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Large-population optimal control with mixed agents: the multi-scale analysis and decentralized control

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***Abstract*— We consider a large-population optimal control problem involving a major agent and a large number of minor agents. By starting with a centralized optimal control problem, we employ a re-scaling method to derive decentralized control laws. This re-scaling method is further used to obtain a tight upper bound of $O(1/N)$ for the performance loss resulting from decentralized control. This improves upon known results of $O(1/\sqrt{N})$ in the literature for similar models.**

I. INTRODUCTION

Mean field game (MFG) theory provides a powerful methodology to over the curse of dimensionality in large population noncooperative decision problems [16], [14], [25], [8]. A significant extension the theory is to introduce one or a few major players interacting with a large number of minor players [13]. These MFG models of major and minor players (i.e. mixed players/agents) have attracted considerable interest and led to extensive generalizations [7], [10], [28], [4], [27]. The major player serves as a common source of randomness for all players, which has connections with mean field games with common noise [5], [9].

On the other hand, cooperation in dynamic multi-agent decision problems is traditionally a well studied subject. For general cooperative differential games, see [32]. Naturally, cooperative decision-making in mean field models is of interest, especially from the point of view of addressing complexity [15]. Such decision problems may be referred to as mean field teams. The work [15] introduced an linear-quadratic (LQ) social optimization problem where all the agents cooperatively minimize a social cost as the sum of their individual costs, and it shows that the consistency based approach [14] in mean field games may be extended to this model by combining with a person-by-person optimality principle in team decision theory [22]. The central result is the so-called social optimality theorem which states that the optimality loss of the obtained decentralized strategies becomes negligible when the population size goes to infinity [15]. A mean field team is studied in [31] where a Markov jump parameter appears as a common source of randomness for all agents. Optimal control of McKean–Vlasov dynamics is analyzed in [24] and under some conditions it is shown that the optimal solution may be interpreted as the limit of the social optimum solution of N-players as $N \to \infty$. Cooperative mean field control has applications in economic theory [29] and power grids [11]. Furthermore, social optima are useful for studying efficiency of mean field games by providing a performance benchmark [3].

For mean field teams with mixed players, the analysis in an LQ framework has been formulated in our earlier work [17], where partial analysis was presented by applying a state space augmentation technique to characterize the dynamics of the random mean field evolution. Later, [18], [19] re-examined the problem by applying the person-by-person optimality principle adopted in [15], and showed an optimality loss of $O(1/\sqrt{N})$ under decentralized control. In a mixed player setting, [7] considers a nonlinear diffusion model and assumes that all minor players act as a team to minimize a common cost against the major player. More recently, optimal control of large-populations of mixed players is analyzed to treat LQ non-Gaussian models [2] and design deep learning algorithms [1].

In this paper we apply a re-scaling technique to the high dimensional optimal control problem with mixed agents. This technique was initially developed for LQ mean field games [21], and has been applied to social optimization with indefinite cost weights but no major player [20]. Our model involves random coefficients, which leads to in-depth analysis of high dimensional backward stochastic differential equations (BSDEs). The optimal control nature of our problem shares some similarity with mean field type optimal control [12], [33]. However, the later involves only a single decision-maker which directly controls the state mean.

Throughout this paper, we use $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ to denote an underlying filtered probability space. Let S^n be the Euclidean space of $n\times n$ real and symmetric matrices, S^n_+ its subset of positive semi-definite matrices, and I_k the $k\times k$ identity matrix. The Banach space $L^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$ consists of all $\mathbb{R}^k$-valued $\mathcal{F}_t$-adapted square integrable processes $\{v(t), 0\leq t\leq T\}$ with norm $\|v\|_{L^2_{\mathcal{F}}} = (\mathbb{E}\int_0^T |v(t)|^2 dt)^{1/2}$. The Banach space $L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^k)$ consists of all $\mathbb{R}^k$-valued $\mathcal{F}_t$-adapted essentially bounded processes $\{v(t), 0\leq t\leq T\}$ with norm $\|v\|_{L^\infty_{\mathcal{F}}} = \operatorname{ess\,sup}_{t,\omega} |v(t)|$. We can similarly define such spaces with other choices of the filtration and the Euclidean space. Given a symmetric matrix $M\geq 0$, the quadratic form $z^T M z$ may be denoted as $|z|^2_M$. Let $\{\mathcal{F}^W_t, t\geq 0\}$ be the filtration by a Brownian motion $\{W(t), t\geq 0\}$.

The paper is organized as follows. Section II formulates the large population optimal control problem (i.e. mean field social optimization) with a major player. Section III develops the multi-scale analysis to derive mean field limit of the solution and resulting decentralized control laws. Section IV derives some prior bounds for a high dimensional BSDE.

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Section V obtains bounds of the optimality loss of the decentralized control. Section VI concludes the paper.

II. The Mean Field Social Optimization Model

Consider the LQ mean field decision model with a major player $\mathcal{A}_0$ and minor players $\{\mathcal{A}_i, 1 \le i \le N\}$. At time $t \ge 0$, the states of $\mathcal{A}_0$ and $\mathcal{A}_i$ are, respectively, denoted by $X_0^N(t)$ and $X_i^N(t)$, $1 \le i \le N$. The state processes satisfy the linear stochastic differential equations (SDEs):

$$\begin{aligned} dX_0^N(t) =& \big[A_0(t)X_0^N(t) + B_0(t)u_0^N(t) + F_0(t)X^{(N)}(t)\big]dt \\ &+ D_0(t)dW_0(t), \qquad (1) \\ dX_i^N(t) =& \big[A(t)X_i^N(t) + B(t)u_i^N(t) + F(t)X^{(N)}(t) \\ &+ G(t)X_0^N(t)\big]dt + D(t)dW_i(t) \\ &+ D_1(t)dW_0(t), \quad 1 \le i \le N, \qquad (2) \end{aligned}$$

where $X^{(N)}(t) = (1/N)\sum_{i=1}^N X_i^N(t)$ is the coupling term and W_0 is the common noise. The states X_0^N, X_i^N and controls u_0^N, u_i^N are, respectively, n and n_1 dimensional vectors. The initial states $X_j^N(0)$, $0 \le j \le N$, are independent with finite second moment, and also independent of the Brownian motions. The coefficients in the dynamics are random. The noise processes W_0, W_i are, respectively, n_2 and n_3 dimensional independent standard Brownian motions adapted to $\mathcal{F}_t$. We choose $\mathcal{F}_t$ as the σ-algebra $\mathcal{F}_t^{W,X} := \sigma(X_j^N(0), W_j(\tau), 0 \le j \le N, \tau \le t)$. Denote $\mathcal{F}_t^{W_0} := \sigma(W_0(\tau), \tau \le t)$.

For $0 \le j \le N$, denote $u_{-j}^N = \big(u_0^N, \dots, u_{j-1}^N, u_{j+1}^N, \dots, u_N^N\big)$. The cost for $\mathcal{A}_0$ is given by

$$\begin{aligned} J_0(u_0^N, u_{-0}^N) = \mathbb{E}\int_0^T &\Big\{\big|X_0^N(t) - H_0(t)X^{(N)}(t)\big|^2_{Q_0(t)} \\ &+ (u_0^N(t))^T R_0(t)u_0^N(t)\Big\}dt \\ &+ \mathbb{E}|X_0^N(T) - H_{0,f}X^{(N)}(T)|^2_{Q_{0f}}, \qquad (3) \end{aligned}$$

where $\Psi_0(X^{(N)}(t)) = H_0(t)X^{(N)}(t)$. The cost for $\mathcal{A}_i$, $1 \le i \le N$, is given by

$$\begin{aligned} &J_i(u_i^N, u_{-i}^N) \\ &= \mathbb{E}\int_0^T \Big\{\big|X_i^N(t) - H_1(t)X_0^N(t) - H_2(t)X^{(N)}(t)\big|^2_{Q(t)} \\ &\qquad + (u_i^N(t))^T R(t)u_i^N(t)\Big\}dt \\ &\quad + \mathbb{E}|X_i^N(T) - H_{1f}X_0^N(T) - H_{2f}X^{(N)}(T)|^2_{Q_f}, \qquad (4) \end{aligned}$$

The terms $H_1(t)X_0^N(t)$ and $H_{1f}X_0^N(T)$ indicate the strong influence of the major agent. Also, the parameters in the two costs are random.

We introduce the standing assumptions for this paper.

(A1) We have

$$\begin{aligned} &A_0, F_0, A, F, G, H_0, H_1, H_2 \in L^\infty_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^{n\times n}), \\ &B_0, B \in L^\infty_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^{n\times n_1}), \\ &D_0, D, D_1 \in L^2_{\mathcal{F}^{W_0}}(0,T;\mathbb{R}^{n\times n_2}), \\ &Q_0, Q \in L^\infty_{\mathcal{F}^{W_0}}(0,T;S^n), \quad Q_0(t) \in S^n_+,\ Q(t) \in S^n_+, \\ &R_0, R \in L^\infty_{\mathcal{F}^{W_0}}(0,T;S^{n_1}),\ R_0(t) \ge c_1 I_{n_1},\ R(t) \ge c_1 I_{n_1}, \end{aligned}$$

where $t \in [0,T]$ and $c_1 > 0$ is a fixed deterministic constant.

(A2) The terminal cost parameters $H_{0f}, Q_{0f}, H_{1f}, H_{2f}, Q_f$, are $\mathcal{F}_T^{W_0}$-measurable and essentially bounded, and Q_{0f}, Q_f are S^n_+-valued.

The stochastic control literature [6], [23], [30] has considered a similar randomness structure where the system coefficients depend on a smaller filtration.

For a stochastic process $\{Z(t), 0 \le t \le T\}$, we will often write Z for $Z(t)$ by suppressing the time variable t.

A. The mean field social optimization problem

For the mean field social optimization problem, we attempt to minimize the following social cost

$$J_{\rm soc}^{(N)}(u) = J_0 + \frac{\lambda}{N}\sum_{k=1}^N J_k, \qquad (5)$$

where $u^N = (u_0^N, u_1^N, \dots, u_N^N)$ and $\lambda > 0$. It is necessary to introduce the scaling factor λ/N in order to obtain a well defined limiting problem when N tends to infinity.

III. The Multi-scale Approach

For notational simplicity, we take $n_2 = 1$ so that W_0 is a scalar. The general case does not cause essential difficulty.

A. The high dimensional vector model

Denote

$$\boldsymbol{X}_t = \begin{bmatrix} X_0^N \\ X_1^N \\ \vdots \\ X_N^N \end{bmatrix}, \boldsymbol{u}_t = \begin{bmatrix} u_0^N \\ u_1^N \\ \vdots \\ u_N^N \end{bmatrix}, \ \boldsymbol{W}_t = \begin{bmatrix} W_1 \\ \vdots \\ W_N \end{bmatrix},$$

We write the system dynamics in a compact form

$$d\boldsymbol{X}_t = (\boldsymbol{A}\boldsymbol{X}_t + \boldsymbol{B}\boldsymbol{u}_t)dt + \boldsymbol{D}d\boldsymbol{W}_t + \boldsymbol{D}_0 dW_0(t), \qquad (6)$$

where

$$\boldsymbol{A}(t) = \begin{bmatrix} A_0 & \frac{F_0}{N} & \frac{F_0}{N} & \cdots & \frac{F_0}{N} \\ G & A + \frac{F}{N} & \frac{F}{N} & \cdots & \frac{F}{N} \\ G & \frac{F}{N} & A + \frac{F}{N} & \cdots & \frac{F}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G & \frac{F}{N} & \frac{F}{N} & \cdots & A + \frac{F}{N} \end{bmatrix},$$

and it is straightforward to determine $\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{D}_0$. For the representation of the cost, we denote

$$\boldsymbol{Q}(t) = \begin{bmatrix} \bar{Q}_0 & \bar{Q}_2 & \bar{Q}_2 & \cdots & \bar{Q}_2 \\ \bar{Q}_2^T & \bar{Q}_1 & \bar{Q}_3 & \cdots & \bar{Q}_3 \\ \bar{Q}_2^T & \bar{Q}_3 & \bar{Q}_1 & \cdots & \bar{Q}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_2^T & \bar{Q}_3 & \bar{Q}_3 & \cdots & \bar{Q}_1 \end{bmatrix}, \qquad (7)$$

where

$$\bar{Q}_0 = Q_0 + \lambda H_1^T Q H_1,$$
$$\bar{Q}_1 = \frac{H_0^T Q_0 H_0}{N^2} + \frac{\lambda}{N}[(I - \frac{H_2^T}{N})Q(I - \frac{H_2}{N}) + \frac{(N-1)H_2^T Q H_2}{N^2}],$$
$$\bar{Q}_2 = -\frac{Q_0 H_0}{N} + \frac{\lambda}{N}[\frac{(N-1)H_1^T Q H_2}{N} - H_1^T Q(I - \frac{H_2}{N})],$$
$$\bar{Q}_3 = \frac{H_0^T Q_0 H_0}{N^2} + \frac{\lambda}{N^2}[\frac{(N-2)H_2^T Q H_2}{N} - (I - \frac{H_2^T}{N})QH_2 - H_2^T Q(I - \frac{H_2}{N})].$$

We can similarly define $\boldsymbol{Q}_f$ with the same structure and its submatrices $\bar{Q}_{kf}$, $k = 0, 1, 2, 3$. The social cost is written in the form

$$J_{\rm soc}^{(N)} = \mathbb{E}\int_0^T (\boldsymbol{X}_t^T \boldsymbol{Q} \boldsymbol{X}_t + \boldsymbol{u}_t^T \boldsymbol{R} \boldsymbol{u}_t)dt + \mathbb{E}\boldsymbol{X}_T^T \boldsymbol{Q}_f \boldsymbol{X}_T.$$

where $\boldsymbol{R}$ can be easily determined.

Denote the value function

$$V(t, \mathbf{x}) = \mathbb{E}_{t,\mathbf{x}}^{\mathcal{F}_t^{W_0}}[\int_t^T (\boldsymbol{X}_s^T \boldsymbol{Q} \boldsymbol{X}_s + \boldsymbol{u}_s^T \boldsymbol{R} \boldsymbol{u}_s)ds + \boldsymbol{X}_T^T \boldsymbol{Q}_f \boldsymbol{X}_T] \tag{8}$$

as a random field. The subscript in the expectation indicates the initial condition $(t, \mathbf{x})$. Write $V(t, \mathbf{x}) = \mathbf{x}^T \boldsymbol{P}_t \mathbf{x} + 2\mathbf{x}^T \boldsymbol{S}_t + r_t$. We use V, Φ to write a stochastic Hamilton-Jacobi-Bellman (SHJB) equation as a BSDE as in [30]. Let $\Phi = \mathbf{x}^T \boldsymbol{\Psi}_t \mathbf{x} + 2\mathbf{x}^T \boldsymbol{\Upsilon}_t + \gamma_t$.

The stochastic Riccati equation is given in the form

$$\begin{cases} 0 = d\boldsymbol{P}_t + (\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q})dt \\ \qquad -\boldsymbol{\Psi}_t dW_0(t), \\ \boldsymbol{P}_T = \boldsymbol{Q}_f. \end{cases}$$

By Lemma [19, Lemma A.1], we obtain a unique solution $(\boldsymbol{P}, \boldsymbol{\Psi})$, where $\boldsymbol{P} \in L^\infty_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^{n(N+1)})$. Next, $\boldsymbol{S}$ and r satisfy the BSDEs

$$\begin{cases} d\boldsymbol{S}_t = -[(\boldsymbol{A}^T - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T)\boldsymbol{S}_t + \boldsymbol{\Psi}_t \boldsymbol{D}_0]dt + \boldsymbol{\Upsilon}_t dW_0, \\ \boldsymbol{S}_T = 0, \end{cases}$$

$$\begin{cases} dr_t = -\{\mathrm{Tr}(\boldsymbol{P}[\boldsymbol{D}\boldsymbol{D}^T + \boldsymbol{D}_0\boldsymbol{D}_0^T]) \\ \qquad -\boldsymbol{S}^T\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{S} + 2\boldsymbol{\Upsilon}^T\boldsymbol{D}_0\}dt + \gamma_t dW_0(t), \\ r_T = 0. \end{cases}$$

We denote

$$\boldsymbol{P}_t = \begin{bmatrix} \Pi_0^N & \Pi_2^N & \Pi_2^N & \cdots & \Pi_2^N \\ \Pi_2^{NT} & \Pi_1^N & \Pi_3^N & \cdots & \Pi_3^N \\ \Pi_2^{NT} & \Pi_3^N & \Pi_1^N & \cdots & \Pi_3^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_2^{NT} & \Pi_3^N & \Pi_3^N & \cdots & \Pi_1^N \end{bmatrix}, \tag{9}$$

$$\boldsymbol{\Psi}_t = \begin{bmatrix} \Psi_0^N & \Psi_2^N & \Psi_2^N & \cdots & \Psi_2^N \\ \Psi_2^{NT} & \Psi_1^N & \Psi_3^N & \cdots & \Psi_3^N \\ \Psi_2^{NT} & \Psi_3^N & \Psi_1^N & \cdots & \Psi_3^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Psi_2^{NT} & \Psi_3^N & \Psi_3^N & \cdots & \Psi_1^N \end{bmatrix},$$

and

$$\boldsymbol{S}_t = \begin{bmatrix} S_0^N \\ S^N \\ \vdots \\ S^N \end{bmatrix}, \qquad \boldsymbol{\Upsilon}_t = \begin{bmatrix} \Upsilon_0^N \\ \Upsilon^N \\ \vdots \\ \Upsilon^N \end{bmatrix}.$$

The above structural properties (i.e. symmetry among the minor agents) of $(\boldsymbol{P}, \boldsymbol{\Psi}, \boldsymbol{S}, \boldsymbol{\Upsilon})$ may be established using the permutation method in [26, Lemma A.1] and uniqueness of the solution of the BSDEs of $(\boldsymbol{P}, \boldsymbol{S})$. To proceed, we will write the BSDEs of Π_0^N, Π_1^N, Π_2^N, and Π_3^N. It is easily seen that, due to the properties of $\boldsymbol{P}_T$ and $\boldsymbol{Q}_f$, we immediately have the terminal conditions

$$\Pi_k^N(T) = \bar{Q}_{kf}, \quad k = 0, 1, 2, 3.$$

Denote $M_0 = B_0 R_0^{-1} B_0^T$ and $M_\lambda = \lambda^{-1} B R^{-1} B^T$. For Π_0^N, we have the BSDE

$$\begin{aligned} 0 = d\Pi_0^N(t) + [&\Pi_0^N A_0 + N\Pi_2^N G + A_0^T \Pi_0^N \\ &+ NG^T \Pi_2^{NT} - \Pi_0^N M_0 \Pi_0^N - N^2 \Pi_2^N M_\lambda \Pi_2^{NT} \\ &+ Q_0 + \lambda H_1^T Q H_1]dt - \Psi_0^N dW_0(t). \end{aligned}$$

The BSDEs of Π_k^N, $1 \le k \le 3$, are not displayed due to limited space. Next, we similarly write the BSDEs of S_0^N and S^N with terminal conditions $S_0^N(T) = 0$, $S^N(T) = 0$. Finally, the BSDE of r_t with $r_T = 0$ reads

$$\begin{aligned} dr_t = \; -\{&\mathrm{Tr}[\Pi_0^N D_0 D_0^T + N\Pi_1^N(DD^T + D_1 D_1^T) \\ &+ 2N\Pi_2^N D_0 D_1^T + N(N-1)\Pi_3^N D_1 D_1^T] \\ &- (S_0^{NT} M_0 S_0^N + N^2 S^{NT} M_\lambda S^N) \\ &+ 2(\Upsilon_0^{NT} D_0 + N\Upsilon^{NT} D_1)\}dt + \gamma_t dW_0(t). \end{aligned}$$

B. The re-scaling method

Next define

$$\Lambda_0^N = \Pi_0^N,\ \Lambda_1^N = N\Pi_1^N,\ \Lambda_2^N = N\Pi_2^N,\ \Lambda_3^N = N^2\Pi_3^N,$$
$$\Phi_0^N = \Psi_0^N,\ \Phi_1^N = N\Psi_1^N,\ \Phi_2^N = N\Psi_2^N,\ \Phi_3^N = N^2\Psi_3^N.$$

Letting $N \to \infty$ in the equations of Λ_k^N, $0 \le k \le 3$, we formally obtain the following BSDEs

$$\begin{aligned}0 =& d\Lambda_0(t) + (\Lambda_0 A_0 + A_0^T \Lambda_0 + \Lambda_2 G + G^T \Lambda_2^T \\ &- \Lambda_0 M_0 \Lambda_0 - \Lambda_2 M_\lambda \Lambda_2^T + Q_0 + \lambda H_1^T Q H_1)dt \\ &- \Phi_0 dW_0(t), \\ 0 =& d\Lambda_1(t) + (\Lambda_1 A + A^T \Lambda_1 - \Lambda_1 M_\lambda \Lambda_1 + \lambda Q)dt \\ &- \Phi_1 dW_0(t), \\ 0 =& d\Lambda_2(t) + [\Lambda_0 F_0 + \Lambda_2(A+F) + A_0^T \Lambda_2 \\ &+ G(\Lambda_1 + \Lambda_3) - \Lambda_0 M_0 \Lambda_2 - \Lambda_2 M_\lambda(\Lambda_1 + \Lambda_3) \\ &+ \lambda(H_1^T Q H_2 - H_1^T Q) - Q_0 H_0]dt - \Phi_2 dW_0(t), \\ 0 =& d\Lambda_3(t) + [\Lambda_2^T F_0 + F_0^T \Lambda_2 + \Lambda_1 F + F^T \Lambda_1 \\ &+ \Lambda_3(A+F) + (A+F)^T \Lambda_3 - \Lambda_2^T M_0 \Lambda_2 \\ &- \Lambda_1 M_\lambda \Lambda_3 - \Lambda_3 M_\lambda \Lambda_1 - \Lambda_3 M_\lambda \Lambda_3 + H_0^T Q_0 H_0 \\ &+ \lambda(H_2^T Q H_2 - Q H_2 - H_2^T Q)]dt - \Phi_3 dW_0\end{aligned}$$

with the terminal conditions

$$\begin{aligned}&\Lambda_0(T) = Q_{0f} + \lambda H_{1f}^T Q_f H_{1f},\ \Lambda_1(T) = \lambda Q_f + H_{2f}^T Q_f H_{2f}, \\ &\Lambda_2(T) = -Q_{0f} H_{0f} + \lambda H_{1f}^T Q_f H_{2f} - \lambda H_{1f}^T Q_f, \\ &\Lambda_3(T) = H_{0f}^T Q_{0f} H_{0f} + \lambda(H_{2f}^T Q_f H_{2f} - Q_f H_{2f} - H_{2f}^T Q_f).\end{aligned}$$

By a similar argument, as $N \to \infty$, (S_0^N, NS^N) and $r = r^N$, respectively, have the limiting forms (φ_0, φ) and ρ, which satisfy the following equations

$$\begin{aligned}0 =& d\varphi_0 + [(A_0^T - \Lambda_0 M_0)\varphi_0 + (G^T - \Lambda_2 M_\lambda)\varphi \\ &+ \Phi_0 D_0 + \Phi_2 D_1]dt - \eta_0 dW_0(t), \\ 0 =& d\varphi + \{(F_0^T - \Lambda_2 M_0)\varphi_0 + [A + F - (\Lambda_1 + \Lambda_3)M_\lambda]\varphi \\ &+ \Phi_2^T D_0 + (\Phi_1 + \Phi_3) D_1\}dt - \eta dW_0(t), \\ 0 =& d\rho + \{\mathrm{Tr}[\Lambda_0 D_0 D^T + \Lambda_1(DD^T + D_1 D_1^T) \\ &+ 2\Lambda_2 D_0 D_1^T + \Lambda_3 D_1 D_1^T] - (\varphi_0^T M_0 \varphi_0 + \varphi^T M_\lambda \varphi) \\ &+ 2(\eta_0^T D_0 + \eta^T D_1)\}dt - \zeta dW_0(t),\end{aligned}$$

where $\varphi_0(T) = \varphi(T) = 0$ and $\rho(T) = 0$.

C. Existence and uniqueness of the BSDEs

Theorem 3.1: (i) There exists a unique solution (Λ_1, Φ_1) on $[0, T]$.

(ii) There exists a unique solution $(\Lambda_0, \Lambda_2, \Lambda_3, \Phi_0, \Phi_2, \Phi_3)$ on $[0, T]$.

(iii) There exists a unique solution $(\varphi_0, \varphi, \rho, \eta_0, \eta, \zeta)$ on $[0, T]$.

Proof: Part (i) follows easily from [19, Lemma A.1]. Now we consider the Riccati equation of $\mathbf{P}$ (where $\mathbf{P}$ has a unique solution) in the proof of Theorem 5.2 in [19], for which we take the partition

$$\mathbf{P}(t) = \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix}.$$

By comparing the equations of $(\Lambda_0, \Lambda_2, \Lambda_3)$ and these of (P_1, P_2, P_3), we obtain $\Lambda_0 = P_1,\ \Lambda_2 = P_2^T,\ \Lambda_1 + \Lambda_3 = P_3$. Then the existence of $(\Lambda_0, \Lambda_2, \Lambda_3)$ follows, and uniqueness holds since $\mathbf{P}$ is unique.

After uniquely determining the solution in part (ii), we uniquely obtain (φ_0, φ) from linear BSDEs, and finally get (ρ, ζ) on $[0, T]$. ■

D. Decentralized control laws for $N+1$ agents

The major agent's control law is given by

$$u_0^N = -R_0^{-1} B_0^T (\Lambda_0 X_0^N(t) + \Lambda_2 m(t) + \varphi_0(t)), \qquad (10)$$

and the minor agent's control law is given by

$$u_i^N = -\lambda^{-1} R^{-1} B^T (\Lambda_1 X_i(t) + \Lambda_2^T X_0^N(t) + \Lambda_3 m(t) + \varphi(t)).$$

In the above, $m(t)$ approximates $X^{(N)}(t)$ and is given by

$$\begin{aligned}dm(t) =& \{[A + F - M_\lambda(\Lambda_1 + \Lambda_3)]m + (G - M_\lambda \Lambda_2^T) X_0^N \\ &- M_\lambda \varphi\}dt + D_1 dW_0(t).\end{aligned}$$

IV. Prior Bounds on $\boldsymbol{P}$

In order to obtain more specific bound information on the matrix $\boldsymbol{P}_t$ in (9), we introduce an auxiliary optimal control problem, which has state dynamics and cost:

$$\begin{aligned}&d\boldsymbol{X}_t = (\boldsymbol{A}\boldsymbol{X}_t + \boldsymbol{B}\boldsymbol{u}_t)dt, \\ &J_{\mathrm{soc}}^{(N)} = \mathbb{E}\int_0^T (\boldsymbol{X}_t^T \boldsymbol{Q} \boldsymbol{X}_t + \boldsymbol{u}_t^T \boldsymbol{R} \boldsymbol{u}_t)dt + \mathbb{E}\boldsymbol{X}_T^T \boldsymbol{Q}_f \boldsymbol{X}_T.\end{aligned}$$

We still denote the state components by $X_k^N(t)$.

Now let the initial condition be $\mathbf{x} = (x_0^T, x_1^T, \cdots, x_N^T)^T \in \mathbb{R}^{n(N+1)}$ at time s. Conditioning on $\mathcal{F}_s^{W_0}$, we determine the optimal cost as $\mathbf{x}^T \boldsymbol{P}(s)\mathbf{x}$. By elementary ODE estimates, for the particular control $\boldsymbol{u}_t = 0$ for all $t \in [s, T]$, we have

$$\sup_N \sup_{|x_0|\le 1, \cdots, |x_N| \le 1} \sup_{0 \le k \le N, s \le t \le T} |X_k^N(t)| \le C. \qquad (11)$$

with probability one. Since s is arbitrary, by use of the individual costs, we obtain the bound

$$\sup_t \sup_{|x_0|\le 1, \cdots, |x_N| \le 1} \mathbf{x}^T \boldsymbol{P}(t)\mathbf{x} \le C. \qquad (12)$$

In particular, if we take $\hat{\mathbf{x}} = (x_0, y, \cdots, y)$, then

$$\sup_t \sup_{|x_0|\le 1, |y| \le 1} \hat{\mathbf{x}}^T \boldsymbol{P}(t)\hat{\mathbf{x}} \le C. \qquad (13)$$

Taking particular values of (x_0, y) in (13), we may obtain bound information on Π_i^N and prove the following lemma.

Lemma 4.1: We have

$$\sup_{0 \le t \le T} \{|\Pi_0^N| + N|\Pi_1^N| + N|\Pi_2^N| + N^2|\Pi_3^N|\} = O(1).$$

The bound on the right hand side is deterministic.

A. Approximation error estimate

Notice that we can compare the BSDEs of $(\Lambda_0^N, \Lambda_1^N, \Lambda_2^N, \Lambda_3^N)$ and those of $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ by viewing the former as the latter being perturbed by some small error terms of $O(1/N)$, where the error bound is due to Lemma 4.1. Since both solution processes stay in a prior compact set, the Lipschitz property (B) in [34, Theorem 7.3.3] is satisfied. Subsequently, in view of Lemma 4.1 and [34, Theorem 7.3.3], for $0 \le k \le 3$, we have

$$\mathbb{E} \sup_{t \in [0,T]} \|\Lambda_k^N(t) - \Lambda_k(t)\| = O(1/N). \qquad (14)$$

B. The asymptotic value of the social optimum

Denote the optimal control in (8) by $\mathbf{u}^{\mathrm{opt}}$. To evaluate the asymptotic value of $J_{\mathrm{soc}}^{(N)}(\mathbf{u}^{\mathrm{opt}})$, we suppose

$$\begin{cases}\mathbb{E}X_0^N(0)=\mu_0, \quad \mathbb{E}X_i^N(0)=\mu,\ i\geq 1,\\ \mathrm{Cov}(X_0^N(0),X_0^N(0))=\Sigma_0,\\ \mathrm{Cov}(X_i^N(0),X_i^N(0))=\Sigma,\ i\geq 1.\end{cases} \tag{15}$$

Denote

$$\begin{aligned}J_{\mathrm{soc}}^{\infty}&=\mu_0^T\Lambda_0(0)\mu+2\mu_0^T\Lambda_2(0)\mu+\mu^T[\Lambda_1(0)+\Lambda_3(0)]\mu\\&\quad+\mathrm{Tr}(\Lambda_0(0)\Sigma_0+\Lambda_1(0)\Sigma)\\&\quad+2\mu_0^T\varphi_0(0)+2\mu^T\varphi(0)+\rho(0).\end{aligned}$$

Theorem 4.2: Under (15), we have

$$|J_{\mathrm{soc}}^{(N)}(\mathbf{u}^{\mathrm{opt}})-J_{\mathrm{soc}}^{\infty}|=O(1/N). \tag{16}$$

Proof: The optimal social cost is given by

$$J_{\mathrm{soc}}^{(N)}(\mathbf{u}^{\mathrm{opt}})=\mathbb{E}\boldsymbol{X}^T(0)\boldsymbol{P}_0\boldsymbol{X}(0)+2\mathbb{E}\boldsymbol{X}^T(0)\boldsymbol{S}_0+r_0.$$

Note that $\boldsymbol{P}_0$, $\boldsymbol{S}_0$, and r_0 are all deterministic for $t=0$. We can show similar convergence rate for S_0^N to φ_0, NS^N to φ, and r to ρ, as in (14). The theorem follows. ■

V. Closed-loop Performance Analysis

Under the decentralized control laws, the closed-loop state processes are

$$\begin{aligned}dX_0^N=&\big[A_0X_0^N-M_0(\Lambda_0X_0^N+\Lambda_2m+\varphi_0)+F_0X^{(N)}\big]dt\\&+D_0dW_0(t),\end{aligned} \tag{17}$$

$$\begin{aligned}dX_i^N=&\big[AX_i^N-M_\lambda(\Lambda_1X_i+\Lambda_2^TX_0^N+\Lambda_3m+\varphi)\\&+FX^{(N)}+GX_0^N(t)\big]dt\\&+DdW_i(t)+D_1dW_0(t),\quad 1\leq i\leq N,\end{aligned} \tag{18}$$

where

$$\begin{aligned}dm(t)=&\{[A+F-M_\lambda(\Lambda_1+\Lambda_3)]m+(G-M_\lambda\Lambda_2^T)X_0^N\\&-M_\lambda\varphi\}dt+D_1dW_0(t),\quad m(0)=\mu.\end{aligned}$$

Denote

$$\widehat{\boldsymbol{A}}=\begin{bmatrix}A_0-M_0\Lambda_0 & \frac{F_0}{N} & \cdots & \frac{F_0}{N} & -M_0\Lambda_2\\ G-M_\lambda\Lambda_2^T & A_c & \cdots & \frac{F}{N} & -M_\lambda\Lambda_3\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ G-M_\lambda\Lambda_2^T & \frac{F}{N} & \frac{F}{N} & A_c & -M_\lambda\Lambda_3\\ G-M_\lambda\Lambda_2^T & 0 & 0 & 0 & A+F-M_\lambda(\Lambda_1+\Lambda_3)\end{bmatrix}$$

where $A_c=A-M_\lambda\Lambda_1+\frac{F}{N}$. Denote

$$\widehat{\boldsymbol{D}}=\begin{bmatrix}\boldsymbol{D}\\ \mathbf{o}\end{bmatrix},\qquad \widehat{\boldsymbol{D}}_0=\begin{bmatrix}D_0\\ \mathbf{1}_{(N+1)\times 1}\otimes D_1\end{bmatrix}, \tag{19}$$

where $\mathbf{o}$ is the $n_3\times Nn_3$ zero matrix and

$$\widehat{\boldsymbol{b}}=\begin{bmatrix}-M_0\varphi_0\\ -M_\lambda\varphi\\ \vdots\\ -M_\lambda\varphi\end{bmatrix},\qquad \widehat{\boldsymbol{\varphi}}=\begin{bmatrix}\hat{R}_0\varphi_0\\ \hat{R}_\lambda\varphi\\ \vdots\\ \hat{R}_\lambda\varphi\end{bmatrix}. \tag{20}$$

We write the closed-loop dynamics in the form

$$d\boldsymbol{Z}_t=(\widehat{\boldsymbol{A}}\boldsymbol{Z}+\widehat{\boldsymbol{b}})dt+\widehat{\boldsymbol{D}}d\boldsymbol{W}_t+\widehat{\boldsymbol{D}}_0dW_0(t). \tag{21}$$

Denote

$$\boldsymbol{x}=\begin{bmatrix}x_0\\x_1\\ \vdots\\ x_N\end{bmatrix},\quad \boldsymbol{z}=\begin{bmatrix}\boldsymbol{x}\\ m\end{bmatrix},\quad \boldsymbol{u}=\begin{bmatrix}u_0\\u_1\\ \vdots\\ u_N\end{bmatrix},$$

$\hat{R}_0=-R_0^{-1}B_0^T$, $\hat{R}_\lambda=-R_\lambda^{-1}B^T$, and

$$\widehat{\boldsymbol{M}}=\begin{bmatrix}\hat{R}_0\Lambda_0 & 0 & \dots & 0 & \hat{R}_0\Lambda_2\\ \hat{R}_\lambda\Lambda_2^T & \hat{R}_\lambda\Lambda_1 & \dots & 0 & \hat{R}_\lambda\Lambda_3\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \hat{R}_\lambda\Lambda_2^T & 0 & \dots & \hat{R}_\lambda\Lambda_1 & \hat{R}_\lambda\Lambda_3\end{bmatrix}.$$

In addition, denote $\widehat{\boldsymbol{Q}}_f=\mathrm{diag}[\boldsymbol{Q}_f,\mathbf{o}_{n\times n}]$ and

$$\widehat{\boldsymbol{Q}}=\mathrm{diag}[\boldsymbol{Q},\mathbf{o}_{n\times n}]+\widehat{\boldsymbol{M}}^T\boldsymbol{R}\widehat{\boldsymbol{M}}.$$

We can write $\boldsymbol{u}=\widehat{\boldsymbol{M}}\boldsymbol{Z}+\widehat{\boldsymbol{\varphi}}$ and

$$\begin{aligned}\hat{V}(t,\boldsymbol{z})=\mathbb{E}_{t,\boldsymbol{z}}^{\mathcal{F}_t^{W_0}}\Big[\int_t^T&(\boldsymbol{Z}_s^T\widehat{\boldsymbol{Q}}\boldsymbol{Z}_s+2\boldsymbol{Z}_s^T\widehat{\boldsymbol{M}}^T\boldsymbol{R}\widehat{\boldsymbol{\varphi}}_s\\&+\widehat{\boldsymbol{\varphi}}_s^T\boldsymbol{R}\widehat{\boldsymbol{\varphi}}_s)ds+\boldsymbol{Z}_T^T\widehat{\boldsymbol{Q}}_f\boldsymbol{Z}_T\Big].\end{aligned}$$

We have

$$\begin{aligned}&-d\hat{V}-[(\widehat{\boldsymbol{A}}z)^TD_z\hat{V}+\frac{1}{2}\mathrm{tr}[(\widehat{\boldsymbol{D}}\widehat{\boldsymbol{D}}^T+\widehat{\boldsymbol{D}}_0\widehat{\boldsymbol{D}}_0^T)D_{zz}\hat{V}]\\&\quad+\widehat{\boldsymbol{D}}_0^TD_z\hat{\Phi}+z^T\widehat{\boldsymbol{Q}}z+2z^T\widehat{\boldsymbol{M}}^T\boldsymbol{R}\widehat{\boldsymbol{\varphi}}\\&\qquad+\widehat{\boldsymbol{\varphi}}^T\boldsymbol{R}\widehat{\boldsymbol{\varphi}}]dt+\hat{\Phi}dW_0(t)=0,\\&\hat{V}(T,z)=z^T\widehat{\boldsymbol{Q}}_fz.\end{aligned}$$

Then given the initial condition $\boldsymbol{Z}_t=z$, the social cost may be represented as

$$\begin{aligned}\widehat{V}(t,z)&=z^T\widehat{\boldsymbol{P}}_tz+2z^T\widehat{\boldsymbol{S}}_t+\widehat{r}_t,\\ \widehat{\Phi}(t,z)&=z^T\widehat{\boldsymbol{\Psi}}_tz+2z^T\widehat{\boldsymbol{\Upsilon}}_t+\widehat{\gamma}_t,\end{aligned}$$

where

$$\begin{aligned}0&=d\widehat{\boldsymbol{P}}+(\widehat{\boldsymbol{A}}^T\widehat{\boldsymbol{P}}+\widehat{\boldsymbol{P}}\widehat{\boldsymbol{A}}+\widehat{\boldsymbol{Q}})dt-\widehat{\boldsymbol{\Psi}}dW_0,\\ 0&=d\widehat{\boldsymbol{S}}+[\widehat{\boldsymbol{A}}^T\widehat{\boldsymbol{S}}+\widehat{\boldsymbol{\Psi}}\widehat{\boldsymbol{D}}_0+\widehat{\boldsymbol{P}}\widehat{\boldsymbol{b}}+\widehat{\boldsymbol{M}}^T\boldsymbol{R}\widehat{\boldsymbol{\varphi}}]dt\\&\quad-\widehat{\boldsymbol{\Upsilon}}dW_0,\\ 0&=d\widehat{r}+\{\mathrm{tr}[(\widehat{\boldsymbol{D}}\widehat{\boldsymbol{D}}^T+\widehat{\boldsymbol{D}}_0\widehat{\boldsymbol{D}}_0^T)\widehat{\boldsymbol{P}}]+2\widehat{\boldsymbol{b}}^T\widehat{\boldsymbol{S}}+2\widehat{\boldsymbol{D}}_0^T\boldsymbol{\Upsilon}\\&\quad+\widehat{\boldsymbol{\varphi}}^T\boldsymbol{R}\widehat{\boldsymbol{\varphi}}\}dt-\hat{\gamma}dW_0,\end{aligned}$$

with $\widehat{\boldsymbol{P}}_T=\widehat{\boldsymbol{Q}}_f,\widehat{\boldsymbol{S}}_T=0,\widehat{r}_T=0$.

We look for $\widehat{\boldsymbol{P}}$ and $\widehat{\boldsymbol{S}}$ with the representations

$$\widehat{\boldsymbol{P}}=\begin{bmatrix}\hat{\Pi}_0^N & \hat{\Pi}_2^N & \hat{\Pi}_2^N & \cdots & \hat{\Pi}_2^N & \Pi_a^N\\ \hat{\Pi}_2^{NT} & \hat{\Pi}_1^N & \hat{\Pi}_3^N & \cdots & \hat{\Pi}_3^N & \Pi_b^N\\ \hat{\Pi}_2^{NT} & \hat{\Pi}_3^N & \hat{\Pi}_1^N & \cdots & \hat{\Pi}_3^N & \Pi_b^N\\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \hat{\Pi}_2^{NT} & \hat{\Pi}_3^N & \hat{\Pi}_3^N & \cdots & \hat{\Pi}_1^N & \Pi_b^N\\ \Pi_a^{NT} & \Pi_b^{NT} & \Pi_b^{NT} & \cdots & \Pi_b^{NT} & \Pi_m^N\end{bmatrix},$$

$$\widehat{\boldsymbol{S}}=[\hat{S}_0^{NT},\hat{S}^{NT},\dots,\hat{S}^{NT},\hat{S}_m^{NT}]^T,$$

and similar decomposition of $\widehat{\boldsymbol{\Psi}}$ and $\widehat{\boldsymbol{\Upsilon}}$. We obtain

$$\begin{aligned}0 = & d\hat{\Pi}_0^N + [\hat{\Pi}_0^N (A_0 - M_0 \Lambda_0) + (A_0 - M_0 \Lambda_0)^T \hat{\Pi}_0^N \\ & + N \hat{\Pi}_2^N (G - M_\lambda \Lambda_2^T) + N (G - M_\lambda \Lambda_2^T)^T \hat{\Pi}_2^{NT} \\ & + \hat{\Pi}_a^N (G - M_\lambda \Lambda_2^T) + (G - M_\lambda \Lambda_2^T)^T \hat{\Pi}_a^{NT} \\ & + \Lambda_0^T M_0 \Lambda_0 + \Lambda_2 M_\lambda \Lambda_2^T + \bar{Q}_0] dt - \hat{\Psi}_0^N dW_0(t),\end{aligned}$$

with terminal conditions $\hat{\Pi}_k^N(T) = \hat{Q}_{kf}$, $0 \leq k \leq 3$, and

$$\hat{\Pi}_a^N(T) = \hat{\Pi}_b^N(T) = \hat{\Pi}_m^N(T) = 0.$$

Next, we similarly have equations for $\hat{S}_0^N, \hat{S}^N, \hat{S}_m^N$ and $\hat{r}$.

Denote $\hat{\Lambda}_0^N = \hat{\Pi}_0^N, \hat{\Lambda}_1^N = N\hat{\Pi}_1^N, \hat{\Lambda}_2^N = N\hat{\Pi}_2^N, \hat{\Lambda}_3^N = N^2\hat{\Pi}_3^N$, and $\hat{\Lambda}_a^N = \hat{\Pi}_a^N, \hat{\Lambda}_n^N = N\hat{\Pi}_n^N, \hat{\Lambda}_m^N = \hat{\Pi}_m^N$. We further obtain a set of limiting BSDEs, which are omitted here due to limited space.

Theorem 5.1: Assume (15). Then we have

$$0 \leq \mathbb{E}\widehat{V}(0, \boldsymbol{X}(0)) - J_{\rm soc}^{(N)}(\mathbf{u}^{\rm opt}) = O(1/N). \qquad (22)$$

Proof: (Sketch) We first obtain the limiting linear BSDEs of $(\hat{\Lambda}_k^N, \hat{\Lambda}_a^N, \hat{\Lambda}_b^N, \hat{\Lambda}_m, 0 \leq k \leq 3)$ and $\widehat{\boldsymbol{S}}_t$, $\widehat{r}_t$. By comparing the above limiting BSDEs with those of $(\Lambda_0, \cdots, \rho)$, we further show $|\mathbb{E}\widehat{V}(0, \boldsymbol{X}(0)) - J_{\rm soc}^{\infty}| = O(1/N)$. Recalling Theorem 4.2, we complete the proof. ∎

The above performance estimate improves upon the bound $O(1/\sqrt{N})$ in [18], [19]. For the model without a major player, a similar bound of $O(1/\sqrt{N})$ was obtained in [15].

VI. Conclusion

We analyze an LQ mean field social optimization problem with mixed agents. We adopt a re-scaling method to derive decentralized control laws and further obtain tight bound of $O(1/N)$ for optimality loss.

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