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Second Order Approximation of Reachable Sets of LTI Systems

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Abstract— **We present a novel method to approximate reachable sets at time points, of continuous-time LTI systems, in which initial states are subject to compact convex uncertainty and the input may arbitrarily vary over time within a zonotopic uncertainty set. We prove a priori bounds on the approximation error, which are of second order depending on a discretization parameter and can be used to subsequently obtain over- and under-approximations rather than mere approximations. In contrast to competing approaches, our method does not iteratively propagate over- or under-approximations, and it does not reduce the complexity of any of the zonotopes internally produced at intermediate stages. We compare the performance of our method to that of competing approaches on examples.**

I. Introduction

Reachable sets represent a central concept in systems and control theory and have applications in numerous fields; see, e.g. [1]–[5] and the references given there. This is why the problem of efficiently and accurately approximating these sets has been generating research interest for decades. In this note, we consider continuous-time LTI systems of the form

$$\dot{x}(t) = Ax(t) + u(t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $x(0) \in X_0$ and the input signal u may arbitrarily vary over time within the set U. The matrix A, the uncertainty sets $X_0, U \subseteq \mathbb{R}^n$ as well as a duration T are given. The form of the system (1) covers the variant $\dot{x}(t) = Ax(t) + c + Bu(t)$ for a vector c and a matrix B of suitable dimensions, with obvious correspondence between the two problem data. Assuming that X_0 is nonempty compact and convex, U is a zonotope, and $T > 0$, we seek to approximate the reachable set of (1) at time T. For a definition of reachable sets, we refer to Section II-C.

Early attempts have approximated reachable sets by intersections of supporting half-spaces and by convex hulls of support points [1], [6]. Veliov has proposed to use Minkowski sums of linearly transformed copies of the uncertainty set U instead [7]. His method can be interpreted as numerically approximating the set-valued integral representing reachable sets. It applies to reachable sets at time points of LTV systems and to any compact convex uncertainty set U. The method shows second order convergence, i.e., the approximation error is of second order depending on a discretization parameter, and this is best possible for linear quadrature methods unless the uncertainty set U is smooth [7], [8]. While the aforementioned research has focused on qualitative, asymptotic results, specific error bounds allowing to obtain under- and over-approximations have also been presented [7], [9]. The thesis [10] provides an excellent survey of research in linear systems reachability up to the year 1995.

In an influential paper, Girard has proposed a method to over-approximate reachable sets over compact time intervals (*reachable tubes*), which applies to LTI systems under zonotopic uncertainties [11]. Important variants and extensions have been proposed in, e.g. [5], [12]–[16]; see the surveys [17], [18]. The method has been extended to LTV systems [19]. An effective method to automatically tune the various algorithm parameters to the reachability problem at hand has been presented in [20] and has been implemented in the software CORA [21]. Other variants of Girard's original method have also been implemented [16], [22]–[24].

All the aforementioned contributions [5], [11]–[16], [19] show first order convergence and follow, at the algorithmic level, a two-step procedure, in which a reachable set over a small time interval is over-approximated and subsequently and iteratively propagated and extended to obtain a sequence of over-approximations over longer time intervals. Extremely complex intermediate results may be produced which are subsequently reduced to remove information not needed in the approximation that is actually sought.

Ellipsoidal techniques use intersections and unions of ellipsoids to enclose reachable sets at time points [3]. In theory, arbitrarily accurate under- and over-approximations can be obtained by increasing the number of ellipsoids. However, error estimates are not available, numerical errors are neglected, and in practice, neither over-approximations nor under-approximations can be guaranteed. See, e.g. [3, Sec. 7.2, Fig. 4.3]. The methods in [4], [25]–[28] compute interval over-approximations and are not convergent. Finally, some methods numerically solve partial differential equations characterizing reachable sets [3], which requires discretizing the state space and so the computational effort would scale exponentially with the state space dimension.

In this paper, we present a novel method to over-approximate reachable sets of the system (1) at time points. Similar to Veliov [7], we interpret the problem as one of approximating a set-valued integral, and we approach it using the set-valued analog of the Trapezoidal Method. The resulting method converges quadratically depending on a discretization parameter, as proved in Section III, and avoids propagating over-approximations and reducing zonotopes. Our convergence result is based on two contributions which may be of independent interest, namely, a novel error bound

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for the (classical, real-valued) Trapezoidal Method, and a novel result regarding the structure of set-valued integrands arising in linear reachability problems. In Section IV we compare the performance of our method to that of competing approaches on several examples.

II. Preliminaries

A. Notation

The relative complement of the set A in the set B is denoted by $B \setminus A$. $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the sets of real numbers, non-negative real numbers, integers and non-negative integers, respectively, and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$. $[a,b]$, $]a,b[$, $[a,b[$, and $]a,b]$ denote closed, open and half-open, respectively, intervals with end points a and b, e.g. $[0,\infty[= \mathbb{R}_+$. $[a;b]$, $]a;b[$, $[a;b[$, and $]a;b]$ stand for discrete intervals, e.g. $[a;b] = [a,b] \cap \mathbb{Z}$, $[1;4[= \{1,2,3\}$, and $[0;0[= \emptyset$. For a nonempty subset M of the extended reals $[-\infty,\infty]$, $\max M$, $\min M$, $\sup M$ and $\inf M$ denote the maximum, minimum, supremum and infimum, respectively, of M and we adopt the convention that $\sup \emptyset = 0$.

$A \times B$ is the Cartesian product of sets A and B. $f\colon A \to B$ denotes a map from A into B, and the image of a subset $C \subseteq A$ under f is denoted $f(C)$, $f(C) = \bigcup_{a \in C} f(a)$.

Arithmetic operations involving subsets of a linear space X are defined pointwise, e.g. $\alpha M = \{\alpha y \,|\, y \in M\}$ and $M + N = \{y + z \,|\, y \in M, z \in N\}$ whenever $\alpha \in \mathbb{R}$ and $M, N \subseteq X$. If X is endowed with a norm $\|\cdot\|$, the norm of a nonempty subset $M \subseteq X$ is defined by $\|M\| = \sup_{x \in M} \|x\|$, $\operatorname{rad}(M)$ is the (Chebyshev) radius of M, the closed ball with radius $r > 0$ centered at $c \in X$ is denoted $\bar{B}(c,r) = \{x \in X \,|\, \|x - c\| \le r\}$, and we shall abbreviate $\bar{B} = \bar{B}(0,1)$. See [34]. The notation is used without change in the case $X = \mathbb{R}^n$, but if $\|\cdot\|$ is the usual p-norm, we add $p \in [1,\infty]$ as a subscript, e.g. $\|\cdot\|_p$, rad_p, and $\bar{B}_p$. $\langle x|y \rangle$ denotes the standard Euclidean inner product of $x, y \in X$, $\langle x|y \rangle = \sum_{k=1}^n x_k y_k$. The support function $\sigma(\cdot, M)$ of a nonempty subset $M \subseteq X$ is defined by $\sigma(v, M) = \sup\{\langle v|x \rangle \,|\, x \in M\}$ for all $v \in X$. If additionally $Y = \mathbb{R}^m$, the set of linear maps $X \to Y$ is identified with the set of real $m \times n$ matrices and is denoted $\mathbb{R}^{m \times n}$. $L^* \in \mathbb{R}^{n \times m}$ is the transpose of $L \in \mathbb{R}^{m \times n}$. Given norms on X and Y, $\mathbb{R}^{m \times n}$ is endowed with the norm given by $\|L\| = \max\left\{\|Lx\| \,\middle|\, x \in \bar{B} \subseteq X\right\}$, and if the norms on X and Y are the respective p-norms, we denote $\|L\|_p = \|L\|$.

A map is of class C^k if it is continuous and k times continuously differentiable, $k \in \mathbb{Z}_+$. The class of maps in C^k with locally Lipschitz kth derivative is denoted $C^{k,1}$.

B. Functions of Bounded Variation and Related Concepts

Integration and measure are always understood in the sense of Lebesgue, and *a.e.* is to abbreviate both *almost every* and *almost everywhere*. For the concepts in this section, we refer to [35], [36].

Let $I \subseteq \mathbb{R}$ be nonempty and $f\colon I \to \mathbb{R}$. The *variation* $\operatorname{var}(f)$ of f is defined by

$$\operatorname{var}(f) = \sup \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|, \tag{2}$$

where the supremum in (2) is taken over all $k \in \mathbb{N}$ and all $t_0, \dots, t_k \in I$ satisfying $t_{i-1} < t_i$ whenever $i \in [1;k]$, and f is of *bounded variation (b.v.)* if $\operatorname{var}(f) < \infty$.
The function f is *absolutely continuous (a.c.)* if $I = [a,b]$, $a, b \in \mathbb{R}$, $a < b$, and f is the indefinite integral of an integrable function $[a,b] \to \mathbb{R}$. In this case, the derivative $f'(t)$ exists for a.e. $t \in [a,b]$, i.e., $f'\colon [a,b] \setminus E \to \mathbb{R}$ for a well-defined null set $E \subseteq [a,b]$. In particular, if f is a.c., then $\operatorname{var}(f')$ is well-defined and coincides with the essential variation in the sense of [36] of any extension of f' to $[a,b]$.

C. Reachable Sets

Let $I \subseteq \mathbb{R}$ be an interval containing the origin. If $u\colon I \to \mathbb{R}^n$ is locally integrable, we denote by $\varphi(\cdot, p, u)$ the unique solution of (1) defined on I and satisfying $\varphi(0, p, u) = p$ for all $p \in \mathbb{R}^n$,

$$\varphi(t, p, u) = \mathrm{e}^{At} p + \int_0^t \mathrm{e}^{A(t-s)} u(s)\, ds \text{ for all } t \in I. \tag{3}$$

If $t \in \mathbb{R}$ and $X, U \subseteq \mathbb{R}^n$, then

$$R(t, X, U) = \{\varphi(t, p, u) \,|\, p \in X, u\colon \mathbb{R} \to U \text{ loc. int.}\}$$

is the *reachable set at time* t of the system (1). If X and U are nonempty, compact and convex, then so are the reachable sets, which are conveniently written using a set-valued integral,

$$R(t, X, U) = \mathrm{e}^{At} X + \int_0^t \mathrm{e}^{As} U\, ds \quad (t \ge 0). \tag{4}$$

See, e.g. [37].

III. Main results

Consider the system (1), where both the initial state $x(0)$ and the input u are subject to uncertainty as detailed in Section I. We assume the following.

($\mathbf{H_1}$) The uncertainty set $U \subseteq \mathbb{R}^n$ is a zonotope,

$$U = \bar{Z}(u_0, G),$$

where $u_0 \in \mathbb{R}^n$, $G \in \mathbb{R}^{n \times m}$, and $m \in \mathbb{Z}_+$. The uncertainty set $X_0 \subseteq \mathbb{R}^n$ is nonempty, compact and convex, $T > 0$, and $A \in \mathbb{R}^{n \times n}$, where $n \in \mathbb{N}$.

We propose to approximate the reachable set $R(T, X_0, U)$ as follows. Given a discretization parameter $N \in \mathbb{N}$, we produce a sequence $\Omega_0, \dots, \Omega_N$ of subsets Ω_k given by $\Omega_0 = X_0$ and by

$$\Omega_k = \Phi^k X_0 + \xi + \frac{h}{2}\left(U_0 + \Phi^k U_0\right) + \sum_{i=1}^{k-1} \Phi^i (\xi + h U_0) \tag{5}$$

for all $k \in [1;N]$, where $h = T/N$, $U_0 = \bar{Z}(0, G)$, and $\xi \in \mathbb{R}^n$ and $\Phi \in \mathbb{R}^{n \times n}$ are given by the identity

$$\begin{pmatrix} \Phi & \xi \\ 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} Ah & u_0 h \\ 0 & 0 \end{pmatrix}. \tag{6}$$

We here assume the ability to compute the set $\Phi^k X_0$. By suitably representing X_0 and possibly requiring X_0 to belong to a specific class of sets, the right hand side of (5) can be evaluated on a computer. In addition, we assume throughout that the exponential of a matrix can be computed exactly, and we note that (6) implies

$$\Phi = \exp(Ah) \qquad \text{and} \qquad \xi = \int_0^h \exp(As) u_0 \, ds. \tag{7}$$

As it will turn out, each set Ω_k approximates $R(kh, X_0, U)$, and in particular, Ω_N is an approximation of $R(T, X_0, U)$. We emphasize that, in contrast to existing methods [11], [13], [19], the sets Ω_k are not, in general, over- or under-approximations. Over-approximations $\widehat{\Omega}_k$ will be produced from Ω_k only later, see (9), and will never be propagated.

In order to quantify the approximation error, we shall rely on the bound

$$\|\exp(At)\|_\infty \leq M \exp(\eta t). \tag{8}$$

We summarize our assumptions as follows.

($\mathbf{H_2}$) $N \in \mathbb{N}$ and $h = T/N$. The constants $M, \eta \in \mathbb{R}$ satisfy (8) for all $t \in [0, T]$, and the sequence $\Omega_0, \ldots, \Omega_N$ is given by $\Omega_0 = X_0$ and by (5) for all $k \in [1; N]$.

We next present the main result of this note.

III.1 Theorem. *Assume (H_1) and (H_2), denote by $\|\cdot\|$ the infinity norm, and let the radius* rad*, the Hausdorff distance d_H and the closed unit ball $\bar{B}$ be defined w.r.t. this norm. Define*

$$\vartheta = \frac{M\|A\| \operatorname{rad}(U)}{24}, \; \varepsilon(t) = \|A\| \int_0^t \mathrm{e}^{\eta s} \, ds, \; \delta(t) = 1 + \mathrm{e}^{\eta t}$$

for all $t \in \mathbb{R}$, and define

$$\widehat{\Omega}_k = \Omega_k + 2\vartheta\varepsilon(kh)h^2\bar{B} \tag{9}$$

for all $k \in [1; N]$. Then we have

$$R(kh) \subseteq \widehat{\Omega}_k, \tag{10}$$

$$\Omega_k \subseteq R(kh) + \vartheta \left(3\delta(kh) + 5\varepsilon(kh)\right) h^2 \bar{B}, \tag{11}$$

$$d_H(R(kh), \widehat{\Omega}_k) \leq \vartheta \left(3\delta(kh) + 7\varepsilon(kh)\right) h^2 \tag{12}$$

for all $k \in [1; N]$, where we have denoted $R(kh) = R(kh, X_0, U)$.

We briefly discuss the result. The only non-trivial assumption is the growth bound (8), which is satisfied, e.g. by any pair of constants

$$(M, \eta) \in \left\{ (n, \mu_1(A)), (n^{1/2}, \mu_2(A)), (1, \mu_\infty(A)) \right\}, \tag{13}$$

where $\mu_p(A)$ denotes the logarithmic norm of A w.r.t. the p-norm [38]. The quantities ϑ, $\varepsilon(t)$ and $\delta(t)$ in the theorem depend of our choice of M and η and define the amount by which the set Ω_k is enlarged to obtain the over-approximation $\widehat{\Omega}_k$ of $R(kh, X_0, U)$. Importantly, for any choice of M and η, the functions ε and δ are bounded on the interval $[0, T]$, and hence, the bound (12) on the approximation error is of order h^2 as $h \to 0$, uniformly for $k \in [0; N]$. While in this paper we focus on over-approximations, under-approximations of second order accuracy may be obtained from the inclusion (11) in a similar fashion.

Theorem III.1 has been established with the help of two contributions presented in Sections III-A and III-B.

A. Error bounds for the Trapezoidal Method

We consider integrable functions $f\colon [a, a+T] \to \mathbb{R}$ for some $a \in \mathbb{R}$ and some $T > 0$. To approximate the integral $\int_a^{a+T} f(t) \, dt$, we divide the interval $[a, a+T]$ into $N \in \mathbb{N}$ sub-intervals of length h and apply the Trapezoidal Method $Q_{a,T}^N$ defined by

$$Q_{a,T}^N(f) = \frac{h}{2}\left(f(a) + f(a+T)\right) + h \sum_{k=1}^{N-1} f(a + kh), \tag{14}$$

where $h = T/N$ is the *step size*. The error functional $E_{a,T}^N$ for this quadrature method is defined by

$$E_{a,T}^N(f) = \int_a^{a+T} f(t) \, dt - Q_{a,T}^N(f).$$

In our proof of Theorem III.1, f will be the integrand in the identity

$$\sigma(v, R(T, X_0, U)) = \int_0^T \sigma(v, \mathrm{e}^{At} U) \, dt, \tag{15}$$

which is why we are interested in accurate bounds on the quadrature error $E_{a,T}^N(f)$ for functions f satisfying an exponential growth bound similar to the one in (8).

III.2 Proposition. *Let $a, \eta, M \in \mathbb{R}$, $T > 0$ and $N \in \mathbb{N}$, and denote $h = T/N$. Then the following holds for every function $f\colon [a, a+T] \to \mathbb{R}$.*

(i) $E_{a,T}^N(f) \leq 0$ whenever f is convex.

(ii) $|E_{a,T}^N(f)| \leq \varepsilon h^2 MT/12$ whenever f is of class $C^{1,1}$ and $|f''(t)| \leq M\mathrm{e}^{\eta(t-a)}$ for a.e. t, where $\varepsilon = 1$ if $\eta = 0$ and otherwise

$$\varepsilon = \frac{\mathrm{e}^{\eta T} - 1}{\eta T} \cdot \frac{12}{(\eta h)^2} \left(\frac{\eta h}{2\tanh(\eta h/2)} - 1 \right). \tag{16}$$

(iii) $|E_{a,T}^N(f)| \leq h^2 \operatorname{var}(f')/8$ whenever f' is of b.v..

The bounds given in (i) through (iii) are attained. Moreover, we have $\varepsilon \leq T^{-1} \int_0^T \exp(\eta t) \, dt \leq (1 + \exp(\eta T))/2$ in (ii).

The bounds in (i) and (iii), and their sharpness, are well known, and the same holds for the one given in (ii) in the special case $\eta = 0$, $\varepsilon = 1$. See [39]. If $\eta \neq 0$, the latter estimate would have to be applied with $\varepsilon = \max\{1, \exp(\eta T)\}$ instead, which is where our estimate in (ii) improves upon the known one.

B. Regularity of the integrand in (15)

The integrand in (15) equals $\langle v | \exp(At) u_0 \rangle + \|G^* \exp(A^* t) v\|_1$. Our next result shows that we can bound the quadrature error resulting from the second summand, using a combination of (i)-(iii) in Proposition III.2.

III.3 Proposition. *Let* $f\colon [a,b] \to \mathbb{R}^n \in C^{1,1}$ *and define*

$$V = \|f'(a)\|_1 + \|f'(b)\|_1 + \int_a^b \|f''(t)\|_1 \, dt, \tag{17}$$

where $a, b \in \mathbb{R}$, $a < b$, *and* $n \in \mathbb{N}$. *Then there exist functions* $\alpha, \beta\colon [a,b] \to \mathbb{R}$ *satisfying the following conditions.*

(i) β *is convex and Lipschitz, and* $\beta'(b) - \beta'(a) \leq V$.

(ii) $\alpha \in C^{1,1}$ *and* $|\alpha''(t)| \leq \|f''(t)\|_1$ *for a.e.* $t \in [a,b]$.

(iii) $\alpha(t) + \beta(t) = \|f(t)\|_1$ *for every* $t \in [a,b]$, *and for a.e.* $t \in [a,b]$ *we have* $|\alpha'(t) + \beta'(t)| \leq \|f'(t)\|_1$.

The above result implies that

$$\operatorname{var}(g') \leq V + \int_a^b \|f''(t)\|_1 \, dt, \tag{18}$$

where g is given by $g(t) = \|f(t)\|_1$, and hence, allows bounding the error of the Trapezoidal Method applied to $\int_a^b \|f(t)\|_1 \, dt$ using Proposition III.2(iii). Bounds similar to (18) are known [8], [40]–[42] and have been applied to bound quadrature errors [7]–[9], [41]. The main advantage of the decomposition established in Proposition III.3 is that due to the convexity of the summand β the corresponding quadrature error can be ignored when we enlarge the approximation Ω_k to obtain the over-approximation $\widehat{\Omega}_k$ in (9). This reduces the error of our over-approximation of the reachable set and explains the difference between the upper and lower bounds in (10) and (11), respectively.

IV. Numerical Experiments

In this section, we compare the performance of the four methods listed in Tab. I. In all examples, the uncertainty set X_0 is a zonotope, and hence, the sets Ω_k and $\widehat{\Omega}_k$ in Theorem III.1 are zonotopes. We over-approximate the reachable set $R = R(T, X_0, U)$ by the set $\widehat{\Omega}_N$ and report the approximation error, i.e., the Hausdorff distance between R and $\widehat{\Omega}_N$, or a bound on that error, as well as the computational time to obtain $\widehat{\Omega}_N$ and the number of generators of $\widehat{\Omega}_N$. We are particularly interested in the dependence of the error on the number of generators in the over-approximation, and on the computational time to obtain the over-approximation.

TABLE I

Methods compared in Section IV

Method	Description
CORAa (□)	CORA [21]; automatic parameter tuning; theory in [11], [14], [15], [20]
CORAs (◇)	CORA [21]; standard algorithm without zonotope reduction, 9 Taylor terms; theory in [11], [14], [15]
KK (○)	method from [9], using [9, equ. (2.1)] to obtain over-approximation[1]; theory in [7], [9]; implemented by present author
GR (●)	method proposed in present paper

The software CORA (v2024.1.1) [21] is run under MATLAB (R2023a) [43], GR and KK are run under Mathematica (13.2.1) [44]. All tests are run in serial mode on a computer equipped with AMD EPYC 7452 32-core processors and 1 TB of RAM under Linux x86 (64-bit).

[1]We have replaced $\|\exp(AT)\|_2$ in [9, equ. (2.1)], an obvious misprint, by $\exp(\|A\|_2 T)$.

A. Oscillator

Consider the reachability problem from [21, Sect. 9.3.1] given by

$$A = \begin{pmatrix} -7/10 & -2 \\ 2 & -7/10 \end{pmatrix},$$

$X_0 = (10, 5) + [-1/2, 1/2]^2$, $U = [3/4, 5/4]^2$, and $T = 5$.

For GR, Theorem III.1 is applied with $M = 2^{1/2}$ and $\eta = \mu_2(A) = -7/10$. The approximation error ε is the Hausdorff distance w.r.t. the Euclidean norm, which has been computed with relative error not exceeding 1% using the method from [10, Sect. 3.1.4]. From the results presented in Fig. 1 we can clearly distinguish between the two second order methods GR and KK, and the two first order methods CORAa and CORAs. In addition, the results for CORAs show the huge complexity of the over-approximations generated by the algorithms in CORA [21], which is subsequently reduced by a post-processing step. For this example, GR shows the best performance and KK shows the worst, in terms of both the dependence of the error on the number of generators and in terms of the dependence on the computational time. The performance of KK is the result of the conservativeness of the a priori error bound in [9, equ. (2.1)] that we have used to obtain over-approximations, and KK would be expected to outperform CORA only if very accurate approximations of the reachable set are to be computed.

We would like to add that the relatively large computational times for CORA compared to GR should not be interpreted as a deficit of CORA, as CORA additionally approximates reachable tubes while GR does not. However, the computational times of CORA for ever stricter error requirements seem to grow faster than those of both KK and GR.

B. Platoon of vehicles

We consider a reachability problem proposed in [45], which is also used in [46], [47]. The system (1) describes a controlled platoon of three vehicles with a manually driven leader whose acceleration is uncertain. We focus on the case without any communication between the vehicles, for which the matrix $A \in \mathbb{R}^{9\times 9}$ is given in [45, p. 40], $U = \{0\} \times [-9, 1] \times \{0\}^7$, $X_0 = \{0\}$, and $T = 20$.

We compare GR to CORAa. As for GR, all pairs (M, η) in (13) lead to overly conservative estimates of the norm of the exponential matrix, which is why we have resorted to $M = 300$ and $\eta = -0.4$, which are easily demonstrated to satisfy the condition (8).

The results presented in Fig. 2 show that for over-approximations of comparable complexity, the approximations produced by CORA are more precise than those produced by GR. The second order method GR can be expected to outperform the first order method CORAa only if extremely accurate approximations are required. In addition, the presentation Fig. 2 gives a slight advantage to GR over CORAa, since $\hat{\varepsilon}$ is a bound on $d_H(R, \widehat{\Omega}_N)$ w.r.t. the 2-norm for CORAa and w.r.t. the ∞-norm (GR). Similarly to the case of KK in Example IV-A, this is due to a very poor a priori

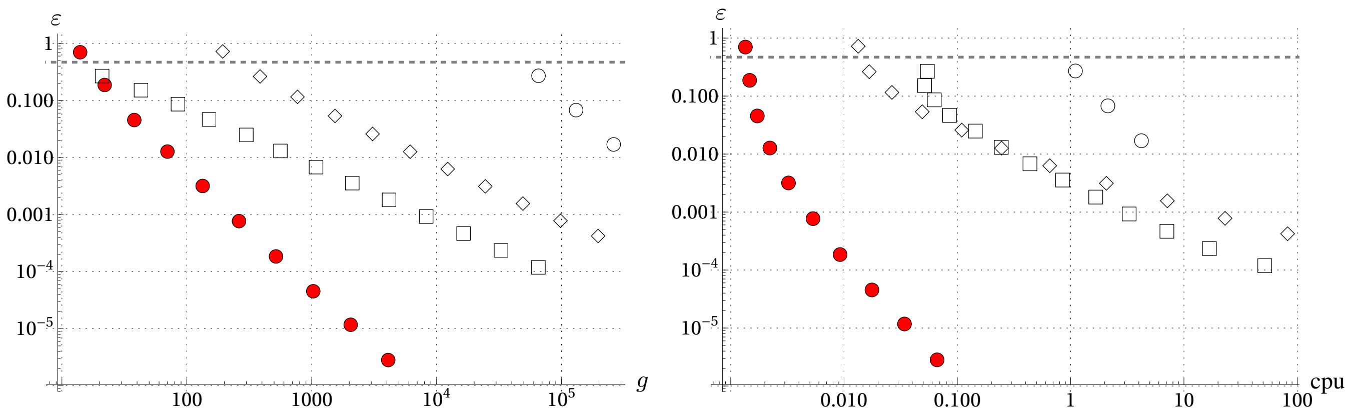

Fig. 1. Results for the example in Section IV-A. Approximation error ε over the number of generators g in the zonotopic over-approximation $\widehat{\Omega}_N$ of the reachable set R (left) and over the computational time in seconds (right). ε is the Hausdorff distance between R and $\widehat{\Omega}_N$ w.r.t the Euclidean norm, and $---$ marks $\mathrm{rad}_2(R)$. See also Tab. I.

bound for the exponential matrix used by GR to obtain over-approximations. Computational times are comparable, but again, CORAa additionally approximates reachable tubes. The computational times of CORAa for ever stricter error requirements seem to grow faster than those of GR.

V. Conclusions

We have proposed a novel method to over-approximate reachable sets of continuous-time LTI systems. Our method is convergent of second order and outperforms the second order method from [9] in terms of accuracy and speed by several orders of magnitude. In general, the proposed method performs very well, and even extremely well when highly accurate over-approximations are to be computed, provided that the norm of the exponential matrix can be effectively bounded by an a priori estimate. Otherwise, the method may perform rather poorly, when compared to, e.g. the excellent performance of CORA [21]. To combine the proposed method with less conservative, and in particular, with a posteriori bounds is a subject of our current research.

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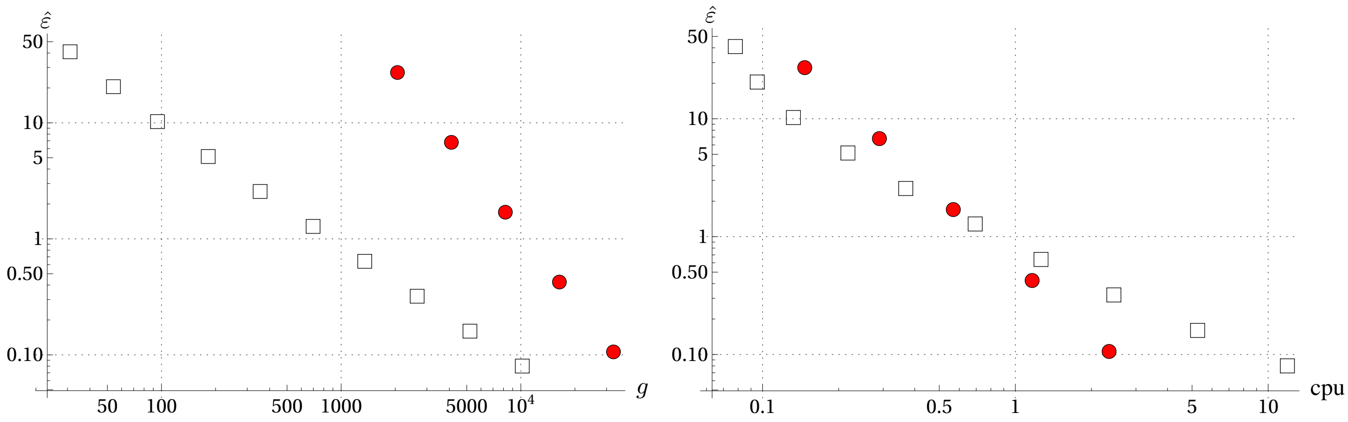

Fig. 2. Results for the example in Section IV-B. Bound $\hat{\varepsilon}$ on approximation error over the number of generators g in the zonotopic over-approximation $\widehat{\Omega}_N$ of the reachable set R (left) and over the computational time in seconds (right). ε is the Hausdorff distance between R and $\widehat{\Omega}_N$ w.r.t the Euclidean norm, and — — — marks $\mathrm{rad}_2(R)$. See also Tab. I.

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