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An Integral Sliding–Mode–based Robust Interval Predictive Control for Perturbed Unicycle Mobile Robots

Héctor Ríos[†,⋆], Manuel Mera[‡], Tarek Raïssi[∗] and Denis Efimov[§,⋄]

Abstract—**This paper contributes to the design of a robust control strategy for the trajectory tracking problem in perturbed unicycle mobile robots. The proposed strategy comprises the design of a robust control law, which is based on an Integral Sliding–Mode Control (ISMC) approach together with an interval predictor–based state–feedback controller and a Model Predictive Control (MPC) scheme. The robust controller deals with some perturbations in the kinematic model, and with state and input constraints that are related to restrictions on the workspace and saturated actuators, respectively. The proposed approach guarantees the exponential convergence to zero of the tracking error. Furthermore, the performance of the proposed approach is validated through some simulations.**

Index Terms—**Unicycle Mobile Robots, Sliding–Mode Control, Model Predictive Control.**

I. Introduction

THE unicycle mobile robots (UMRs) have been widely studied during the last decades due to their capability of moving freely from one point to another one and to the broad diversity of possible real applications (see, *e.g.,* [1]). One of the main difficulties regarding the control design is the fact that the kinematic model of this class of systems does not fulfill Brockett's necessary condition for smooth state–feedback stabilization [2]. Therefore, the design of non–smooth or time–varying feedback controllers is a requirement for this class of mobile robots. Moreover, as it is highlighted in [3], even if external forces cannot be considered in the kinematic model, there exist some other signals or non–modeled phenomena, *e.g.,* the skidding and slipping of the wheels and corrupt control signals, that must be taken into account for the design of a controller. Additionally, it is well–known that such UMRs must move in restricted work–spaces and have energy limitations. These effects can be seen as state and input constraints, and should also be considered in the control design. In this sense, the trajectory tracking control design, considering the non–holonomic constraints, external perturbations and system constraints, is still a challenging problem.

Regarding the literature taking into account external disturbances in the kinematic model, in [4], a first–order sliding–mode control (SMC) approach is proposed to deal with the trajectory tracking problem in perturbed UMRs. This approach considers some skidding and slipping effects on the wheels, input saturation constraints and guarantees the asymptotic convergence to zero of the tracking error. A robust controller, based on the Super–Twisting algorithm, was presented in [5], and guarantees asymptotic convergence of the tracking error to zero, despite the presence of some skidding and slipping effects. The proposed controller is continuous but local. However, these works do not consider state constraints.

In the context of trajectory tracking control for constrained UMRs, the Model Predictive Control (MPC) approach (see, *e.g.,* [6] and [7]) is quite popular. For instance, in [8], a neural network–based robust MPC is proposed to stabilize a state and input constrained mobile robot in the presence of some additive disturbances. In the same sense, a couple of MPC approaches; namely, tube–MPC and nominal robust MPC, are proposed in [9] for tracking unicycle robots. Such approaches are able to deal with input constraints and bounded additive disturbances. In [10], the authors propose a trajectory tracking controller for UMRs based on an MPC approach with adaptive prediction horizon. The proposed controller is able to deal with some additive disturbances and some system constraints. However, for all of the above–mentioned works, the considered additive disturbances lack physical meaning since the skidding and slipping effects cannot be represented by additive disturbances but by multiplicative disturbances. Moreover, all of the above–mentioned works only provide some input–to–state stability properties.

Motivated by the above–mentioned issues, *i.e.,* meaningful external disturbances and system constraints, in this paper, a robust control strategy is proposed to solve the trajectory tracking problem in perturbed unicycle mobile robots, tak-

[†]Tecnológico Nacional de México/I.T. La Laguna, C.P. 27000, Torreón, Coahuila, México. Email: hriosb@lalaguna.tecnm.mx
[‡]Instituto Politécnico Nacional, Section of Graduate Studies and Research, ESIME–UPT, C.P. 07340, CDMX, México. Email: mlmera@ipn.mx
[⋆]CONAHCYT IxM, C.P. 03940, CDMX, México.
[∗]Conservatoire National des Arts et Métiers (CNAM), Cedric 292, Rue St–Martin, 75141 Paris, France. Email: tarek.raissi@cnam.fr
[§]Inria, Univ. Lille, CNRS, UMR 9189–CRISTAL, F–59000, Lille, France. Email: Denis.Efimov@inria.fr
[⋄]Department of Control Systems and Informatics, Information Technologies Mechanics and Optics University, Saint Petersburg 197101, Russia.

ing into account state and input constraints. The proposed strategy comprises the design of a robust control law, which is based on an ISMC approach together with an interval predictor–based state–feedback controller and an MPC scheme[1]. The robust controller deals with some perturbations in the kinematic model, and with state and input constraints that are related to restrictions on the workspace and saturated actuators, respectively. The proposed approach guarantees the exponential convergence to zero of the tracking error despite the considered perturbations and the system constraints, and the computational burden is relaxed due to the switching structure of the controller.

Notation: Denote the trigonometric functions $\sin(\theta)$, $\cos(\theta)$ and $\mathrm{sinc}(\theta)$ as $\mathrm{s}(\theta)$, $\mathrm{c}(\theta)$ and $\mathrm{sc}(\theta)$, respectively. Let us denote a sequence of integers $1, ..., n$ as $\overline{1,n}$, for any $n \in \mathbb{N}$. The absolute value is represented as $|\cdot|$ while the Euclidian norm of a vector $z \in \mathbb{R}^n$ is denoted as $||z||$, and for a matrix $A \in \mathbb{R}^{m\times n}$, the induced norm is the spectral norm, *i.e.,* $||A|| = \sqrt{\lambda_{\max}(A^\top A)}$. The set of all inputs $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ such that its L_∞ norm on $[0,\infty]$ is less that infinity, *i.e.,* $||u||_\infty := ||u||_{[0,\infty]} = ess\sup_{t\geq 0}\|u(t)\| < \infty$, is denoted as $\mathcal{L}_\infty$. For a couple of vectors $x_1, x_2 \in \mathbb{R}^n$ and a couple of matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood in the component–wise sense. In the same sense, for a matrix $A \in \mathbb{R}^{n\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ and $|A| = A^+ + A^-$, similarly for a vector. For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, the notation $P \prec 0$ ($P \succeq 0$) means that P is negative (positive) definite. A matrix $A \in \mathbb{R}^{n\times n}$ is called Metzler when all its non–diagonal elements are non–negative. The term $\mathrm{He}(A)$ denotes $A + A^\top$, for a matrix $A \in \mathbb{R}^{n\times n}$.

II. Problem Statement

Consider the perturbed kinematic model of an UMR:

$$\dot{\theta} = [1 + d_1(t)]\omega, \tag{1a}$$

$$\dot{x} = [1 + d_2(t)]\mathrm{c}(\theta)v, \tag{1b}$$

$$\dot{y} = [1 + d_2(t)]\mathrm{s}(\theta)v, \tag{1c}$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}$ denote the midpoint between the wheels and $\theta \in \mathbb{R}$ represents the orientation angle of the UMR. The terms v and ω contain the linear and angular velocities of the UMR, and represent the control inputs. The terms d_1 and d_2 represent some time–varying perturbations, which are multiplicative to the inputs and that may come from the settling time of the internal controller that translates the velocity commands in current/voltage inputs and sends them to the motors [15] or non–modeled kinematics phenomena proportional to the control inputs, such as skidding and slipping of the wheels [3]. It is assumed that such time–varying perturbations $d_i(t)$ are unknown but bounded, *i.e.,* $-1 < d_i(t) \leq d_{\max} < 1$, for $i = 1, 2$, with a known positive constant $d_{\max}$. Note that the constraint $d_i(t) > -1$ ensures that the perturbations do not cause a change of sign in the control inputs.

The aim of this work is to design a trajectory tracking control for the UMR able to compensate some multiplicative perturbations and reach the desired trajectory taking into account some state and input constraints, *i.e.,* $x(t) \in \mathbb{X} = [x_{\min}, x_{\max}] \subset \mathbb{R}$, $y(t) \in \mathbb{Y} = [y_{\min}, y_{\max}] \subset \mathbb{R}$, $v(t) \in \mathbb{V} = [-v_{\max}, v_{\max}] \subset \mathbb{R}$ and $\omega(t) \in \mathbb{W} = [-\omega_{\max}, \omega_{\max}] \subset \mathbb{R}$, for all $t \geq t_0$, for some given sets $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{V}$ and $\mathbb{W}$.

III. LPV Tracking Error Dynamics

Let us define the tracking errors as follows

$$e_1 = \theta_d - \theta, \tag{2a}$$

$$e_2 = \mathrm{c}(\theta)(x_d - x) + \mathrm{s}(\theta)(y_d - y), \tag{2b}$$

$$e_3 = \mathrm{c}(\theta)(y_d - y) - \mathrm{s}(\theta)(x_d - x), \tag{2c}$$

where x_d, y_d and θ_d come from a reference kinematic model for the UMR, *i.e.,*

$$\dot{\theta}_d = \omega_d, \tag{3a}$$

$$\dot{x}_d = \mathrm{c}(\theta_d)v_d, \tag{3b}$$

$$\dot{y}_d = \mathrm{s}(\theta_d)v_d, \tag{3c}$$

where v_d and ω_d are the linear and angular reference velocities, respectively. These are assumed continuous and bounded by some positive constants $\underline{v}_d$, $\overline{v}_d$ and $\overline{\omega}_d$, *i.e.,* $0 < \underline{v}_d < v_d(t) \leq \overline{v}_d$, and $||\omega_d||_\infty \leq \overline{\omega}_d$, and such that $v_d(t) \in \mathbb{V}$ and $\omega_d(t) \in \mathbb{W}$, for all $t \geq 0$. Moreover, the trajectories of the reference model also hold the state constraints, *i.e.,* $x_d(t) \in \mathbb{X}$ and $y_d(t) \in \mathbb{Y}$, for all $t \geq 0$.

Therefore, the tracking error dynamics can be calculated as

$$\dot{e}_1 = -\omega d_1(t) + \tau_1, \tag{4a}$$

$$\dot{e}_2 = [1 + d_1(t)]\omega e_3 - v d_2(t) + \tau_2, \tag{4b}$$

$$\dot{e}_3 = -[1 + d_1(t)]\omega e_2 + v_d \mathrm{s}(e_1), \tag{4c}$$

with the virtual control inputs τ_1 and τ_2 satisfying

$$\tau_1 = \omega_d - \omega, \tag{5a}$$

$$\tau_2 = v_d \mathrm{c}(e_1) - v. \tag{5b}$$

Note that the tracking error dynamics (4) can be rewritten as follows:

$$\dot{e} = A(\rho)e + B[\tau + F(\rho)d], \tag{6}$$

where $e = (e_1, e_2, e_3)^\top \in \mathbb{R}^3$, $\tau = (\tau_1, \tau_2)^\top \in \mathbb{R}^2$, $d = (d_1, d_2)^\top \in \mathbb{R}^2$ and

$$A(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & [1 + d_1(t)]\omega \\ v_d \mathrm{sc}(e_1) & -[1 + d_1(t)]\omega & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \; F(\rho) = \begin{pmatrix} -\omega & 0 \\ 0 & -v \end{pmatrix},$$

[1]In the literature, there have been several proposals for the combination of MPC and ISMC. For instance, one can see [11], [12], [13], and [14]. However, these works are not applied to UMRs and most of them only provide ISS properties.

with the vector of scheduling variables as $\rho = (v_d \text{sc}(e_1), [1+d_1(t)]\omega)^\top \in \mathbb{R}^2$, which is bounded due to the physical and actuator limitations, and the fact that $0 < \underline{v}_d < v_d(t) \le \overline{v}_d$ and $||d_1||_\infty \le d_{\max} < 1$. It is clear that system (6) is in an LPV form and the input and state constraint sets are given now as follows:

$$\mathbb{E} = \{e \in \mathbb{R}^3 : (e_1, e_2, e_3) \in \mathbb{R} \times [-\overline{xy}, \overline{xy}] \times [-\overline{xy}, \overline{xy}]\}, \quad (7)$$

$$\mathbb{U} = \{\tau \in \mathbb{R}^2 : (\tau_1, \tau_2) \in [-\overline{\tau}_1, \overline{\tau}_1] \times [-\overline{\tau}_2, \overline{\tau}_2]\}, \quad (8)$$

where $\overline{xy} = (x_{\max} - x_{\min}) + (y_{\max} - y_{\min})$, $\overline{\tau}_1 = \overline{\omega}_d + \omega_{\max}$ and $\overline{\tau}_2 = \overline{v}_d + v_{\max}$. Moreover, note that there always exist a Metzler matrix $A_0 \in \mathbb{R}^{3\times 3}$, and some matrices $A_j \in \mathbb{R}^{3\times 3}$, for $j = \overline{1,4}$, such that the following equations

$$A(\rho) = A_0 + \sum_{j=1}^{4} \alpha_j(\rho) A_j, \; \sum_{j=1}^{4} \alpha_j(\rho) = 1, \quad (9)$$

with $\alpha_j(\rho) \in [0, 1]$, hold for the system (6). Therefore, the problem now is to design a robust control law τ such that the trajectories of the system (6) converge to zero despite the disturbances d taking into account the state and input constraints (7) and (8), *i.e.,* $e(t) \in \mathbb{E}$ and $\tau(t) \in \mathbb{U}$, for all $t \ge 0$.

Thus, the idea is to design a controller in the following form:

$$\tau = u_0 + u_1. \quad (10)$$

The controller (10) is divided into two parts. The nonlinear element u_1 is the ISMC that will compensate the effect of the matched disturbances, without reaching phase, taking into account only the input constraints; whereas u_0 is the nominal control part, which is composed of a interval predictor–based state–feedback controller and the MPC scheme that will deal with the state and input constraints.

According to (8), $||\tau|| \le \tau_{\max}$, for a given $\tau_{\max} > 0$; then, a specific control effort can be assigned to each part of the controller (10), *i.e.,* $||u_0|| \le u_{0\max}$ and $||u_1|| \le u_{1\max}$, with some constants $u_{0\max}, u_{1\max} > 0$ such that $u_{0\max} + u_{1\max} \le \tau_{\max}$.

IV. Robust Control Design

The proposed controller comprises the design of a robust control law, which is based on an ISMC approach together with an interval predictor–based state–feedback controller and the MPC, which deals with state and input constraints. The proposed approach will guarantee exponential stability for the tracking error dynamics. In the following sections, we describe the proposed methodology.

A. Integral Sliding–Mode Control Design

Let us define the following sliding variable

$$s(e(t)) = G[e(t) - e(0)] - G\int_0^t [A_0 e(\sigma) + B u_0(\sigma)] d\sigma, \quad (11)$$

where $G \in \mathbb{R}^{2\times 3}$ is such that $\det(GB) \ne 0$. The optimal way to design G is $G = B^\top$ or $G = (B^\top B)^{-1} B^\top$, for more details see *e.g.,* [16] and [17]. Note that the dynamics of the sliding variable satisfies

$$\dot{s} = GB[u_1 + F(\rho)d] + G\sum_{j=1}^{4} \alpha_j(\rho) A_j e. \quad (12)$$

Then, the ISMC u_1 is proposed as

$$u_1 = -\zeta(e) \frac{(GB)^\top s}{||(GB)^\top s||}, \quad (13)$$

with some positive gain $\zeta(e) > 0$, for all $e \in \mathbb{E}$. The following lemma provides the conditions to ensure the finite–time convergence of the sliding variable to zero fulfilling the input constraint.

Lemma 1. *Let the ISMC* (13)*, with* $G = B^\top$ *or* $G = (B^\top B)^{-1} B^\top$*, be applied to the system* (12)*, for a given* $u_{1\max} > 0$*. If the gain* $\zeta(e)$ *is selected as*

$$\zeta(e) = \gamma + F_{\max} + A_{\max} ||e||, \quad (14)$$

with $F_{\max} = d_{\max}\sqrt{v_{\max}^2 + \omega_{\max}^2}$*,* $A_{\max} = \sum_{j=1}^{4} ||A_j||$*, and some* $\gamma > 0$ *such that* $0 < \gamma \le u_{1\max} - F_{\max} - A_{\max} e_{\max}$*, with* $e_{\max} = \sqrt{\pi^2 + 2\overline{xy}^2}$*, is satisfied for a given* $u_{1\max} > 0$*; then,* $s = 0$ *is* UFTS.

Therefore, the robust controller u_1 will deal with the perturbations $F(\rho)d$ and part of the parametric uncertainty $\sum_{j=1}^{n} \alpha_j(\rho) A_j e$, satisfying the input constraint $||u_1|| \le u_{1\max}$. Note that the term $F(\rho)d$ is matched with the control u_1, and hence, it is completely compensated from the beginning. However, since the parametric uncertainty $\sum_{j=1}^{n} \alpha_j(\rho) A_j e$ is not matched with the control u_1, only its projection into the matched space of $B^\top$, *i.e.,* the space directly affected by u_1, could be compensated by u_1.

Additionally, considering that $u_{0\max} + u_{1\max} \le \tau_{\max}$, it is possible to fix $u_{1\max}$ according to the upper bound of the disturbances, *e.g.,* $u_{1\max} = 1.1(F_{\max} + A_{\max} e_{\max})$; and then, to assign the rest of the control effort to u_0, *i.e.,* $u_{0\max} \le \tau_{\max} - 1.1(F_{\max} + A_{\max} e_{\max})$.

B. Interval Predictor

Since the sliding–mode takes place, from (12) and recalling that $GB = I_2$, it follows that the equivalent control is

$$u_{1eq} = -G\sum_{j=1}^{4} \alpha_j(\rho) A_j e - F(\rho)d. \quad (15)$$

Therefore, taking into account (9) and substituting (15) in (6), the dynamics of the system on the sliding surface is given by

$$\dot{e} = \left[A_0 + \sum_{j=1}^{4} \alpha_j(\rho)\tilde{A}_j\right] e + B u_0, \quad (16)$$

where $\tilde{A}_j = (I_3 - BG) A_j$. Then, according to [18] and [19], it follows that

$$-\overline{A}\underline{e}^- - \underline{A}\overline{e}^+ \le \sum_{j=1}^{4} \alpha_j(\rho)\tilde{A}_j e \le \overline{A}\overline{e}^+ + \underline{A}\underline{e}^-,$$

where $\overline{A}=\sum_{j=1}^{4}\tilde{A}_j^+$, $\underline{A}=\sum_{j=1}^{4}\tilde{A}_j^-$ and with a couple of vectors $\underline{e},\overline{e}\in\mathbb{R}^3$ such that $\underline{e}\leq e\leq\overline{e}$. Thus, it is possible to design the following interval predictor [20] for system (16)

$$\dot{z}=\mathcal{A}_0z+\mathcal{A}_1z^++\mathcal{A}_2z^-+\mathcal{B}u_0, \tag{17}$$

where $z=(\underline{e}^\top,\overline{e}^\top)^\top\in\mathbb{R}^6$, and the system matrices given as follows

$$\mathcal{A}_0=\begin{pmatrix} A_0 & 0\\ 0 & A_0\end{pmatrix},\ \mathcal{A}_1=\begin{pmatrix} 0 & -\underline{A}\\ 0 & \overline{A}\end{pmatrix},$$
$$\mathcal{A}_2=\begin{pmatrix} -\overline{A} & 0\\ \underline{A} & 0\end{pmatrix},\ \mathcal{B}=\begin{pmatrix} B\\ B\end{pmatrix}.$$

Then, in order to stabilize the tracking error dynamics (6), we need to design a state–feedback u_0 to take the trajectories of the system (17) to zero (see, *e.g.,* [21]).

C. State–Feedback Control Design

The control signal u_0 is proposed as

$$u_0(t)=\begin{cases}\mathcal{U}_0(t), & z(t_i)\notin\mathbb{E}_f,\\ \bar{u}_0(t), & z(t_i)\in\mathbb{E}_f,\end{cases} \tag{18}$$

where $\mathcal{U}_0$ is the control signal provided by the MPC scheme, for all $t\in[t_i,t_i+h)$, with $h\in(0,T)$ and T as the application time and the prediction interval for the MPC, respectively; and $\bar{u}_0$ is the state–feedback controller. The switching set $\mathbb{E}_f$ is defined further on.

The state–feedback controller $\bar{u}_0$ is designed, based on (17), as

$$\bar{u}_0=K_0z+K_1z^++K_2z^-, \tag{19}$$

where $K_0,K_1,K_2\in\mathbb{R}^{2\times6}$ are the matrix gains to be designed. Note that the above controller is nonlinear. The following lemma provides a constructive way to design the state–feedback gains in order to ensure the convergence of the trajectories of the system (17) to zero.

Lemma 2. *Let the state–feedback control law* (19) *be applied to the system* (17), i.e., $u_0(t)=\bar{u}_0(t)$. *Suppose that there exist two vectors $\underline{e}_0,\overline{e}_0\in\mathbb{R}^3$, such that $\underline{e}_0\leq e(0)\leq\overline{e}_0$, that there also exist diagonal matrices $0<X_l\in\mathbb{R}^{6\times6}$, $0\leq R_1,R_2\in\mathbb{R}^{6\times6}$, some diagonal matrices $Q_l,R_0\in\mathbb{R}^{6\times6}$ and some matrices $Y_l\in\mathbb{R}^{2\times6}$, for $l=\overline{0,2}$, such that the following LMIs*

$$\Omega=\begin{pmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13}\\ \star & \Omega_{22} & \Omega_{23}\\ \star & \star & \Omega_{33}\end{pmatrix}\preceq0, \tag{20}$$

$$Q_0+\min\{Q_1,Q_2\}+2\min\{R_1,R_2\}>0, \tag{21}$$
$$\Omega_{11}=\mathrm{He}(\mathcal{A}_0X_0+\mathcal{B}Y_0)+Q_0,$$
$$\Omega_{12}=\mathcal{A}_1X_1+\mathcal{B}Y_1+X_0\mathcal{A}_0^\top+Y_0^\top\mathcal{B}^\top+R_1,$$
$$\Omega_{13}=\mathcal{A}_2X_2+\mathcal{B}Y_2-X_0\mathcal{A}_0^\top-Y_0^\top\mathcal{B}^\top-R_2,$$
$$\Omega_{22}=\mathrm{He}(\mathcal{A}_1X_1+\mathcal{B}Y_1)+Q_1,$$
$$\Omega_{23}=\mathcal{A}_2X_2+\mathcal{B}Y_2-X_1\mathcal{A}_1^\top-Y_1^\top\mathcal{B}^\top+R_0,$$
$$\Omega_{33}=Q_2-\mathrm{He}(\mathcal{A}_2X_2+\mathcal{B}Y_2),$$

are feasible. If the state–feedback gains are designed as $K_l=Y_lX_l^{-1}$, for $l=\overline{0,2}$; then, the trajectories of the system (17) *exponentially converge to zero.*

It is worth saying that the diagonal structure required for $P_0=X_0^{-1}$ is feasible since the existence of a diagonal matrix P_0, as a solution of the Lyapunov equation $\mathrm{He}(P_0\bar{\mathcal{A}}_0)\prec0$, is equivalent to the stability of a Meztler matrix $\bar{\mathcal{A}}_0=\mathcal{A}_0+\mathcal{B}K_0$.

Therefore, the control (19) provides exponential convergence to zero for the interval predictor (17). Moreover, due to the fact that $\mathcal{A}_0$ is a Metzler matrix and $\underline{e}_0\leq e(0)\leq\overline{e}_0$, for two vectors $\underline{e}_0,\overline{e}_0\in\mathbb{R}^3$, the inclusion property $\underline{e}(t)\leq e(t)\leq\overline{e}(t)$ is satisfied (see, for more details, *e.g.,* [21]); hence, the trajectories of system (4) will also converge to zero and the considered problem will be properly solved provided that $x(t)\in\mathbb{X}$, $y(t)\in\mathbb{Y}$, $v(t)\in\mathbb{V}$ and $\omega(t)\in\mathbb{W}$, for all $t\geq0$, hold.

Now, we are able to define the set $\mathbb{E}_f$ as follows

$$\mathbb{E}_f=\{z\in\mathbb{R}^6: V_z(z)\leq\beta^{-1}\varepsilon\}, \tag{22}$$

where $V_z=z^\top P_0z+z^\top P_1z^+-z^\top P_2z^-$, ε is a positive constant, and $\beta=\min_{\forall i=\overline{1,6}}\lambda_i\left[\Phi(P_0+P_1^++P_2^+)^{-1}\right]$, with $\Phi=\bar{Q}_0+\min\{\bar{Q}_1,\bar{Q}_2\}+2\min\{\bar{R}_1,\bar{R}_2\}$, and $X_l=P_l^{-1}$, $\bar{Q}_l=P_lQ_lP_l$, $\bar{R}_l=P_lR_lP_l$, for $l=\overline{0,2}$. It is clear that for any $\varepsilon>0$, $\mathbb{E}_f$ is an invariant set for the system (17); and moreover, all the trajectories, outside of $\mathbb{E}_f$, are attracted inside it and converge to zero. Therefore, it is always possible to select ε such that $\mathbb{E}_f\subset\mathbb{E}\times\mathbb{E}$, *i.e.,* such that $x(t)\in\mathbb{X}$ and $y(t)\in\mathbb{Y}$ hold, for all $t\geq t_f\geq0$.

D. Model Predictive Control Design

Before proceeding with the description of the MPC scheme, we need to introduce the following assumption.

Assumption 1. *There exist $\zeta(e)$, K_0, K_1 and K_2, satisfying the conditions of Lemmas 1 and 2, such that*

$$-\zeta(e(t))\frac{(GB)^\top s(t)}{||(GB)^\top s(t)||}+K_0z(t)+K_1z^+(t)+K_2z^-(t)\in\mathbb{U},$$

for any $z\in\mathbb{E}_f$ and all $t\geq t_f\geq0$.

The previous assumption implies that there always exists a controller (10), given by (13) and (19), such that system (6) is stabilized, and inside $\mathbb{E}_f$, the state and input constraints hold.

Let us now describe the application of the MPC, which will deal with the state and input constraints for all $z(t_i)\notin\mathbb{E}_f$. Define T and $h\in(0,T)$ as the prediction interval and the application time for the MPC, respectively. Therefore, the optimal control problem for the MPC algorithm is given as follows (see, *e.g.*, [6], [7], and [20]):

Problem 1. *For given matrices $0 \preceq W_l \in \mathbb{R}^{6\times 6}$, $l = \overline{0,1}$, a matrix $0 \preceq W_2 \in \mathbb{R}^{2\times 2}$, and $t_i = ih$, with $i \in \mathbb{N}_+$, to find the control signals*

$$\mathcal{U}_0 = \underset{u:[t_i, t_i+T]\to\mathbb{R}^2}{\operatorname{argmin}} z^\top(t_i+T)W_0 z(t_i+T) + \int_{t_i}^{t_i+T} \left[z^\top(\sigma)W_1 z(\sigma) + u_0^\top(\sigma)W_2 u_0(\sigma)\right] d\sigma, \quad (23)$$

such that the following constraints hold: a) $z : [t_i, t_i+T] \to \mathbb{R}^6$ *is a solution of* (17)*;* b) $z(\sigma) \in \mathbb{E}\times\mathbb{E}$ *and* $u_0(\sigma) \in \mathbb{U}$ *for* $\sigma \in [t_i, t_i+T]$*; and* c) $z(t_i+T) \in \mathbb{E}_f$.

Thus, if the above–mentioned optimal control problem is feasible, the trajectories of the system (17) will convergence to the terminal set $\mathbb{E}_f$, when $u_0(t) = \mathcal{U}_0(t)$; and then, inside $\mathbb{E}_f$, when $u_0(t) = \bar{u}_0(t)$, the trajectories will converge to zero satisfying the state and input constraints given in (7) and (8).

Finally, the statements of Lemmas 1 and 2, and the solution of the Problem 1 will provide the main result of this paper, which is described by the following Theorem.

Theorem 1. *Let Assumption 1, the conditions of Lemmas 1 and 2 be satisfied and Problem 1 be feasible. If the control law* (10)*, given by* (13) *and* (18)*, is applied to the system* (4) *and designed according to Lemmas 1, 2 and the solution of Problem 1; then, the tracking error dynamics* (4) *is* UES *and satisfies the state and input constraints given in* (7) *and* (8).

V. Simulation Results

The initial conditions for the kinematics are $x_0 = 1.5$[m], $y_0 = 1.5$[m] and $\theta_0 = 1.5$[rad], and the state and input constraints sets are given as $\mathbb{X} = [-1.6, 1.6]$, $\mathbb{Y} = [-1.6, 1.6]$, $\mathbb{V} = [-5, 5]$ and $\mathbb{W} = [-9, 9]$. The desired trajectory is given by $\omega_d(t) = (\dot{x}_d\ddot{y}_d - \dot{y}_d\ddot{x}_d)/(\dot{x}_d^2 + \dot{y}_d^2)$, $v_d(t) = \sqrt{\dot{x}_d^2 + \dot{y}_d^2}$, $x_d(t) = \mathrm{c}(0.21t)$, $y_d(t) = \mathrm{s}(0.42t)$, and $\theta_d(t) = \int_0^t \omega_d(\tau)d\tau$; and thus, $\overline{\omega}_d = 0.3094$[rad/s] and $\overline{v}_d = 0.6918$[m/s]. For robustness purposes, the external perturbations are taken as $d_1(t) = 0.1\mathrm{s}(3t) + 0.5$ and $d_2(t) = 0.1\mathrm{c}(t) + 0.5$, and hence, $d_{\max} = 0.6$. It is possible to fix the matrix A_0 as

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0.3914\overline{v}_d & 0 & -1 \end{pmatrix},$$

and then, A_i, with $i = \overline{1,4}$, can be computed by a convex polytopic approach, which guarantees the fulfillment of (9), *i.e.,* such that $\sum_{j=1}^4 \alpha_j(\rho)A_j = A(\rho) - A_0$. For the interval predictor, the initial conditions are $\underline{z}_0 = (0.0508, -1.5516, 0.3726)^\top$ and $\overline{z}_0 = (0.0908, -1.5116, 0.4126)^\top$. The parameters for the ISMC are fixed, following the statements of Lemma 1, as $F_{\max} = 0.6177$, $A_{\max} = 8.1458$ and $\gamma = 0.5$. The gains of the state–feedback controller are assessed with the LMIs of Lemma 2, *i.e.,*

$$K_0 = \begin{pmatrix} -0.05, 0.00, -0.036, -0.05, 0.00, -0.036 \\ 0.00, -0.024, 0.00, 0.00, -0.024, 0.00 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} -0.211, 0.00, -0.09, -126.479, -0.055, -0.09 \\ 0.00, -0.205, 0.00, -0.059, -116.542, 0.00 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 126.426, -0.078, 0.089, 0.211, 0, 0.089 \\ -0.082, 116.441, 0.00, 0.00, 0.204, 0.00 \end{pmatrix}.$$

The prediction horizon is selected as $N = 5$, with weight matrices $W_0 = I$, $W_1 = 10000I$ and $W_2 = 0.1I$, and we compute the corresponding switching set $\mathbb{E}_f$ according to (22), with $\beta = 5.0911$ and $\varepsilon = 0.0011$.

All the simulations have been done in MATLAB with the Euler discretization method, sampling–time equal to 0.001, and the solutions for the corresponding LMIs have been found by means of SDPT3 solver among YALMIP in MATLAB while the MPC has been implemented using the `nlmpc` toolbox in MATLAB.

The results are shown in Figs. 1 and 2. The results depicted by Figs. 1 and 2 show that the proposed control strategy is able to ensure the trajectory tracking task without transgressing the state and input constraints, respectively.

VI. Conclusions

In this paper, we propose the design of a robust control strategy for the trajectory tracking problem in perturbed unicycle mobile robots, considering state and input constraints. The proposed robust controller is based on an ISMC approach together with an interval predictor–based state–feedback controller and an MPC scheme. The robust controller deals with some perturbations in the kinematic model, and with state and input constraints that are related to restrictions on the workspace and saturated actuators, respectively; and it guarantees the exponential convergence to zero of the tracking error. Moreover, the synthesis of the proposed controller is constructive since is based on LMIs. Furthermore, the performance of the proposed approach is validated through some simulations.

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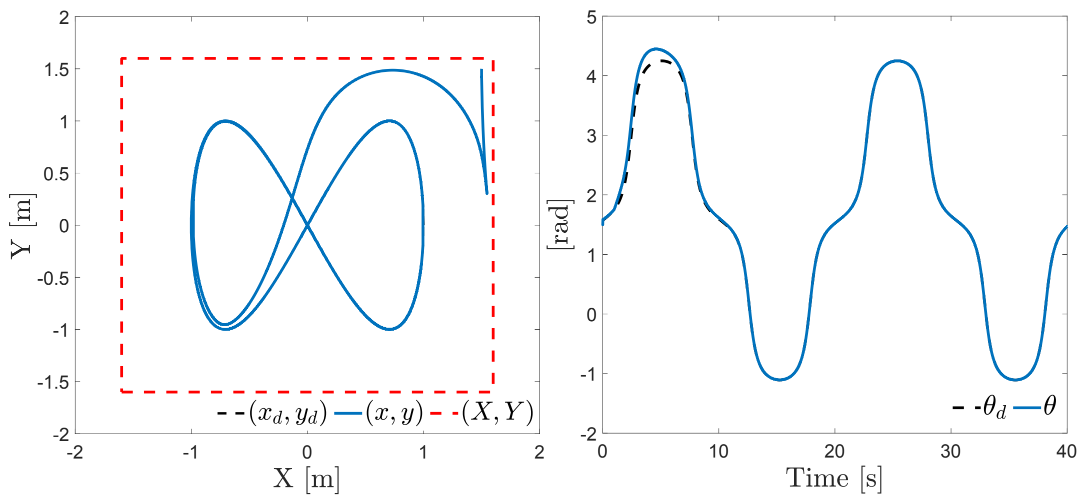

Figure 1. System Trajectories

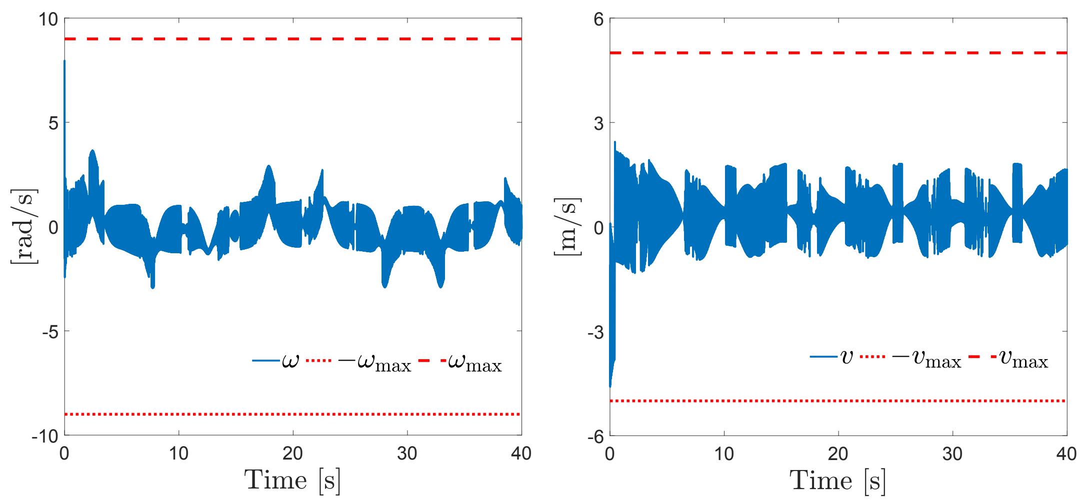

Figure 2. Control Signals

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