2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

On state-space characterisation for output negative imaginary systems with possible poles at the origin and their internal stability result

S. Athoibi Devi, Parijat Bhowmick, *Member, IEEE*, and Alexander Lanzon, *Senior Member, IEEE*

***Abstract*— This paper derives a new and unifying state-space characterisation for the entire class of real, rational, proper Output Negative Imaginary (ONI) systems, allowing poles on the imaginary axis even at the origin. The proposed result captures the existing versions of the NI state-space characterisations, particularly the ones that apply to the NI systems with poles at the origin. A necessary and sufficient LMI condition has been derived to test the strict/non-strict ONI properties of an LTI system with a given minimal state-space realisation. The LMI-based characterisation offers easy and convenient execution due to the easily accessible SDP solver packages. Finally, a necessary and sufficient internal stability theorem is also derived for a positive feedback ONI systems interconnection containing pole(s) at the origin. The proposed stability result specialises to the earlier versions when the earlier assumptions are imposed. Numerical examples are given to show the usefulness of the proposed theoretical results.**

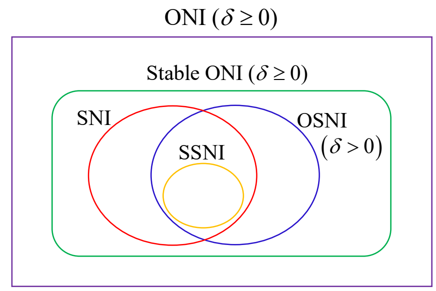

Fig. 1: Relationship among the strict and non-strict subsets within the ONI systems class.

I. Introduction

The theory of Negative Imaginary (NI) systems was first proposed in [1] and it was originally motivated by the vibration control of mechanical systems and flexible structures with high-frequency modes. Such systems with collocated position outputs and force inputs exhibit NI property. Position sensors are sometimes more useful than velocity sensors as they do not aid in injecting sensor noise. Besides the vibration control [1], NI theory finds other significant applications in control of flexible robotic arms [2]–[4], control of Nanopositioning systems [5], [6], cooperative control of various multi-agent systems (e.g. UAVs, UGVs, vehicle & train platoons) [7]–[11], etc. In contrast to Positive-real (or Passive) systems, NI systems allow relative degree up to two and accept RHP zeros [1], [12]. At its inception, NI theory was developed only for asymptotically stable systems [1]. It was later extended in [13] to include poles on the $j\omega$ axis but excluding the origin. The article [4] further extended the NI definition to allow up to two poles at $s = 0$ (e.g. $\frac{1}{s}$, $\frac{s+4}{s(s+2)}$, $\frac{1}{s^2}$, etc.). Such systems possess free-body dynamics (e.g. $\frac{1}{s^2}$) [4] and arise when a rigid body moves freely in space due to Newton's second law of motion. During the last five years, NI theory has been intrigued in the direction of dissipativity and energy-based control approaches [14]–[18].

NI theory became appealing due to its simple internal stability condition [$\lambda_{\max}\big(N(0)M(0)\big) < 1$] that depends on the loop gain only at $\omega = 0$ [1]. In a SISO setting, an NI transfer function's imaginary part remains non-positive for all $\omega \geq 0$. Among the strict subclasses within the NI class, Strictly NI (SNI [1]), Strongly Strict NI (SSNI [19]) and Output Strictly NI (OSNI [12], [14]–[17], [20]) appear quite often in the literature. The internal stability condition for a stable NI-SNI interconnection was first derived in [1]. The result was generalized later to capture the NI systems with poles on the $j\omega$ axis excluding the origin [13]; and finally, to include NI systems with poles (up to two) at the origin [4], [21]. The NI-OSNI stability results have also been enriched consistently [12], [16], [17] since its foundation in [12]. However, none of these articles addressed the internal stability problem of an ONI systems interconnection containing poles at the origin. Although [4] and [21] derived a set of stability criteria for an NI-SNI interconnection containing poles at the origin, it does not apply to an NI-OSNI interconnection because the OSNI and SNI sets are not identical, as illustrated through the Venn diagram in Fig. 1.

This paper deals with the internal stability problem of an ONI-OSNI interconnection allowing poles at the origin. The proposed result (see Theorem 2) also covers all the existing versions of the NI-OSNI stability result as special cases. This paper has also derived a new state-space characterisation (see Lemma 4) for the full class of ONI systems considering poles on the entire $j\omega$ axis, even at the origin. This result can be regarded as a non-trivial extension of the generalised NI lemma in [22] to capture NI systems with poles at the origin.

This work was supported by the Engineering and Physical Sciences Research Council (EPSRC) [grant number EP/R008876/1] and the Science and Engineering Research Board (SERB), DST, India [grant number SRG/2022/000892]. All research data supporting this publication are directly available within this publication. For the purpose of open access, the authors have applied a Creative Commons Attribution (CC BY) licence to any Author Accepted Manuscript version arising.

S. Athoibi Devi and P. Bhowmick are with the Dept. of EEE, IIT Guwahati, Assam–781039, India. A. Lanzon is with the Dept. of EEE, School of Engineering, University of Manchester, Manchester M13 9PL, UK. Emails: d.salam@iitg.ac.in, parijat.bhowmick@iitg.ac.in, Alexander.Lanzon@manchester.ac.uk.

II. Technical background

Section II caters the definitions and properties of the class of ONI[1] systems and its strict subsets. Then it portrays the relationship among the ONI, OSNI, NI and SNI systems.

The class of finite-dimensional, square and causal systems with no poles in $\{s \in \mathbb{C} : \Re[s] > 0\}$ to be studied in this paper is governed by the following state-space equations:

$$M : \begin{cases} \dot{x} = Ax + Bu, \quad x_0 = x(0) = 0; \\ y = Cx + Du. \end{cases} \tag{1}$$

Let M have a minimal state-space realisation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $M(s) \in \mathscr{R}^{m\times m}$ be its transfer function representation. We will first define the ONI and OSNI systems in the frequency domain. A parameter $\delta \geq 0$ is involved with the ONI definition which is an index that classifies the strict (i.e. $\delta > 0$) and non-strict (i.e. $\delta = 0$) subsets under the ONI class. We also set a new notation $\bar{M}(s) = M(s) - M(\infty)$, which is nothing but the strictly proper version of $M(s)$.

Definition 1: (**ONI System**) An LTI system M, as defined in (1), is said to be Output Negative Imaginary (ONI) with a level of strictness $\delta \geq 0$ if

i) $j[M(j\omega) - M(j\omega)^*] - \delta\,\omega \bar{M}(j\omega)^* \bar{M}(j\omega) \geq 0$ for all $\omega \in (0, \infty)$ except the values of ω where $s = j\omega$ is a pole of $M(s)$;

ii) If $s = j\omega_0$ with $\omega_0 \in (0, \infty)$ is a pole of $M(s)$, then it is at most a simple pole and the residue matrix $\lim_{s\to j\omega_0} (s - j\omega_0) jM(s)$ is Hermitian and positive semidefinite;

iii) If $s = 0$ is a pole of $M(s)$, then $\lim_{s\to 0} s^k M(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 M(s)$ is Hermitian and positive semidefinite.

Note that the point-wise frequency-domain condition given in (i) can equivalently be expressed as:

$$j\omega[M(j\omega) - M(j\omega)^*] - \delta\,\omega^2 \bar{M}(j\omega)^* \bar{M}(j\omega) \geq 0 \tag{2}$$

for all $\omega \in \mathbb{R}$ where $s = j\omega$ is not a pole of $M(s)$. It can be readily observed that for $\delta = 0$, the frequency-domain condition given in (i) reduces to $j[M(j\omega) - M(j\omega)^*] \geq 0$ and Definition 1 boils down to the definition of NI systems [4], [21]. Below, we define the OSNI systems, which are defined only for asymptotically stable systems and characterised by $\delta > 0$. This definition resembles Definition 5 of [14].

Definition 2: (**OSNI system**) An LTI system M, as defined in (1), is said to be Output Strictly Negative Imaginary (OSNI) with a level of strictness $\delta > 0$ if M is asymptotically stable [i.e., $M(s) \in \mathscr{RH}_\infty^{m\times m}$] and

$$j[M(j\omega) - M(j\omega)^*] - \delta\,\omega \bar{M}(j\omega)^* \bar{M}(j\omega) \geq 0 \tag{3}$$

for all $\omega \in (0, \infty)$.

After defining the ONI and OSNI systems, we will now recapitulate the definition of SNI systems.

[1]Although the ONI system property is defined in this paper for LTI systems only, it can be readily extended to LTV and nonlinear input-affine systems by exploiting a particular time-domain dissipative supply rate $w(u, \dot{y}) = 2\dot{y}^\top u \quad \delta\,\dot{y}^\top \dot{y}$, as developed in [15]–[18].

Definition 3: (**SNI System**) [1], [21] An LTI system M, as defined in (1), is said to be Strictly Negative Imaginary (SNI) if M is asymptotically stable [i.e., $M(s) \in \mathscr{RH}_\infty^{m\times m}$] and $j[M(j\omega) - M(j\omega)^*] > 0$ for all $\omega \in (0, \infty)$.

It is already revealed and explained in [12], [14], [17] that SNI and OSNI systems are not identical. These two strict subsets under the parent ONI class intersect each other as illustrated in the Venn diagram shown in Fig. 1. Note also that the class of Strongly Strict Negative Imaginary (SSNI) systems defined in [19] belongs to the intersection of the SNI and OSNI subsets.

To this end, we will recall the state-space characterisation for Input-Output Passive systems as it will be required to derive the intended ONI lemma in the next section.

Lemma 1: (**Input-Output Passive lemma**) [12] The system M as defined in (1) is said to be Input-Output Passive (IOP) if and only if there exist a real matrix $P = P^\top > 0$ and two real parameters $\delta \geq 0$ and $\varepsilon \geq 0$ such that

$$\begin{bmatrix} (-PA - A^\top P - \delta C^\top C) & (-PB + C^\top - \delta C^\top D) \\ (-PB + C^\top - \delta C^\top D)^\top & (D + D^\top - \varepsilon I_m - \delta D^\top D) \end{bmatrix} \geq 0. \tag{4}$$

The following lemma is pretty interesting and it reveals the relationship between the ONI and Output Passive systems properties.

Lemma 2: An LTI system M, as defined in (1), is ONI with a level of output strictness $\delta \geq 0$ if and only if it is Output Passive with respect to the same δ.

Proof. The proof readily follows from Lemma 5 of [12]. ■

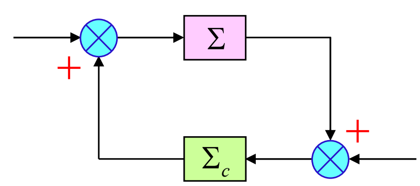

Fig. 2: A positive feedback interconnection of ONI systems.

We will now recall the well-known internal stability result of an NI-SNI interconnection as shown in Fig. 2.

Theorem 1: [21] Let $\Sigma(s)$ be an NI system without poles at the origin and $\Sigma_c(s)$ be an SNI system. Then, the positive feedback interconnection of $\Sigma(s)$ and $\Sigma_c(s)$ shown in Figure 2 is internally stable if and only if

$$\begin{cases} \det[I - \Sigma(\infty)\Sigma_c(\infty)] \neq 0, \\ \lambda_{\max}\left[(I - \Sigma(\infty)\Sigma_c(\infty))^{-1}(\Sigma(\infty)\Sigma_c(0) - I)\right] < 0, \\ \lambda_{\max}\left[(I - \Sigma_c(0)\Sigma(\infty))^{-1}(\Sigma_c(0)\Sigma(0) - I)\right] < 0. \end{cases} \tag{5}$$

III. Main result: ONI lemma considering poles on the imaginary axis including origin

This section derives a new version of the Output Negative Imaginary (ONI) lemma that gives a complete state-space characterisation for the full class of ONI systems (including the strict subsets) allowing poles on the $j\omega$ axis including the origin. This lemma can be seen as an extension of the generalised NI lemma in [22] and an unification of all the earlier NI lemmas reported in [1], [4], [12]–[14], [17], [21] since its inception.

The following two lemmas are essential pre-requisites for proving the main results. Lemma 3 provides a set of necessary and sufficient conditions for a system with/without double poles at the origin to have NI property.

Lemma 3: Let an LTI system Σ be defined via (1) with no poles at $s=0$ and have a transfer function matrix $\Sigma(s)$. Let there exist a $\delta \geq 0$. Then, $M(s) \triangleq \Sigma(s)+R_1/s+R_2/s^2$ is ONI if and only if $N(s) \triangleq \Sigma(s)+R_1/s$ is ONI and $R_2 = R_2^\top \geq 0$, where R_1 and R_2 are the residue and quadratic residue of the pole at $s=0$.

Proof. The proof has been divided into the necessary and sufficiency parts. We declare $\bar{M}(s) = M(s) - M(\infty)$ and $\bar{N}(s) = N(s) - N(\infty)$.

($\Rightarrow$) We begin this proof on noting that the set of poles of $M(s)$ is the union of the set of poles of $N(s)$ and a double pole at $s=0$. Hence, $N(s)$ has no poles in the open RHP as $M(s)$ has no poles there. Now, $M(s)$ being ONI with a $\delta_m \geq 0$ gives $j[M(j\omega)-M(j\omega)^*]-\delta_m\,\omega\bar{M}(j\omega)^*\bar{M}(j\omega) \geq 0$ for all $\omega \in (0,\infty)$ where $s=j\omega$ is not a pole of $M(s)$ via condition (i) of Definition 1. It immediately implies $j[M(j\omega)-M(j\omega)^*] \geq 0$ since $\delta_m \geq 0$. Then we have $j[N(j\omega)-N(j\omega)^*] = j[M(j\omega)-M(j\omega)^*] \geq 0$ for all $\omega \in (0,\infty)$ where $s=j\omega$ is not a pole of either $N(s)$ or $M(s)$ as $R_2 = R_2^\top \geq 0$ via assumption. This inequality condition will eventually imply $j[N(j\omega)-N(j\omega)^*]-\delta_n\,\omega\bar{N}(j\omega)^*\bar{N}(j\omega) \geq 0$ as we can always find a sufficiently small $\delta_n \geq 0$.

It can be readily asserted that $s=j\omega_0$ with $\omega_0 > 0$ is a pole of $M(s) = N(s)+R_2/s^2$ if and only if $s=j\omega_0$ is a pole of $N(s)$. Therefore, the residue matrix of $M(s)$ at $s=j\omega_0$ is $\lim_{s\to j\omega_0}(s-j\omega_0)jM(s) \geq 0$ via condition (ii) of Definition 1. Then, the residue matrix of $N(s)$ at the same complex pole is given by $\lim_{s\to j\omega_0}(s-j\omega_0)jN(s) = \lim_{s\to j\omega_0}(s-j\omega_0)jM(s) \geq 0$. Also, if $s=0$ is a pole of $N(s)$, then $\lim_{s\to 0}s^kN(s)=0$ for all integer $k \geq 2$ due to the construction of $\Sigma(s)$ having no poles at the origin.

Hence, we can conclude that $N(s)$ is ONI with a $\delta_n \geq 0$ via Definition 1 under the assumption $R_2 = R_2^\top \geq 0$.

($\Leftarrow$) $M(s) = N(s)+R_2/s^2$ is ONI via Lemma 3 of [13] because both $N(s)$ and R_2/s^2 with $R_2 = R_2^\top \geq 0$ are ONI.

Hence the proof is done. ■

Now, we will derive an ONI lemma that gives an LMI-based state-space characterisation for the full class of ONI systems allowing poles on the $j\omega$ axis including the origin. This lemma can be regarded as an extension and unification of the NI and OSNI lemmas reported in the earlier literature [1], [4], [12]–[14], [17], [21].

Lemma 4: (**ONI lemma**) Consider an LTI system Σ as defined in (1) with a minimal state-space realisation $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ with $\det(A_1) \neq 0$. Then, $M(s) \triangleq \Sigma(s) + R_1/s+R_2/s^2$ is ONI if and only if $D_1 = D_1^\top$, $R_2 = R_2^\top \geq 0$ and there exist a matrix $Y = Y^\top > 0$ and a δ such that

$$\begin{bmatrix} A_1Y+YA_1^\top+\delta(C_1A_1Y)^\top(C_1A_1Y) & B_1+A_1YC_1^\top \\ (B_1+A_1YC_1^\top)^\top & -(R_1+R_1^\top) \end{bmatrix} \leq 0. \tag{6}$$

Proof. Define $N(s) \triangleq \Sigma(s) + R_1/s$. According to Lemma 3, $N(s)$ is ONI with a $\delta_n \geq 0$ when $M(s)$ is ONI with a $\delta_m \geq 0$. δ_m and δ_n may not be equal. We choose a $\delta = \min\{\delta_m,\ \delta_n\}$. Then, the following statements are equivalent.

$M(s)$ is ONI with $\delta \geq 0$

$\Leftrightarrow$ $N(s)$ is ONI with $\delta \geq 0$ and $R_2 = R_2^\top \geq 0$ via Lemma 3

$\Leftrightarrow$ $[N(s)-D_1]$ is ONI with $\delta \geq 0$, $R_2 = R_2^\top \geq 0$ and $D_1 = D_1^\top$ (via part I of [21, Lemma 7])

$\Leftrightarrow$ $s[N(s)-D_1] = C_1A_1(sI-A_1)^{-1}B_1 + C_1B_1 + R_1$ is Output Passive with a $\delta \geq 0$ via Lemma 1 since $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1A_1 & C_1B_1+R_1 \end{array}\right]$ is minimal due to $\det(A_1) \neq 0$, $R_2 = R_2^\top \geq 0$ and $D_1 = D_1^\top$

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist matrices $P = P^\top > 0$, L, W and a real scalar $\delta \geq 0$ such that
$PA_1 + A_1^\top P + \delta A_1^\top C_1^\top C_1A_1 = -L^\top L$
$PB_1 - A_1^\top C_1^\top + \delta A_1^\top C_1^\top C_1B_1 = -L^\top W$ and
$C_1B_1 + B_1^\top C_1^\top - \delta B_1^\top C_1^\top C_1B_1 + R_1 + R_1^\top = W^\top W$

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist matrices $P = P^\top > 0$, L, W and a real scalar $\delta \geq 0$ such that
$A_1P^{-1} + P^{-1}A_1^\top = -P^{-1}L^\top LP^{-1}$
$B_1 = P^{-1}(A_1^\top C_1^\top - L^\top W)$ and
$C_1B_1 + B_1^\top C_1^\top + R_1 + R_1^\top = W^\top W$

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist matrices $P = P^\top > 0$, L, W and a real scalar $\delta \geq 0$ such that
$A_1P^{-1} + P^{-1}A_1^\top + P^{-1}(\delta A_1^\top C_1^\top C_1A_1)P^{-1} = -P^{-1}L^\top LP^{-1}$
$B_1 = P^{-1}(A_1^\top C_1^\top - \delta A_1^\top C_1^\top C_1B_1 - L^\top W)$ and
$R_1 + R_1^\top = (W + LP^{-1}C_1^\top)^\top(W + LP^{-1}C_1^\top) + (C_1B_1 + C_1A_1P^{-1}C_1^\top)^\top(\delta I)(C_1A_1P^{-1}C_1^\top)$ [via the completion of squares]

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist matrices $P = P^\top > 0$, L, W and a real scalar $\delta \geq 0$ such that
$A_1P^{-1} + P^{-1}A_1^\top + P^{-1}(\delta A_1^\top C_1^\top C_1A_1)P^{-1} = -P^{-1}L^\top LP^{-1}$
$B_1 + A_1P^{-1}C_1^\top = -\delta P^{-1}A_1^\top C_1^\top(C_1B_1 + C_1A_1P^{-1}C_1^\top)$ and
$R_1 + R_1^\top = (W + LP^{-1}C_1^\top)^\top(W + LP^{-1}C_1^\top) + (C_1B_1 + C_1A_1P^{-1}C_1^\top)^\top(\delta I)(C_1B_1 + C_1A_1P^{-1}C_1^\top)$ [via the completion of squares]

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist matrices $P = P^\top > 0$, L, W and a real scalar $\delta \geq 0$ such that
$A_1P^{-1} + P^{-1}A_1^\top + P^{-1}(\delta A_1^\top C_1^\top C_1A_1)P^{-1} = -P^{-1}L^\top LP^{-1}$
$B_1 + A_1P^{-1}C_1^\top = 0$ and
$R_1 + R_1^\top = (W + LP^{-1}C_1^\top)^\top(W + LP^{-1}C_1^\top)$ and $(C_1B_1 + C_1A_1P^{-1}C_1^\top) = 0$

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist matrices $P = P^\top > 0$, L, W and a real scalar $\delta \geq 0$ such that
$A_1P^{-1} + P^{-1}A_1^\top + P^{-1}(\delta A_1^\top C_1^\top C_1A_1)P^{-1} =$

$-P^{-1}L^\top LP^{-1}$

$B_1 + A_1P^{-1}C_1^\top = P^{-1}L^\top W + P^{-1}L^\top LP^{-1}C^\top$ [since $P^{-1}L^\top W = P^{-1}L^\top(-LP^{-1}C^\top)$] and $R_1 + R_1^\top = (W + LP^{-1}C_1^\top)^\top(W + LP^{-1}C_1^\top)$ and $(C_1B_1 + C_1A_1P^{-1}C_1^\top) = 0$

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist matrices $P = P^\top > 0$, L, W and a real scalar $\delta \geq 0$ such that

$$\begin{bmatrix} A_1P^{-1} + P^{-1}A_1^\top + \\ \delta\,(C_1A_1P^{-1})^\top(C_1A_1P^{-1}) & (B_1 + A_1P^{-1}C_1^\top) \\ \\ (B_1 + A_1P^{-1}C_1^\top)^\top & -(R_1 + R_1^\top) \end{bmatrix}$$

$$= -\begin{bmatrix} P^{-1}L^\top \\ W^\top + C_1P^{-1}L^\top \end{bmatrix}\begin{bmatrix} LP^{-1} & W + LP^{-1}C_1^\top \end{bmatrix}$$

$\Leftrightarrow$ $R_2 = R_2^\top \geq 0$, $D_1 = D_1^\top$ and there exist a $Y \triangleq P^{-1} > 0$ and $\delta \geq 0$ such that

$$\begin{bmatrix} A_1Y + YA_1^\top + \\ \delta\,(C_1A_1Y)^\top(C_1A_1Y) & (B_1 + A_1YC_1^\top) \\ \\ (B_1 + A_1YC_1^\top)^\top & -(R_1 + R_1^\top) \end{bmatrix} \leq 0.$$

This completes the proof. ■

The following example illustrates the above lemma.

Example 1: Consider the transfer function $M(s) = (s^2 + s + 1)/(s^3 + s)$ decomposed as $\Sigma(s) + R_1/s + R_2/s^2$. Here, $R_1 = 1$, $R_2 = 0$, $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $D_1 = 0$. It is easy to verify that the realisation is minimal and $\det(A_1) \neq 0$. Then, via Lemma 4, we can see that $M(s)$ is ONI with a $\delta \geq 0$ because $D_1 = D_1^\top = 0$, $R_2 = R_2^\top = 0 \geq 0$ and there exists a $Y = Y^\top > 0$ that satisfies the LMI given in (6). This can also be confirmed from Definition 1 that, since $M(s)$ has no poles in the open right-half plane, $j[M(j\omega) - M(j\omega)^*] - \delta\,\omega\bar{M}(j\omega)^*\bar{M}(j\omega) \geq 0\ \forall\omega \in (0,\infty)\backslash\{1\}$, the residue matrix $K_0 = \lim_{s\to j1} j(s - j1)M(s) = \frac{1}{2} = K_0^\top > 0$ and $\lim_{s\to 0} s^kM(s) = 0$ for all integer $k \geq 2$.

If Σ is a static system, Lemma 4 cannot be applied because $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ is not minimal. To overcome this minor limitation, we propose the following statement to directly check the ONI property of such systems.

Lemma 5: The transfer function matrix $D + R_1/s + R_2/s^2$ is ONI if and only if $D = D^\top$, $R_2 = R_2^\top \geq 0$ and $R_1 + R_1^\top \geq 0$.

Proof. From Definition 1, conditions (ii) is trivially fulfilled. Condition (iii) therein is equivalent to $R_2 = R_2^\top \geq 0$ and that condition (i) is hence equivalent to $j[D - D^\top] + \frac{1}{\omega}[R_1 + R_1^\top] \geq 0\ \forall\omega \in (0,\infty)$. This in turn is equivalent to $D = D^\top$ and $R_1 + R_1^\top \geq 0$ via a limiting argument as $\omega \to \infty$. ■

The following example is used to illustrate the result given in Lemma 5.

Example 2: Consider the transfer function $M(s) = (s + 1)/s^2$. In this case, $D = 0$, $R_1 = 1$ and $R_2 = 1$. Then via Lemma 5, $M(s)$ is ONI with $\delta = 0$ [or simply NI] since $D = D^\top = 0$, $R_2 = R_2^\top = 1 \geq 0$ and $R_1 + R_1^\top = 2 \geq 0$. This can also be confirmed from Definition 1 as $M(s)$ has no poles in the open right-half plane, nor on the $j\omega$ axis for any $\omega \in (0,\infty)$, $j[M(j\omega) - M(j\omega)^*] - \delta\,\omega\bar{M}(j\omega)^*\bar{M}(j\omega) = \frac{2}{\omega} \geq 0\ \forall\omega \in (0,\infty)$, $\lim_{s\to 0} s^kM(s) = 0\ \forall k \geq 3$ and $\lim_{s\to 0} s^2M(s) = 1 \geq 0$.

Under the suppositions $M(s) \in \mathscr{RH}_\infty^{m\times m}$ and $[M(s) - M^\sim(s)]$ has full normal rank, Lemma 4 reduces to the OSNI lemma [given below as Corollary 1] and re-establishes [12, Lemma 5] and [17, Lemma 16].

Corollary 1: [12], [17] Consider an LTI system M as defined in (1) with a minimal state-space realisation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with A Hurwitz, $D = D^\top$ and $[M(s) - M^\sim(s)]$ has full normal rank. Then, M is OSNI with a level of strictness $\delta > 0$ if and only if there exists a real matrix $Y = Y^\top > 0$ such that

$$AY + YA^\top + \delta\,(CAY)^\top(CAY) \leq 0 \text{ and } B + AYC^\top = 0. \quad (7)$$

Proof. The proof is a straightforward specialisation of the proof of Lemma 4 under the suppositions mentioned in the statement above. ■

It is easy to check that Lemma 4 specialises to [13, Lemma 7], given in Corollary 2, when $M(s)$ is an NI system without poles at the origin.

Corollary 2: [13] Consider an LTI system M as defined in (1) with a minimal state-space realisation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $\det(A) \neq 0$ and $D = D^\top$. Then, M is an NI system without poles at the origin if and only if there exists a matrix $Y = Y^\top > 0$ such that $AY + YA^\top \leq 0$ and $B = -AYC^\top$.

Proof. The proof is a straightforward specialisation of the proof of Lemma 4 under the setting $\delta = 0$, $R_1 = 0$ and $R_2 = 0$. ■

Furthermore, when restricted to asymptotically stable NI systems, Lemma 4 resembles [1, Lemma 5].

Remark 1: The article [22] derived an NI lemma depending on a minimal state-space realisation of the system $M(s)$ without decomposing it into $M(s) \triangleq \Sigma(s) + R_1/s + R_2/s^2$ whereas Lemma 4 decomposes it into three parts as already explained. This decomposition yields possibly smaller dimensions of the matrices A_1, B_1 and C_1. Smaller dimensions can be beneficial for the computational aspects, especially for higher order and complex systems (e.g. flexible structures, distributed-parameter systems). Most importantly, Lemma 4 imposes only the relevant structural properties on the constituent parts of $M(s)$ so that the lemma can capture the earlier results when the earlier assumptions are imposed.

Remark 2: An ONI system $M(s) \triangleq \Sigma(s) + R_1/s + R_2/s^2$ does not necessarily imply that $\Sigma(s)$ is ONI without poles at the origin via Lemma 4. This is because any shortage of ONIness in $\Sigma(s)$ can be compensated by an excess ONIness of R_1/s. For example, $M(s) = -\frac{1}{s+1} + \frac{2}{s}$ is an ONI (or simply an NI) transfer function via Definition 1, but $\Sigma(s) = -\frac{1}{s+1}$ is not ONI/NI. This crucial observation is also revealed by (6) in Lemma 4, which contains the term $(R_1 + R_1^\top) \geq$

0 in its (2,2) blocks. Whereas, the generalised NI lemma LMI derived in [22] to capture the full class of NI systems allowing poles at the origin had 0 in its (2,2) block.

IV. Internal stability of ONI systems

This section will develop a internal stability result for a positive feedback interconnection (shown in Fig. 2) of an ONI (or simply NI) system with possible poles on the $j\omega$ axis including the origin and an OSNI system. Before presenting the main stability theorem, which is a major contribution of this paper, we provide two essential pre-requisite technical lemmas. These results build on a set of similar results derived in [21] in the context of NI-SNI interconnections containing poles at the origin.

Lemma 6 given below is the backbone for proving the internal stability of an ONI interconnection containing pole(s) at the origin. It solely relies on Lemma 18 of [21] which first exploited the classical 'loop transformation' technique to transform an NI system with pole(s) at the origin into an NI system without any pole at the origin. The lemma shows that the internal stability of two ONI systems is equivalent to the internal stability of the *loop-transformed* ONI systems.

Lemma 6: Suppose $\Sigma(s)$ is an ONI system with possible poles on the $j\omega$ axis including the origin and $\Sigma_c(s)$ is an OSNI system. Construct a real matrix $\Upsilon < 0$ such that $\det[I-\Sigma(\infty)\Upsilon] \neq 0$. Define $\hat{\Sigma}(s) \triangleq [I-\Sigma(s)\Upsilon]^{-1}\Sigma(s)$ and $\hat{\Sigma}_c \triangleq \Sigma_c(s)-\Upsilon$. Then,

i) $\det[I-\Sigma(\infty)\Sigma_c(\infty)] \neq 0$ if and only if $\det[I-\hat{\Sigma}(\infty)\hat{\Sigma}_c(\infty)] \neq 0$;
ii) The positive feedback interconnection of $\Sigma(s)$ and $\Sigma_c(s)$ is internally stable if and only if the positive feedback interconnection of $\hat{\Sigma}(s)$ and $\hat{\Sigma}_c(s)$ is internally stable.

Proof. The proof builds on Lemma 18 of [21]. We can readily show that $[I-\hat{\Sigma}(\infty)\hat{\Sigma}_c(\infty)] = I-[I-\Sigma(\infty)\Upsilon]^{-1}\Sigma(\infty)[\Sigma_c(\infty)-\Upsilon] = [I-\Sigma(\infty)\Upsilon]^{-1}[I-\Sigma(\infty)\Sigma_c(\infty)]$. Therefore, $\det[I-\Sigma(\infty)\Sigma_c(\infty)] \neq 0 \Leftrightarrow \det[I-\hat{\Sigma}(\infty)\hat{\Sigma}_c(\infty)] \neq 0$ since $\det[I-\Sigma(\infty)\Upsilon] \neq 0$ via construction. This proves part (i).

Part (ii) can also be readily proved along the lines of [21, Lemma 18] due to the relationship $\hat{\Sigma}(s)\big[I-\hat{\Sigma}_c(s)\hat{\Sigma}(s)\big]^{-1} = \Sigma(s)\big[I-\Upsilon\Sigma(s)\big]^{-1}\big[I-\big(\Sigma_c(s)-\Upsilon\big)\Sigma(s)\big[I-\Upsilon\Sigma(s)\big]^{-1}\big]^{-1} = \Sigma(s)\big[I-\Sigma_c(s)\Sigma(s)\big]^{-1}$. ■

This lemma shows that a static loop transformation technique truly plays the trick to transform an ONI system with pole(s) at the origin to an ONI system without any pole at the origin and it builds on Lemma 20 of [21].

Lemma 7: Suppose $\Sigma(s)$ is an ONI system with possible poles on the $j\omega$ axis including the origin and $\Sigma_c(s)$ is an OSNI system. Construct a real matrix $\Upsilon < 0$ such that $\det[I-\Sigma(\infty)\Upsilon] \neq 0$. Then,

i) $[I-\Sigma(s)\Upsilon]^{-1}$ does not have any poles in the open RHP and at the origin;
ii) $\hat{\Sigma}(s) \triangleq [I-\Sigma(s)\Upsilon]^{-1}\Sigma(s)$ is an ONI system without any pole at the origin.

Proof. This proof readily follows from Lemma 20 of [21]. ■

We will now state the main stability theorem for an interconnection of ONI and OSNI systems, $\Sigma(s)$ and $\Sigma_c(s)$ respectively, where $\Sigma(s)$ is allowed to contain pole(s) at the origin. Note $\Sigma_c(s) \in \mathscr{RH}_\infty^{m\times m}$. As OSNI systems (say $\Sigma_c(s)$) may contain transmission/blocking zeros [$s=\pm j\omega_z$, $\omega_z \in (0,\infty)$] on the $j\omega$ axis, $\det\big[\Sigma_c(j\omega_z)-\Sigma_c(j\omega_z)^*\big]=0$ may occur unlike the case of an SNI system. Therefore the internal stability results of an NI-SNI interconnection containing pole(s) at the origin proposed in [4] and [21] cannot capture Theorem 2. Note also that the existing NI-OSNI stability theorems (given in [12], [14], [16], [17]) have not accounted for poles at the origin.

Theorem 2: Suppose $\Sigma(s)$ is an ONI system with possible poles on the $j\omega$ axis including the origin and $\Sigma_c(s)$ is an OSNI system. Construct a real matrix $\Upsilon < 0$ such that $[I-\Sigma(\infty)\Upsilon] > 0$. Define $\hat{\Sigma}(s) \triangleq [I-\Sigma(s)\Upsilon]^{-1}\Sigma(s)$ and $\hat{\Sigma}_c \triangleq \Sigma_c(s)-\Upsilon$. Let $\Omega \triangleq \{\omega \in (0,\infty) : s=j\omega \text{ is not a pole of } \hat{\Sigma}(s)\}$ and let $j\left[\Sigma_c(j\omega_0)-\Sigma_c(j\omega_0)^*\right] > 0 \ \forall \omega_0 \in (0,\infty)\backslash\Omega$. Suppose there exists no $\omega \in \Omega$ such that $\det[\hat{\Sigma}(j\omega)-\hat{\Sigma}(j\omega)^*]=0$ and $\det[\Sigma_c(j\omega)-\Sigma_c(j\omega)^*]=0$. Then, the positive feedback interconnection of $\Sigma(s)$ and $\Sigma_c(s)$ shown in Fig. 2 is internally stable if and only if

$$\det[I-\Sigma(\infty)\Sigma_c(\infty)] \neq 0, \tag{8a}$$

$$\lambda_{\max}\big[[I-\Sigma(\infty)\Sigma_c(\infty)]^{-1}(\Sigma(\infty)\Sigma_c(0)-I)\big] < 0, \tag{8b}$$

$$\lambda_{\max}\Big[\lim_{s\to 0}\big[(I-\Upsilon\Sigma(\infty))\big[I-\Sigma_c(s)\Sigma(\infty)\big]^{-1}\times \big[\Sigma_c(s)\Sigma(s)-I\big]\big[I-\Upsilon\Sigma(s)\big]^{-1}\big]\Big] < 0. \tag{8c}$$

Proof. We begin the proof on noting that $\Sigma_c(s)$ and $\hat{\Sigma}_c(s)$ have the same set of poles and $j[\hat{\Sigma}(j\omega)-\hat{\Sigma}(j\omega)^*] = j[\Sigma(j\omega)-\Sigma(j\omega)^*]$ as $\Upsilon=\Upsilon^\top$. Also, since the choice of the matrix $\Upsilon<0$ lies with us, the first two assumptions are not overly restrictive. This proof primarily relies on the trick that internal stability of $\Sigma(s)$ and $\Sigma_c(s)$ is equivalent to that of $\hat{\Sigma}(s)$ and $\hat{\Sigma}_c(s)$ for an appropriate choice of $\Upsilon<0$, as established in Lemma 6. Now,

The positive feedback interconnection of $\hat{\Sigma}(s)$ and $\hat{\Sigma}_c(s)$ is internally stable

$\Leftrightarrow$ $\det[I-\hat{\Sigma}(\infty)\hat{\Sigma}_c(\infty)] \neq 0$ [via part (i) of Lemma 6], $\lambda_{\max}\big[\big[I-\hat{\Sigma}(\infty)\hat{\Sigma}_c(\infty)\big]^{-1}\big(\hat{\Sigma}(\infty)\hat{\Sigma}_c(0)-I\big)\big] < 0$ and $\lambda_{\max}\big[\big[I-\hat{\Sigma}_c(0)\hat{\Sigma}(\infty)\big]^{-1}\big(\hat{\Sigma}_c(0)\hat{\Sigma}(0)-I\big)\big] < 0$ [since $\hat{\Sigma}(s)$ is an ONI system without poles at the origin via Lemma 7, $\hat{\Sigma}_c(s)$ is an OSNI system and then direct application of Theorem 5 of [17]]

$\Leftrightarrow$ $\det[I-\hat{\Sigma}(\infty)\hat{\Sigma}_c(\infty)] \neq 0$, (8b) and (8c). [The equivalence can be readily establish by following the lines of algebraic manipulation shown in Theorem 24 of [21] upon expanding the terms $\hat{\Sigma}_c(0)$, $\hat{\Sigma}_c(\infty)$ and $\hat{\Sigma}(\infty)$].

This completes the proof. ■

The first corollary is an immediate consequence of Theorem 2 which offers a significantly simpler checking condition, compared to (8a)–(8c), if $\Sigma(s)$ is strictly proper.

Corollary 3: Suppose $\Sigma(s)$ is a strictly proper ONI system with possible poles on the $j\omega$ axis including the origin and $\Sigma_c(s)$ is an OSNI system. Construct a real matrix $\Upsilon < 0$ such that $[I - \Sigma(\infty)\Upsilon] > 0$. Under the assumptions of Theorem 2, the positive feedback interconnection of $\Sigma(s)$ and $\Sigma_c(s)$ shown in Fig. 2 is internally stable if and only if

$$\lambda_{\max}\big[\lim_{s\to 0}\big[\big[\Sigma_c\Sigma(s)(s) - I\big]\big[I - \Upsilon\Sigma(s)\big]^{-1}\big]\big] < 0. \quad (9)$$

Proof. The proof is a direct specialisation of the proof of Theorem 2 subject to the extra assumption $\Sigma(\infty) = 0$. ■

This corollary gives an elegant stability criterion when the ONI system $\Sigma(s)$ does necessarily contain pole(s) at the origin in all directions (SISO examples: $\frac{1}{s}$, $\frac{1}{s^2}$). In such cases, the final expression is free from the Υ matrix and it depends only on the DC gain of the OSNI system in the interconnection. This result may find potential applications in developing a cooperative control scheme for multi-agent systems having single/double integrator dynamics.

Corollary 4: Suppose $\Sigma(s)$ is a strictly proper ONI system with poles (either single or double) at the origin and $\Sigma_c(s)$ is an OSNI system. Construct a real matrix $\Upsilon < 0$ such that $[I - \Sigma(\infty)\Upsilon] > 0$. Let all the assumptions of Theorem 2 be true and in addition, one of the following conditions holds:

i) $\det[\lim_{s\to 0} s^2\Sigma(s)] \neq 0$;
ii) $\lim_{s\to 0} s^2\Sigma(s) = 0$ and $\det[\lim_{s\to 0} s\Sigma(s)] \neq 0$.

Then, the positive feedback interconnection of $\Sigma(s)$ and $\Sigma_c(s)$ shown in Fig. 2 is internally stable if and only if $\Sigma_c(0) < 0$.

Proof. The proof requires a Laurent Series expansion of the transfer function $\Sigma(s)$ following the condition (i) or (ii) and then obtaining the limit via simplification. ■

V. Conclusion

This paper has brought in a new and unified state-space characterisation for the full class of LTI Output Negative Imaginary (ONI) systems that accepts poles on the $j\omega$ axis even at the origin. This result captures the existing state-space characterisations for NI, ONI and OSNI systems. It also extends the generalised NI lemma, in [22], for handling NI systems with poles at the origin. This approach decomposes a transfer function into three parts to separate out the residue and quadratic residue at the pole(s) at origin. The paper also derives an internal stability result for a positive feedback interconnection of two ONI systems where one of the systems may contain up to two poles at the origin. This result is possibly the first one that deals with the internal stability of ONI systems allowing poles at the origin and it covers the existing NI-OSNI stability results as special cases.

References

[1] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1042–1046, May 2008.

[2] E. Pereira, S. S. Aphale, V. Feliu, and S. O. Reza Moheimani, "Integral resonant control for vibration damping and precise tip-positioning of a single-link flexible manipulator," *IEEE/ASME Transactions on Mechatronics*, vol. 16, no. 2, pp. 232–240, April 2011.

[3] S. B. Choi, S. S. Cho, H. C. Shin, and H. K. Kim, "Quantitative feedback theory control of a single-link flexible manipulator featuring piezoelectric actuator and sensor," *Smart Materials and Structures*, vol. 8, no. 3, pp. 338–349, June 1999.

[4] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2692–2707, Oct 2014.

[5] N. Nikooienejad and S. Moheimani, "Convex synthesis of SNI controllers based on frequency-domain data: MEMS nanopositioner example," *IEEE Transactions on Control Systems Technology*, vol. 30, no. 2, pp. 767–778, March 2022.

[6] S. K. Das, H. R. Pota, and I. R. Petersen, "Resonant controller design for a piezoelectric tube scanner: A 'mixed' negative-imaginary and small-gain approach," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 5, pp. 1899–1906, Jan 2014.

[7] J. Wang, A. Lanzon, and I. R. Petersen, "Robust output feedback consensus for networked negative-imaginary systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2547–2552, Sep 2015.

[8] J. Hu, B. Lennox, and F. Arvin, "Robust formation control for networked robotic systems using negative imaginary dynamics," *Automatica*, vol. 140, no. 110235, pp. 1–9, June 2022.

[9] C. Li, J. Wang, J. Shan, A. Lanzon, and I. R. Petersen, "Robust cooperative control of networked train platoons: A negative-imaginary systems' perspective," *IEEE Transactions on Control of Network Systems*, vol. 8, no. 4, pp. 1743–1753, Dec 2021.

[10] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Output feedback consensus for networked heterogeneous nonlinear negative-imaginary systems with free-body motion," *IEEE Transactions on Automatic Control*, vol. 68, no. 9, pp. 5536–5543, Sep 2023.

[11] P. Bhowmick, A. Ganguly, and S. Sen, "A new consensus-based formation tracking scheme for a class of robotic systems using negative imaginary property," *IFAC-PapersOnLine*, vol. 55, no. 1, pp. 685–690, 2022.

[12] P. Bhowmick and S. Patra, "On LTI output strictly negative-imaginary systems," *Systems & Control Letters*, vol. 100, pp. 32–42, Feb 2017.

[13] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2342–2347, Oct 2010.

[14] P. Bhowmick and A. Lanzon, "Output strictly negative imaginary systems and its connections to dissipativity theory," in *Proceedings of 58th IEEE Conference on Decision and Control*. Nice, France: IEEE, Dec 2019, pp. 6754–6759.

[15] ——, "Time-domain output negative imaginary systems and its connection to dynamic dissipativity," in *Proceedings of 59th IEEE Conference on Decision and Control*. Jeju Islands, South Korea: IEEE, Dec 2020, pp. 5167–5172.

[16] ——, "Dynamic dissipative characterisation of time-domain input-output negative imaginary systems," *accepted in Automatica*, pp. 1–18, Oct 2023.

[17] A. Lanzon and P. Bhowmick, "Characterisation of input-output negative imaginary systems in a dissipative framework," *IEEE Transactions on Automatic Control*, vol. 68, pp. 959–974, Feb 2023.

[18] P. Bhowmick, N. Bordoloi, and A. Lanzon, "Frequency-domain dissipativity analysis for output negative imaginary systems allowing imaginary-axis poles," in *Proceedings of the 2023 European Control Conference*, June 2023, pp. 1–6.

[19] A. Lanzon, Z. Song, S. Patra, and I. R. Petersen, "A strongly strict negative-imaginary lemma for non-minimal linear systems," *Communications in Information and Systems*, vol. 11, no. 2, pp. 139–152, 2011.

[20] P. Bhowmick and S. Patra, "On input-output negative-imaginary systems and an output strict negative-imaginary lemma," in *Proceedings of the 2nd IEEE Indian Control Conference*. Hyderabad, India: IEEE, Jan 2016, pp. 176–181.

[21] A. Lanzon and H.-J. Chen, "Feedback stability of negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5620–5633, Nov 2017.

[22] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "A generalized negative imaginary lemma and Riccati-based static state-feedback negative imaginary synthesis," *Systems & Control Letters*, vol. 77, pp. 63–68, 2015.