

Multi-UAV trajectory planning problem using the difference of convex function programming

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Abstract—Trajectory planning for a swarm of UAVs is known as a challenging nonconvex optimization problem, particularly due to a large number of collision avoidance constraints required for individual pairs of UAVs in the swarm. In this paper, we tackle this nonconvexity by leveraging the difference of convex function (DC) programming. We introduce the slack variables to relax and reformulate the collision avoidance conditions and employ the penalty function term to equivalently convert the problem into the DC form. Consequently, we construct a penalty DC algorithm in which we sequentially solve a set of convex optimization problems obtained by linearizing the collision avoidance constraint. The algorithm iteratively tightens the safety condition and reduces the objective cost of the planning problem and the additional penalty term. Numerical results demonstrate the effectiveness of the proposed approach in planning a large number of UAVs in congested space.

Keywords—Trajectory planning, DC programming, penalty DC algorithm, collision avoidance, non-convex optimization

NOMENCLATURE

A. Set and Indices

\mathcal{N}, i	Set and index of vehicles, $i \in \mathcal{N}$
\mathcal{T}, k	Set and index of time steps, $k = 1, 2, \dots, K \in \mathcal{T}$
\mathcal{S}	Set of initial states, including starting position $\mathcal{S}(x_i^s, y_i^s, z_i^s)$, starting velocity $\mathcal{S}(v_i^x, v_i^y, v_i^z)$, and starting force $\mathcal{S}(f_i^x, f_i^y, f_i^z)$ of vehicle i
\mathcal{G}	Set of goal states, including goal position $\mathcal{G}(x_i^g, y_i^g, z_i^g)$, goal velocity $\mathcal{G}(v_i^x, v_i^y, v_i^z)$, and goal force $\mathcal{G}(f_i^x, f_i^y, f_i^z)$ of vehicle i

B. Parameters

\bar{V}_i	Maximum velocity of vehicle i
\bar{F}_i	Maximum force of vehicle i
d	Minimum distance between two vehicles to avoid a collision
$\hat{\mathbf{A}}_i$	State-space matrix of vehicle i
$\hat{\mathbf{B}}_i$	Input matrix of vehicle i
ρ_f^j	Penalty for O_f of objective function
ρ_g^k	Penalty for O_g of objective function
τ, μ, ϵ	Parameters used in DCA

C. Variables

$x_{i,k}$	x -coordinate of position of vehicle i at time step k
$y_{i,k}$	y -coordinate of position of vehicle i at time step k
$z_{i,k}$	z -coordinate of position of vehicle i at time step k
$v_{i,k}^x$	x -component of velocity vector of vehicle i at time step k
$v_{i,k}^y$	y -component of velocity vector of vehicle i at time step k
$v_{i,k}^z$	z -component of velocity vector of vehicle i at time step k
$\vec{V}_{i,k}$	velocity vector of vehicle i at time step k
$f_{i,k}^x$	x -component of force vector of vehicle i at time step k

$f_{i,k}^y$	y -component of force vector of vehicle i at time step k
$f_{i,k}^z$	z -component of force vector of vehicle i at time step k
$\vec{F}_{i,k}$	force vector of vehicle i at time step k

I. INTRODUCTION

The trajectory planning problem concerns finding an optimal solution of the trajectory for a single aircraft or a group of aircraft to travel from a given starting state over a map of the environment to a goal state. Mixed-integer linear programming (MILP) is the standard method used to solve the trajectory generation problem for many decades [1]. MILP is a powerful optimization method that allows the inclusion of integer variables and discrete logic of linearization for non-convex constraints in a continuous linear trajectory optimization [2]–[4]. These mixed-integer and continuous variables can be used to model logical constraints such as obstacle avoidance and vehicle separation, while the dynamic and kinematic settings of individual aircrafts are bounded by continuous constraints. Concurrently, the magnitudes of the velocity and force vectors are modeled by the spherical geometry-based sampling approximation technique for a 3-D environment, or the edges of an N-sided polygon approximation technique for a 2-D environment [4]. To this extent, the MILP method uses many auxiliary variables and constraints to formulate the trajectory optimization problem.

Recent improvements in aircraft’s capabilities, especially for unnamed aerial vehicles (UAVs), facilitate them to carry out longer and more complex missions in dynamic environments. Moreover, as more vehicles and more targets are involved in a mission, the size of the trajectory optimization problem based on MILP increases exponentially. Consequently, the computation time of the problem to obtain the optimal solution becomes much more expensive. Convex optimization methods can handle well the conic constraints such as bounds on the magnitude of velocity and force vectors without incorporating the approximation techniques [2], [5].

The most key challenge in solving trajectory optimization models with convex cost functions and affine vehicle dynamics is that we often encounter the nonconvex collision avoidance requirement [6], [7]. This nonconvex requirement is enforced for all individual pairs of UAVs in the swarm, thus making the problem computationally challenging. This research proposes the use of the difference of convex function

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(DC) programming [8] to tackle the nonconvexity of the planning problem for a swarm of a large number of UAVs. First, we relax the collision avoidance constraints by slack variables and add the sum of slack variables as a penalty function to the original objective function. Consequently, we obtain the equivalent reformulation of the original problem. We then sequentially linearize the relaxed non-convex collision avoidance constraints while minimizing the reformulated problem with an increasing penalty term. The algorithm is called the penalty DCA [9] or penalty convex-concave procedure [10], which aims to tighten the convexified problem of the original nonconvex one. This paper is organized as follows. Section II presents the mathematical model of the generic trajectory planning problem for a swarm of multiple UAVs. Section III reformulates the problem into Mixed-Integer Convex Program (MICP) and DC forms, respectively. Section IV presents the penalty DCA. The numerical results are shown in Section V, and Section VI concludes the paper.

II. MULTI-UAV TRAJECTORY PLANNING PROBLEM

A. State-space model of a single UAV

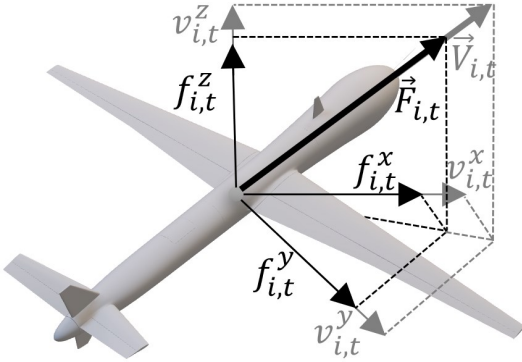


Fig. 1. Velocity and Force vectors in body and fixed axes coordinate system

We consider a fixed-wing UAV modeled as a point mass flying in a predetermined 3-dimensional space with the (x, y, z) coordinates (i.e., forward, side, and vertical directions, respectively) as shown in Figure 1 where i denotes the UAV index in the swarm, (x_i, y_i, z_i) is its location, and m_i is its constant mass. The UAV's velocity \vec{V}_i , by definition, represents the change of the UAV's location and can be decomposed into $\vec{v}_i^x = \frac{d\vec{x}_i}{dt}$ (the forward velocity), $v_i^y = \frac{dy_i}{dt}$ (the side velocity), and $v_i^z = \frac{dz_i}{dt}$ (the vertical velocity) as follows:

$$\vec{V}_i = \vec{v}_i^x + \vec{v}_i^y + \vec{v}_i^z = \frac{d\vec{x}_i}{dt} + \frac{dy_i}{dt} + \frac{dz_i}{dt}. \quad (1)$$

The force \vec{F}_i as the control input alternates the UAV acceleration following Newton's second law:

$$\vec{F}_i = m_i \frac{d(\vec{V}_i)}{dt} = m_i \left(\frac{d\vec{v}_i^x}{dt} + \frac{d\vec{v}_i^y}{dt} + \frac{d\vec{v}_i^z}{dt} \right), \quad (2)$$

which is also decomposed into $\vec{f}_i^x = \frac{d\vec{v}_i^x}{dt}$, $\vec{f}_i^y = m_i \frac{d\vec{v}_i^y}{dt}$, and $\vec{f}_i^z = m_i \frac{d\vec{v}_i^z}{dt}$ (i.e., forward, side, vertical forces).

Equations (1)-(2) together form the following UAV's kinodynamic state-space model:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i \mathbf{u}_i \quad (3)$$

where $\mathbf{x}_i = [x_i, y_i, z_i, v_i^x, v_i^y, v_i^z]^\top$,

$$\mathbf{u}_i = [f_i^x, f_i^y, f_i^z]^\top,$$

$$\mathbf{A}_i = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{B}_i = \frac{1}{m_i} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here, \mathbf{x}_i denotes the vector of state variables, \mathbf{u}_i denotes the control input, \mathbf{A}_i denotes the state matrix, and \mathbf{B}_i denotes the input matrix of UAV i . The kinodynamic model (3) can be converted into the discrete time-variant form as follows:

$$\mathbf{x}_{i,k+1} = \hat{\mathbf{A}}_i \mathbf{x}_{i,k} + \hat{\mathbf{B}}_i \mathbf{u}_{i,k}, \quad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \quad (4)$$

where $\hat{\mathbf{A}}_i = \mathbf{I} + \Delta T \mathbf{A}_i$, $\hat{\mathbf{B}}_i = \Delta T \mathbf{B}_i$, particularly:

$$\hat{\mathbf{A}}_i = \begin{pmatrix} 1 & 0 & 0 & \Delta T & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta T & 0 \\ 0 & 0 & 1 & 0 & 0 & \Delta T \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \hat{\mathbf{B}}_i = \frac{\Delta T}{m_i} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\mathbf{x}_{i,k} = [x_{i,k}, y_{i,k}, z_{i,k}, v_{i,k}^x, v_{i,k}^y, v_{i,k}^z]^\top$ and $\mathbf{u}_{i,k} = [f_{i,k}^x, f_{i,k}^y, f_{i,k}^z]^\top$ respectively represent vectors of state variables and control inputs of UAV i at time step k , and ΔT is the length of the time step.

B. Problem Formulation

We consider the trajectory planning problem for a swarm of N UAVs in which each UAV needs to travel from its initial position to its final destination without colliding with other UAVs. In other words, for each UAV $i \in \mathcal{I}$ in the swarm, we need to determine a sequence of positions $(x_{i,k}, y_{i,k}, z_{i,k})$ forming the UAV's trajectory and the sequence of control action $\mathbf{u}_{i,k}$ at each time step k such that the UAV reaches its final destination without colliding with others in the swarm. This can mathematically be formulated as the following large-scale non-convex optimization problem:

$$\min_{\mathbf{x}, \mathbf{u}} \sum_{i=1}^N \sum_{k=1}^K \left(\rho^f \times \underbrace{\sqrt{(f_{i,t}^x)^2 + (f_{i,t}^y)^2 + (f_{i,t}^z)^2}}_{O_f(\mathbf{u})} + \rho^k \times \underbrace{\sqrt{(x_{i,t} - x_i^g)^2 + (y_{i,t} - y_i^g)^2 + (z_{i,t} - z_i^g)^2}}_{O_g(\mathbf{x})} \right), \quad (5)$$

subject to:

$$(\mathbf{x}_i, \mathbf{u}_i) \in \Omega_i = \left\{ \mathbf{x}_{i,k+1} = \hat{\mathbf{A}}_i \mathbf{x}_{i,k} + \hat{\mathbf{B}}_i \mathbf{u}_{i,k}, \forall k, \right. \quad (6a)$$

$$(x_{i,1}, y_{i,1}, z_{i,1})^\top = (x_i^S, y_i^S, z_i^S)^\top, \quad (6b)$$

$$(v_{i,1}^x, v_{i,1}^y, v_{i,1}^z)^\top = (v_i^{x,S}, v_i^{y,S}, v_i^{z,S})^\top, \quad (6c)$$

$$(x_{i,K}, y_{i,K}, z_{i,K})^\top = (x_i^G, y_i^G, z_i^G)^\top, \quad (6d)$$

$$(v_{i,K}^x, v_{i,K}^y, v_{i,K}^z)^\top = (v_i^{x,G}, v_i^{y,G}, v_i^{z,G})^\top, \quad (6e)$$

$$(f_{i,K}^x, f_{i,K}^y, f_{i,K}^z)^\top = (f_i^{x,G}, f_i^{y,G}, f_i^{z,G})^\top, \quad (6f)$$

$$\sqrt{(v_{i,k}^x)^2 + (v_{i,k}^y)^2 + (v_{i,k}^z)^2} \leq \bar{V}_i, \forall k \quad (6g)$$

$$\left. \sqrt{(f_{i,k}^x)^2 + (f_{i,k}^y)^2 + (f_{i,k}^z)^2} \leq \bar{F}_i, \forall k \right\}, \forall i. \quad (6h)$$

$$\sqrt{(x_{i,k} - x_{j,k})^2 + (y_{i,k} - y_{j,k})^2 + (z_{i,k} - z_{j,k})^2} \geq d, \quad \forall i \neq j, \forall k. \quad (7)$$

The objective function (11) aims to minimize the control effort and the traveling time of UAVs to reach their final destination. The objective consists of two terms, O_f penalizes the force supplying to the vehicle i at time t with a unit fuel cost ρ^f whereas O_g penalizes the remaining distance of each vehicle i to its goal position multiplying with the value ρ^k . Typically, ρ^k is set as an increasing function of the time indexes, e.g., $\rho^k := a \times k$, $a > 0$, so O_g urges UAVs to reach their goal points as soon as possible. The objective function is subject to two sets of constraints, as follows.

Constraint (6) encapsulates all local constraints of state variables and control inputs of individual UAVs $i = 1, \dots, N$ in their corresponding feasible set Ω_i . In particular, the dynamics of each vehicle following discrete-time and linear state-space equation (4) now acts as a linear constraint (6a). The starting position is expressed in (6b)-(6c) whereas the set of final conditions including the goal position, velocity, and force of vehicle are introduced in (6d)-(6f). The physical limits of the UAV's velocity and driving force are captured in (6g) and (6h). The feasible set Ω_i is convex, and (6) is a convex constraint.

The constraint (7) represents the collision avoidance among UAVs in the pair. In particular, the Euclidean separation distance between all pairs of vehicles $i \neq j$ must be equal to or greater than the safety margin d at every time step $k = 1, \dots, K$. The number of collision avoidance conditions is $\frac{N \times (N-1)}{2} \times K$. Since the Euclidean distance norm is a convex function, (7), is a non-convex constraint.

Overall, the multi UAVs' trajectory planning problem can be summarized in the following form:

$$\begin{aligned} [\text{P}] \quad \min \quad & O_f(\mathbf{u}) + O_g(\mathbf{x}) \\ \text{s.t.} \quad & (\mathbf{x}_i, \mathbf{u}_i) \in \Omega_i, \forall i, \\ & \text{non-convex collision avoidance (7)}. \end{aligned}$$

It is worth mentioning that the problem P is generic, as we can tailor Ω_i or the objective function for different

application requirements, e.g., the UAV's trajectory must visit certain locations or stay close as much as possible for certain predetermined paths. Such modifications generally do not affect the convexity of Ω_i , thus not affecting computational performance. The complexity of P stems from a large number of nonconvex collision avoidance conditions (7). However, such a constraint is critical for safety requirements and cannot be ignored. The aim of this paper is to tackle this convex constraint set, thus facilitating the computation of UAV swarm coordination in the form of P.

III. TRADITIONAL MIXED-INTEGER CONVEX PROGRAMMING APPROACH

We can use mixed integer linear programming to capture the nonconvex collision constraint (7). Theoretically, a generic nonconvex constraint can be written in the form $x \notin \mathcal{C}$ where \mathcal{C} is a convex set of variables x . If we can polyhedrally outer approximate \mathcal{C} by a set of L linear constraints [11], [12]

$$\text{Poly}(\mathcal{C}) = \left\{ x \mid a_\nu^\top x \leq b_\nu, \nu = 1, 2, \dots, L \right\}, \quad (8)$$

then the condition $x \notin \mathcal{C}$ will be attained by letting at least one constraint in (8) be violated using the auxiliary binary variable u_m as follows:

$$a_\nu^\top x \geq b_\nu + \epsilon - \bar{U}u_\nu, \quad u_\nu \in \{0, 1\}, \quad \forall m = 1, 2, \dots, M, \quad (9)$$

$$\sum_{\nu=1}^M (1 - u_\nu) \geq 1$$

where \bar{U} is a sufficient large number and ϵ is a small number. Constraint (9) means that at least one value of $u_\nu = 0$, consequently, one inequality $a_\nu^\top x \geq b_\nu + \epsilon$ activates, forcing $x \notin \mathcal{C}$ (the term ϵ is used to prevent the equality $a_\nu^\top x = b_\nu$).

We are now applying (9) to the case of the collision avoidance constraint. Note that we can approximate the 2-D Lorentz cone:

$$\mathbb{L}^2 = \left\{ (\hat{x}, \hat{y}, d) \in \mathbb{R}^2 \times \mathbb{R}_+ \mid \sqrt{\hat{x}^2 + \hat{y}^2} \leq d \right\}$$

by the following linear inequalities of variables α, β :

$$\alpha_0 \geq \hat{x}, \quad \alpha_0 \geq -\hat{x}, \quad \beta_0 \geq \hat{y}, \quad \beta_0 \geq -\hat{y}, \quad (10a)$$

$$\left\{ \beta_{\nu+1} \geq -\sin\left(\frac{\pi}{2\nu}\right) \alpha_\nu + \cos\left(\frac{\pi}{2\nu}\right) \beta_\nu, \right. \quad (10b)$$

$$\left. \beta_{\nu+1} \geq \sin\left(\frac{\pi}{2\nu}\right) \alpha_\nu - \cos\left(\frac{\pi}{2\nu}\right) \beta_\nu, \right. \quad (10c)$$

$$\left. \alpha_{\nu+1} = \cos\left(\frac{\pi}{2\nu}\right) \alpha_\nu + \sin\left(\frac{\pi}{2\nu}\right) \beta_\nu, \right\} \quad (10d)$$

$$\begin{aligned} & \nu = 0, \dots, L-1, \\ & \alpha_L \leq d, \quad \beta_L \leq \tan\left(\frac{\pi}{2L}\right) \alpha_L. \end{aligned} \quad (10e)$$

The approximation (10) basically forms a regular 2^L -sided polygon with $2(L+1)$ additional variables $\alpha_\nu, \beta_\nu, \nu = 1, \dots, L$ as follows:

$$\text{Poly}(\mathbb{L}^2) = \left\{ (\hat{x}, \hat{y}, \alpha, \beta) \in \mathbb{R}^2 \times \mathbb{R}^{2(L+1)} \mid (10a) - (10e) \right\}.$$

Note also that the collision condition, i.e., the distance between two UAV is less than d , is in the form of 3-dimension Lorentz cone \mathbb{L}^3 as follows

$$\mathbb{L}^3 = \left\{ (\hat{x}, \hat{y}, \hat{z}, d) \in \mathbb{R}^2 \times \mathbb{R}_+ \mid \sqrt{\hat{x}^2 + \hat{y}^2 + \hat{z}^2} \leq d \right\},$$

which can be captured by two second-order cone constraints:

$$\hat{x}^2 + \hat{y}^2 \leq \hat{w}^2, \quad \hat{w}^2 + \hat{z}^2 \leq d^2, \quad \hat{w} \geq 0,$$

each is indeed \mathbb{L}^2 and can be polyhedrally approximated using (10). Consequently, we can combine (9) and (10) to construct a set of MILP constraints enforcing the distance between two UAVs outside the collision range d . In particular, we need to write two sets of linear constraints (10) associated with the polyhedral approximation $\text{Poly}(\mathbb{L}^2)$ of two 2-D Lorentz cones and then apply the MILP reformulation trick (9). Due to page limitation, we omit the presentation of the general case with arbitrary L . In the special case $L = 2$, we can compact the set of constraints as follows:

$$x_{i,k} - x_{j,k} \geq d - \bar{U}u_{i,j,k}^1, \quad x_{j,t} - x_{i,k} \geq d - \bar{U}u_{i,j,k}^2 \quad (11a)$$

$$y_{i,k} - y_{j,k} \geq d - \bar{U}u_{i,j,k}^3, \quad y_{j,k} - y_{i,k} \geq d - \bar{U}u_{i,j,k}^4 \quad (11b)$$

$$z_{i,k} - z_{j,k} \geq d - \bar{U}u_{i,j,t}^5, \quad z_{j,k} - z_{i,k} \geq d - \bar{U}u_{i,j,k}^6 \quad (11c)$$

$$\sum_{\nu=1}^6 u_{i,j,k}^\nu \leq 5, \quad \forall k, \forall i \neq j. \quad (11d)$$

which enforces the distance of two UAVs i and j outside the cubic outerly approximating the collision sphere of radius d , i.e., $|x_{i,k} - x_{j,k}| \geq d$ OR $|y_{i,k} - y_{j,k}| \geq d$ OR $|z_{i,k} - z_{j,k}| \geq d$, $\forall k, \forall i \neq j$ (ϵ in (9) is chosen as zero since the distance d satisfies the minimum requirement of safety).

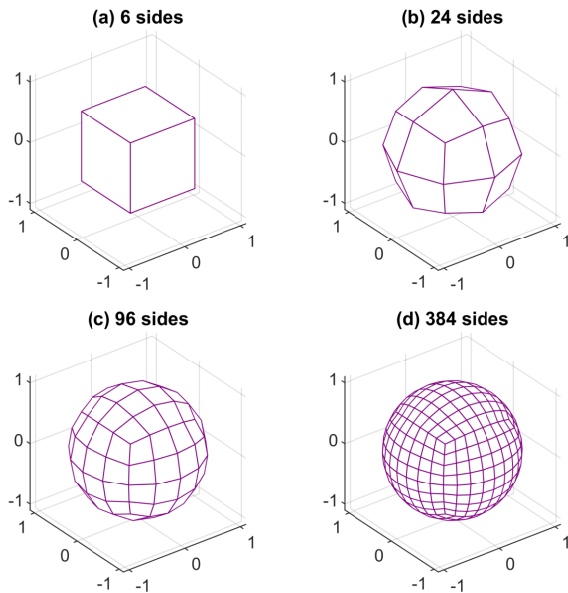


Fig. 2. A polyhedral approximation of the 3-D ball

Remark: Figure 2 represents a polyhedral approximation of the 3-D ball with radius d . While the approximation error

reduces as L increases, the computational demand increases significantly as the number of constraints and binary variables employed increases. Indeed, our examination shows that only $L = 2$ is computationally feasible given that the number of collision avoidance conditions that we need to approximate is $\frac{N \times (N-1)}{2} \times K$. However, the MILP reformulation with $L = 2$ is very conservative, which might result in infeasibility if we coordinate a large swarm of UAVs in a small space.

IV. DC PROGRAMMING APPROACH

A. Problem Reformulation in DC form

Let $g_{i,j,k}(\mathbf{x})$ denote the distance between two UAVs i and j in the time step k :

$$g_{i,j,k}(\mathbf{x}) = \sqrt{(x_{i,k} - x_{j,k})^2 + (y_{i,k} - y_{j,k})^2 + (z_{i,k} - z_{j,k})^2},$$

so the collision avoidance constraints can be rewritten as

$$d - g_{i,j,k}(\mathbf{x}) \leq 0, \quad \forall i \neq j, \forall k \quad (12)$$

We employ the penalty function transformation method to bring the nonconvex constraint (12) into an objective function of the problem P as follows:

$$[\mathbf{P}_\tau] \min \quad \underbrace{O_f(\mathbf{u}) + O_g(\mathbf{x})}_{f_0(\mathbf{u}, \mathbf{x})} + \tau \sum_{i,j,k|i \neq j} s_{i,j,k} \quad (13)$$

$$\text{s.t.} \quad (\mathbf{x}_i, \mathbf{u}_i) \in \Omega_i, \quad \forall i, \quad (14)$$

$$(d - g_{i,j,k}(\mathbf{x})) \leq s_{i,j,k}, \quad \forall i \neq j, \forall k \quad (15)$$

where \mathbf{P}_τ represents the penalty problem of P with the penalty coefficient $\tau \geq 0$ and s represent the relax term for original nonconvex constraint (12). There exists $\tau^* \geq 0$ such that for all $\tau \geq \tau^*$, P and \mathbf{P}_τ have the same optimal solutions and optimal values [9], i.e., $s^* = 0$ and (12) satisfies. The problem \mathbf{P}_τ is indeed a difference of the convex function (DC) programming problem, i.e., the left-hand side of (15) can be considered as the difference of two convex functions on \mathbf{x} : d and $g_{i,j,k}(\mathbf{x})$.

B. The DC Algorithm

We solve the problem \mathbf{P}_τ using the enhanced DCA, namely penalty DCA or DCA2 [9], or penalty convex-concave procedure [10], for tackling nonconvexity appearing in (15). In particular, we sequentially (i) solve a set of convex functions constructed by linearizing the concave term, particularly $-g_{i,j,k}(\mathbf{x})$ in (15) and (ii) increase the penalty coefficient τ until the nonconvex condition is satisfied. The detailed description of the algorithm is as follows:

Step 1: Choose the initial point $\hat{\mathbf{x}}_0$, $\tau_0 \geq 0$, $\bar{\tau} \geq 0$, and $\mu \geq 1$. Set the iteration $m = 0$. Initialize the set $\mathcal{O} := \{\hat{\mathbf{x}}_0\}$.

Step 2: Solve the following optimization problem:

$$[\mathbf{P}_\tau^m] \min \quad \text{objective (13)} \quad (16)$$

$$\text{s.t.} \quad \text{constraint (14)} \quad (16)$$

$$d - g_{i,j,k}(\hat{\mathbf{x}}) - \nabla^T g_{i,j,k}(\hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}}) \leq s_{i,j,k} \quad (17)$$

$$\forall i \neq j, \forall k, \forall \hat{\mathbf{x}} \in \mathcal{O},$$

$$s_{i,j,k} \geq 0, \quad \forall i \neq j, \forall k \quad (18)$$

to obtain the optimal solution \mathbf{x}^* . Mathematically, we replace (15) by a set of linear approximations at a set of points $\hat{\mathbf{x}}$ obtained so far.

Step 3: Let $\hat{\mathbf{x}}_m = \mathbf{x}^*$. Update the set $\mathcal{O} := \mathcal{O} \cup \hat{\mathbf{x}}_m$ and update penalty coefficient $\tau_{m+1} = \min\{\mu\tau_m, \bar{\tau}\}$.

Step 4: Stop if the following criteria satisfy:

- the maximum penalty coefficient reaches

$$\tau_m = \bar{\tau}$$

- the gap between optimal objectives found between two consecutive iterations is small

$$\delta_m = \left(f_0(\mathbf{u}_m^*, \mathbf{x}_m^*) + \tau_m \sum_{i,j,k|i \neq j} s_{i,j,k,m}^* \right) - \left(f_0(\mathbf{u}_{m-1}^*, \mathbf{x}_{m-1}^*) + \tau_{m-1} \sum_{i,j,k|i \neq j} s_{i,j,k,m-1}^* \right) \leq \epsilon,$$

where ϵ is a very small number acting as the tolerance. Note also that $\mathbf{u}_m^*, \mathbf{x}_m^*, s_{i,j,k,m}^*$ are optimal solutions of $\mathbf{u}, \mathbf{x}, s$ found by solving P_τ^m . If the stopping conditions are not satisfied, update $m = m + 1$ and go back to **Step 1**.

The iterative algorithm consists of 4 steps. The key point is that for each iteration we replace the distance between UAV i and j at time k by its linearization at $\hat{\mathbf{x}}_m$,

$$g_{i,j,k}(\mathbf{x}) := g_{i,j,k}(\hat{\mathbf{x}}_m) + \nabla^\top g_{i,j,k}(\hat{\mathbf{x}}_m)(\mathbf{x} - \hat{\mathbf{x}}_m)$$

and consequently obtain the linear approximation of (15) at $\hat{\mathbf{x}}_m$ as follows:

$$d - g_{i,j,k}(\hat{\mathbf{x}}_m) - \nabla^\top g_{i,j,k}(\hat{\mathbf{x}}_m)(\mathbf{x} - \hat{\mathbf{x}}_m) \leq s_{i,j,k} \forall i \neq j, \forall k$$

Consequently, we obtain the convex optimization problem P_τ^m in Step 2. Over iterations, the set of linearized constraints (17) expands to tighten the convexification of the constraint (15) whereas the increasing τ^m due to $\mu > 1$ enforce the slack variables s converge to zero. Together, they try to enforce the feasibility of the obtained solution, i.e., the non-convex collision avoidance (12) is satisfied and the optimal values of P_τ^m converge to the sub-optimal values of P . Thus, we obtain an upper bound of P with a feasible solution \mathbf{x}^* .

Remark: Unlike the MICP formulation, which is NP-hard, in the DC programming approach we sequentially solve a set of convex programs P_τ^m , each can be addressed efficiently by matured convex optimization algorithms such as interior point methods. Thus, the computational performance can be improved significantly. Mathematically, MICP requires approximating the non-convex feasible set (7) beforehand by employing a set of a large number of MILP constraints (11). Many constraints in this set are non-binding at optimum and can be ignored. In contrast, in DC programming, we sequentially add the linearization of the nonconvex constraints at explored points found after each iteration.

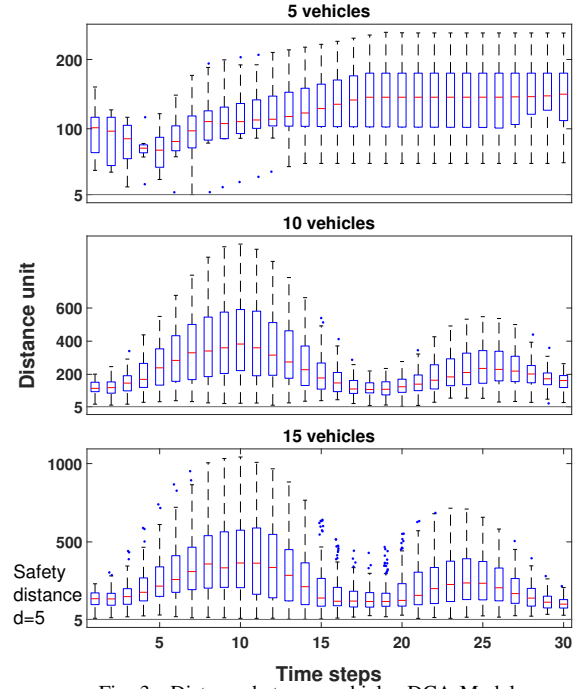


Fig. 3. Distance between vehicles DCA Model

V. NUMERICAL RESULTS

We implemented the DC programming approach on a PC configured with an Intel Xeon and 32GB of RAM. To benchmark the performance of both models, we verify their formulation for 5, 10, and 15 vehicles with the GUROBI solver. Consequently, the number of collision avoidance conditions needed to be satisfied at each time step is 10, 45, and 105. In the three numerical experiments, the minimum safety distance between vehicles is $d = 5$ distance units, and the number of time steps is $\mathcal{T} = 30$ time units. We compare the DC programming results with the ones obtained by using MICP model with the cubic approximation (11) of collision avoidance.

Fig. 3 shows the results of the distance between vehicles at each time step k obtained by solving the UAV planning problem using DC programming approaches. It shows that there is no crash between vehicles throughout the time steps in the DC model in all three experiments. In other words, the DC programming approach guarantees the satisfaction of a large number of nonconvex collision avoidance conditions.

Fig. 4 demonstrates the numerical convergence for penalty DCA used to solve the DC programs in all test cases. The maximum value among all slack variables $s_{i,j,k}^* \geq 0$ converges to zero, which means all collision avoidance constraints are also satisfied at the optimum and the objective value is equal to the original one, i.e., the penalty term $\tau \sum_{i,j,k|i \neq j} s_{i,j,k}^* = 0$. Additionally, the gap between the objective function found between two consecutive iterations δ_m converges to zero, which means we reach the local optimum (sub-optimal solution). In our experiment, the optimal solutions of 5-vehicle, 10-vehicle, and 15-vehicle experiments are

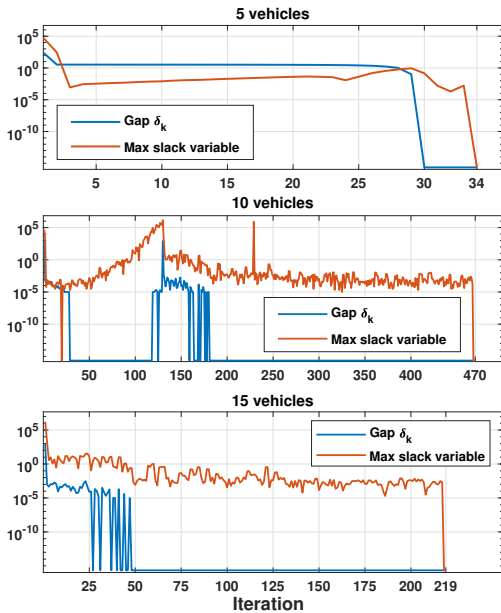


Fig. 4. Convergence analysis of DCA for experiments

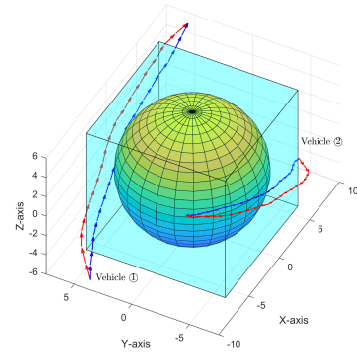
converged at iterations 34, 470, and 219, respectively.

The obtained sub-optimal solution of DC program generally has a very good performance, even surpassing the MICP approach. This is because the DC programming approach employs a less conservative approximation of the nonconvex collision condition, as shown in Fig. 5. In the DCA model, we can utilize the full collision-free space outside the radius sphere d (safety distance). In contrast, the cubic approximation (11) used in the MICP is more conservative. Therefore, the fuel cost of the DC model is lower than that of MICP, as shown in the Table. I. Note also that, while increasing the size of the polyhedral approximation (as shown in Fig. 2) can reduce the conservatives, the MICP easily becomes intractable. Indeed, only the cubic approximation (11) [3] widely used in the literature is computationally feasible in our experiments.

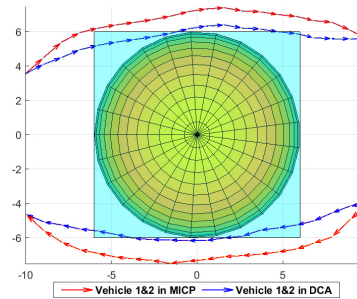
	MICP	DCP	$\Delta(\text{MICP} - \text{DCP})$
5 vehicles	326.18	323.17	3.01
10 vehicles	1594.11	1592.32	1.79
15 vehicles	2855.25	2839.97	15.28

VI. CONCLUSION

This paper presents the DC programming approach to solve the planning problem of a UAV swarm considering the nonconvex collision avoidance requirement. We sequentially approximate this nonconvex constraint by its linearization and adopt the penalty reformulation with slack variables. Thus, the problem is effectively tackled by sequentially solving a set of computationally manageable convex programs. Compared to the traditional mixed integer optimization model with the cubic approximation of collision avoidance constraint, the obtained solution satisfies the safety condition while achieving better cost savings thanks to its less conservative approach.



(a) Three-dimensional view



(b) Top view

Fig. 5. Illustration of the collision avoidance constraint in MICP model and DCA model for trajectory planning of two vehicles

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