(1)}\mathrm{d}t. \tag{32}$$

Since Eq. (32) must hold for each i-th corner and takes the form of integral with respect to t, this effect can be incorporated into Eq. (22c), yielding:

$$\dot{\lambda} = -\left[\nabla_x H(x, \tilde{u}^*, \lambda, \tilde{\mu}^*, t)\right]^\top - 2h(\cdot)\tilde{\delta}(S)S^{(1)}S_x^\top \tag{33}$$

The jump in H can be similarly expressed with a substitution of S_t into S_x, resulting an ordinary differential equation for H that takes into account the jump conditions at corners.

Note that the approximation quality of $\Delta\lambda_i$ via Eq. (33) does not affect the feasibility of the problem. The feasibility is guaranteed by the second property of Lemma 2.

D. Smooth State-constrained Optimal Control Problem

Applying the proposed solution method transforms Problem 1 into Problem 2, which no longer involves discontinuities in control and costate and represents a smooth TPBVP.

Problem 2 (Smooth State-constrained Problem)**.** *For given sharpness parameters* $\{\rho_1, \rho_2, \rho_3\}$*, find the initial costate* $\lambda(0)$ *(and T for variable-time problems) such that satisfy the transversality conditions subject to the state dynamics Eq.* (1) *and costate dynamics Eq.* (33) *under control Eq.* (28)*.*

Due to the smoothness, we can solve Problem 2 by the standard indirect shooting with no need for dividing the problem into sub-arcs. For completeness, the standard indirect shooting procedure can be described as follows: (1) define an augmented state, $X = [x^\top, \lambda^\top]^\top$, which obeys:

$$\dot{X} = \begin{bmatrix} f(x, \tilde{u}^*, t) \\ -[\nabla_x H(x, \tilde{u}^*, \lambda, \tilde{\mu}^*, t)]^\top - 2\tilde{\mu}^*\tilde{\delta}(S)S^{(1)}S_x^\top \end{bmatrix}; \tag{34}$$

(2) set up a shooting function $\Psi(\lambda(0))$ which, for given $\lambda(0)$, integrates Eq. (34) from $t = 0$ to T and evaluates the transversality conditions; (3) use a nonlinear root-finding solver to find $\lambda(0)$ that yields $\Psi(\cdot) = 0$.

To approach the optimal solution of the original problem, we perform continuation over a decreasing sequence of ρ's, where $\rho \triangleq \rho_1 = \rho_2 = \rho_3$ is used for simple algorithm design. That is, after solving Problem 2 with $\rho^{(i)}$, the solution is used as the initial guess of $\lambda(0)$ for the next iteration with $\rho^{(i+1)}$, where $\rho^{(i)}$ decreases as i increases. This approach takes advantage of the third property in Lemma 2 and the convergence property of $\tilde{\delta}(\cdot)$ to $\delta(\cdot)$.

Remark 2. *Each intermediate solution of the continuation procedure respects the state constraint regardless of the values of the sharpness parameters* $\{\rho_1, \rho_2, \rho_3\}$ *(i.e., anytime algorithm) due to the second property of Lemma 2, which is often a desirable property for safety-critical systems.*

V. Numerical Example

We consider a 2-D orbit transfer problem. Let $r, v \in \mathbb{R}^2$ be the position and velocity of the spacecraft, $x \in \mathbb{R}^4$ be the state vector, and $u \in \mathbb{R}^2$ the control vector, defined as:

$$x = \begin{bmatrix} r \\ v \end{bmatrix}, \quad r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{35}$$

Orbital dynamics with gravity parameter μ_g are given by:

$$f(x, u) = f_0(x) + Bu, \ f_0 = \begin{bmatrix} v \\ -\frac{\mu_g}{\|r\|_2^3} r \end{bmatrix}, \ B = \begin{bmatrix} 0_{2\times2} \\ I_2 \end{bmatrix}$$

where the canonical unit is used so that $\|r(t=0)\|_2 = \mu_g = 1$. $\|\cdot\|_2$ represents the 2-norm of a vector. The cost function, boundary conditions, and state constraint are defined as:

$$L = \frac{1}{2}\|u\|_2^2, \ x(0) = x_0, \ x(T) = x_T, \ S = p_{\min} - p(x),$$

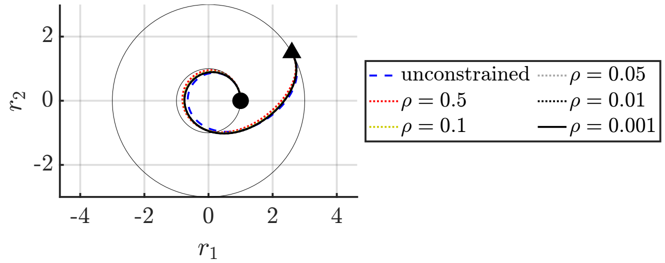
where $p(x) = (r_1 v_2 - r_2 v_1)^2/\mu_g$ is the semilatus rectum of an orbit. $x_0, x_T, T, p_{\min}$ are given and defined as: $x_0 = [1, 0, 0, 1]^\top$, $x_T = [3\sqrt{3}/2, 1.5, -1/(2\sqrt{3}), 0.5]^\top$, $T = 3\pi$, $p_{\min} = 0.9$. The necessary conditions are derived as:

$$u^* = -B^\top(\lambda + \mu^* S_x^\top), \quad \mu^* = \frac{S_x f_0 - S_x BB^\top \lambda}{S_x BB^\top S_x^\top} \tag{36}$$

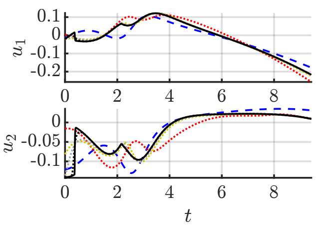
Note that this problem is a special case of the example in Section III-D with $F = B, S_t = 0, c = 0, R = \frac{1}{2}I_2$, and $P = 0$. The transversality condition is $x(T) - x_T = 0$.

We solve this problem in the form of Problem 2 by using Matlab's `fsolve`, where `ode45` is used for the integration of Eq. (34). The tolerances used are $\texttt{Opt/FunTol} = 10^{-8}$ for `fsolve` and $\texttt{Rel/AbsTol} = 10^{-8}$ for `ode45`.

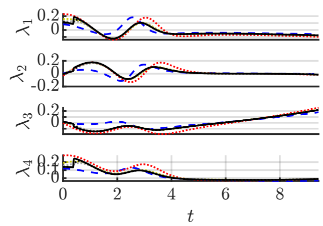
The solutions for $\rho^{(i)} \in \{0.5, 0.1, 5\times10^{-2}, 10^{-2}, 5\times10^{-3}, 10^{-3}\}$ are shown in Fig. 2, including the unconstrained solution for comparison. Figs. 2(b) and 2(c) indicate that control and costate experience a discontinuity



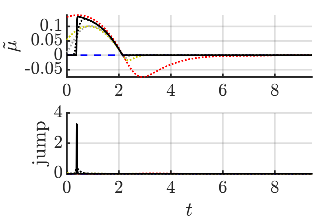
(a) Optimal trajectories for various ρ. The same legend applies to (b)-(e). The filled circle and triangle represent the initial and final positions.



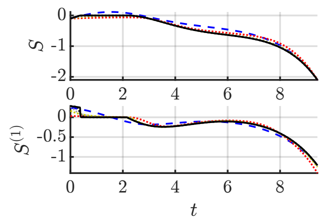
(b) Optimal control profiles



(c) Optimal costate profiles



(d) Smoothed μ and costate jump



(e) State constraint and derivative

Fig. 2. Numerical example results with various ρ.

around $t = 0.55$, which is well-modeled by the smooth approximation. Fig. 2(b) shows a significant difference in the optimal control profiles for the constrained and unconstrained problems. Fig. 2(d) shows the behavior of $\tilde{\mu}^*$ (constraint multiplier) and $2h\tilde{\delta}(S)S^{(1)}$ (time derivative of costate jump; see Eq. (33)), which numerically verifies the analysis in Section IV. Fig. 2(e) plots S and $S^{(1)}$ over t, confirming that intermediate solutions in the continuation respect the state constraint, even for a blunt sharpness parameter $\rho = 0.5$, which demonstrates Remark 2.

VI. Conclusions

In this paper, a new indirect solution method for state-constrained optimal control problems is presented to address the long-standing issue of discontinuous control and costate due to state inequality constraints. The proposed solution method is enabled by re-examining the necessary conditions of optimality for the constrained control problems and deriving a unifying form of necessary conditions based on the uniqueness of the optimal control and constraint multiplier on constrained arcs. In contrast to classical indirect solution methods, the proposed method transforms the originally discontinuous problems into smooth TPBVPs, eliminating the need for *a priori* knowledge about the optimal solution structure, which is usually unknown. A numerical example demonstrates the proposed method and its anytime algorithm-like property. A key next direction of this work is to generalize the framework to higher-order state constraints.

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