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An Interval Predictor–based Robust Control for a Class of Constrained Nonlinear Systems

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***Abstract*—This paper proposes the design of a robust sampled–time controller to stabilize continuous–time nonlinear systems, taking into account state and input constraints. The proposed controller comprises the design of a robust control law, which is based on an interval predictor–based state–feedback controller and a Model Predictive Control (MPC) approach, which deals with the state and input constraints. The interval predictor–based state–feedback controller is designed based on a Lyapunov function approach that provides a safe set, where the state constraints are not transgressed. Out this set, the MPC is activated guaranteeing the fulfillment of the state and input constraints. The proposed switched control strategy guarantees the practical Uniform Asymptotic Stability of the considered nonlinear systems. A constructive method, based on linear matrix inequalities (LMIs), is proposed to compute the controller gains and the state of the system is not required. Some simulation results illustrate the feasibility of the proposed scheme.**

***Index Terms*—Constrained Nonlinear Systems; Interval Predictor; Model Predictive Control.**

I. Introduction

REAL dynamical systems, which in general can be represented by nonlinear models, are constrained due to the structural limitations of the system, *i.e.*, the system dynamics have state and input constraints, (see *e.g.* [1], [2], and [3]). In addition, the systems can be affected by parameter uncertainties, that can modify their behaviour. In order to design a controller that can counteract the effects of parameter uncertainties, taking into account state and input constraints, robust control techniques are required. For instance, the MPC is a technique in which the current control action is obtained by solving on–line, at each sampling instant, a finite horizon open–loop optimal control problem, using the current state of the plant as the initial state. The solution produces an optimal control sequence and the first control in that sequence is applied to the system [4].

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Regarding the control design for constrained systems, in [5], an adaptive MPC for linear systems with unknown parameters and disturbances is proposed. Also, in [6], an MPC for constrained uncertain linear systems is proposed. In this work, the robustness of the system constraints and Input–to–State Stability (ISS) of the closed–loop system is guaranteed. Meanwhile, for linear parameter varying (LPV) systems, the MPC approach is also well–used. For instance, in [7], an output feedback robust MPC for quasi–LPV systems with bounded noise is proposed. This work utilizes both, an ellipsoidal bound and a polyhedral bound in the main optimization problem, and asymptotic convergence of the state trajectories to a neighborhood of the origin is guaranteed. In [8], an output feedback robust MPC is proposed for disturbed LPV systems. In this algorithm single–step dynamic output feedback robust MPC, where the infinite–horizon control moves, are parameterized as a dynamic output feedback law. However, this proposal demands heavy computational burden to achieve the control task. The MPC can also be applied to discrete–time systems, in [9], an output feedback MPC for discrete–time linear systems affected by bounded state and output disturbances is considered. The controller is based on a stable state estimator and an MPC. The proposed robust output feedback controller requires the online solution of a standard quadratic program. The closed–loop system renders a specified invariant set robustly exponentially stable. It is worth mentioning that most of the MPC results given in the literature are devoted to linear systems, and this could be restrictive.

Concerning the application of the MPC approach to nonlinear systems, in [10], a robust learning–based MPC is proposed for nonlinear systems subject to input and output constraints. A nonparametric machine learning technique is used to estimate the prediction model; and thus, an MPC, without terminal constraint, is obtained ensuring the asymptotic stability of the closed–loop system. In [11], a robust MPC is proposed for tracking unicycle robots with input constraint and bounded disturbances. In this work ISS properties are guaranteed and the constraints are ensured by tightening the input domain and the terminal region. The MPC can also be applied to nonlinear discrete–time systems, in [12], a periodic formulation of MPC with a composite self–triggered mechanism is proposed to reduce the communication and computational burden while maintaining the desired control performance for disturbed nonlinear discrete–time systems. However, the algorithm has a relatively low tolerance to disturbances due to the state error bounds are derived by means of Lipschitz properties. In the previous works, a heavy computational burden is needed, due to the MPC being active during the control task.

The MPC can be also combined with other techniques, *e.g.,* sliding–mode control (SMC), and interval observers techniques, in order to improve its behavior. For instance, in [13], a combination of robust MPC with SMC is proposed for nonlinear constrained uncertain systems. The proposed approach provides Input–to–State practically Stable properties. Meanwhile, in [14], an MPC is proposed for nonlinear systems based on a terminal cost function characterized by an implicit SMC law. However, in order to guarantee the asymptotic stability of the closed–loop system, a linearized model is required. In [15], an interval observer and predictor are incorporated into the classic MPC scheme for discrete–time constrained linear systems with bounded state disturbances. Nevertheless, most of the above–mentioned works can only guarantee ISS properties, require the system trajectories and do not take into account sampled control.

Motivated by these issues, this paper proposes the design of a robust sampled controller to stabilize continuous–time nonlinear systems, taking into account state and input constraints. The robust controller is based on a discrete–time interval predictor–based state–feedback controller and an MPC approach. The proposed control approach possesses the following features: 1) Characterization of a safe set, where the state constraints are not transgressed. 2) Low computational cost since the MPC is only activated out of the safe set. 3) Practical Uniform Asymptotic Stability of the constrained nonlinear systems is proved. 4) The synthesis of the controller is constructive since is based on LMIs. 5) The system states are not required [1].

The organization of this work is as follows. In Section II the problem statement is described. Section III presents some preliminary concepts. The robust control design is introduced in Section IV. Simulation results are depicted in Section V. Finally, the conclusions are presented in Section VI.

Notation: The sets of real and natural numbers are defined by $\mathbb{R}$ and $\mathbb{N}$, respectively; $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$, and $\mathbb{N}_+ = \{s \in \mathbb{N} : s \geq 0\}$. For a matrix A, we define $A^+ = \max\{0, A\}$ and $A^- = A^+ - A$ (similarly for vectors). For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood in an element–wise sense. The relation $P \prec 0$ ($P \succeq 0$) means that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive semi–) definite. For a Lebesgue measurable function $w : \mathbb{R}_+ \to \mathbb{R}^q$, define $\|w\|_{(t_0,t_1)} := ess\sup_{t\in(t_0,t_1)} \|w(t)\|$; then, $\|w\|_\infty = \|w\|_{(0,+\infty)}$ and the set of all the functions w with the property $\|w\|_\infty < +\infty$ is denoted as $\mathcal{L}_\infty^q$. A matrix is called Metzler when all its non–diagonal coefficients are non–negative.

[1]Note that its noisy measurements may be used to refine the available information on the set of admissible values.

II. PROBLEM STATEMENT

Consider the following class of nonlinear system

$$\dot{x}(t) = f(x(t)) + Bu(t),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function, $u(t) \in \mathbb{R}^p$ is the control input. The matrix B is known of suitable dimension. It is considered that $u \in \mathscr{U} := \{u \in \mathbb{R}^p | -u_{\max} \leq u \leq u_{\max}\}$, with $0 < u_{\max} \in \mathbb{R}^p$ as the vector containing the maximum value that each control signal can take. Additionally, the control is sampled, *i.e.*, $u(t) = u(t_k)$, for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}_+$.

Moreover, the solutions of the system (1) must be constrained inside the polytope

$$\mathcal{X} := \{x \in \mathbb{R}^n | \mathbf{b}x \leq \mathbf{1}_{k_x}\}, \tag{2}$$

where $\mathbf{b} = (b_1, \dots, b_{k_x})^\top \in \mathbb{R}^{k_x\times n}$, with $b_i \in \mathbb{R}^n$ being some given vectors that characterize the state constraints, $k_x \in \mathbb{N}$ is the number of the polytope faces, and $\mathbf{1}_{k_x} = (1, \dots, 1)^\top \in \mathbb{R}^{k_x}$.

Before proceeding, the following assumptions are imposed on the system (1).

Assumption 1. *There exist a known Metzler matrix $A_0 \in \mathbb{R}^{n\times n}$ and known matrices $A_j \in \mathbb{R}^{n\times n}$, $j = \overline{1,N}$ for some $N \in \mathbb{N}$, such that the following relations are satisfied*

$$f(x) = \left(A_0 + \sum_{j=1}^{N} \alpha_j(x) A_j\right) x,$$
$$\sum_{j=1}^{N} \alpha_j(x) = 1,\ \alpha_j(x) \in (0,1),\ j = \overline{1,N}.$$

Assumption 2. *There exist known vectors $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$ such that $\underline{x}_0 \leq x(0) \leq \overline{x}_0$.*

Assumption 1 characterizes the class of nonlinearities of the system (1), *i.e.*, those that can be expressed by a sum of a linear term, A_0x, and a convex nonlinear function, $\sum_{j=1}^{N} \alpha_j(x) A_j x$. On the other hand, Assumption 2 and the condition that A_0 must be Metzler, are required to design an interval predictor (IP) for the system (1).

The aim of this manuscript is to design a sampled control law in order to ensure the stabilization of the system (1), taking into account the input and state constraints, *i.e.*, $x \in \mathcal{X}$ and $u \in \mathscr{U}$, under Assumptions 1 and 2.

III. PRELIMINARIES

Lemma 1. *[16] Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$, then*

- *If $A \in \mathbb{R}^{m\times n}$ is a constant matrix then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{3}$$

- *If $A \in \mathbb{R}^{m\times n}$ is a variable matrix and $\underline{A} \le A \le \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then*

$$A_m \le Ax \le A_M. \quad (4)$$

with $A_m = \underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^-$ and $A_M = \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-$.

Lemma 2. *[17] Let $\lambda : [a,b] \to \mathbb{R}$ be a continuous function and $\mu : [a,b] \to \mathbb{R}$ be a continuous and non–negative function. If a continuous function $y : [a,b] \to \mathbb{R}$ satisfies*

$$y(t) \le \lambda(t) + \int_a^t \mu(s)y(s)ds,$$

for $a \le t \le b$; then, on the same interval, it holds that

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s)e^{\left[\int_s^t \mu(r)dr\right]}ds.$$

IV. CONTROL DESIGN

The proposed controller comprises the design of a robust control law, which is composed by an interval predictor–based state–feedback controller and the MPC, which deals with state and input constraints. We will show that there exists a safe set for the interval–predictor based state–feedback controller, where the state constraints are not transgressed, that characterizes the regions where each controller is activated: the MPC is activated outside the safe set while the state–feedback controller is applied inside of it. Since the control is sampled, for the continuous–time system (1), a discrete–time interval predictor is designed, which takes into account the error of discretization and guarantees a careful treatment of the constraints.

The following section addresses the design of the IP for the nonlinear system (5) satisfying Assumptions 1 and 2.

A. Interval Predictor

According to Assumption 1, the system (1) has the following representation

$$\dot{x}(t) = \left(A_0 + \sum_{j=1}^{N} \alpha_j(x)A_j\right) x(t) + Bu(t). \quad (5)$$

Based on [18], applying (4) to $\alpha_j(x)x$, the relation $-\underline{x}^- \le \alpha_j(x)x \le \overline{x}^+$ holds. Thus, according to (3), the following inequalities can be obtained

$$\Delta A_m \le \sum_{j=1}^{N} \alpha_j(x)A_j x \le \Delta A_M,$$

where $\Delta A_m = -\Delta A^+\underline{x}^- - \Delta A^-\overline{x}^+$ and $\Delta A_M = \Delta A^+\overline{x}^+ + \Delta A^-\underline{x}^-$, with $\Delta A^+ = \sum_{i=1}^N A_i^+$ and $\Delta A^- = \sum_{i=1}^N A_i^-$.

Then, a continuous–time IP for system (5) can be stated as follows [19]

$$\dot{z}(t) = \mathcal{A}_0 z(t) + \mathcal{A}_1 z^+(t) + \mathcal{A}_2 z^-(t) + \mathcal{B}u(t), \quad (6)$$

where $z = [\underline{x}^\top, \overline{x}^\top]^\top \in \mathbb{R}^{2n}$ is the extended state vector of the predictor, the extended system matrices $\mathcal{A}_0 \in \mathbb{R}^{2n\times 2n}$, $\mathcal{A}_1 \in \mathbb{R}^{2n\times 2n}$, $\mathcal{A}_2 \in \mathbb{R}^{2n\times 2n}$, and $\mathcal{B} \in \mathbb{R}^{2n\times p}$ are given as

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}, \ \mathcal{A}_1 = \begin{pmatrix} 0 & -\Delta A^- \\ 0 & \Delta A^+ \end{pmatrix},$$
$$\mathcal{A}_2 = \begin{pmatrix} -\Delta A^+ & 0 \\ \Delta A^- & 0 \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} B \\ B \end{pmatrix}.$$

Being properly initialized by $\underline{x}(0) = \underline{x}_0$ and $\overline{x}(0) = \overline{x}_0$, this IP ensures the desired interval inclusion property [19]:

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \ \forall t \ge 0.$$

Then, for any $t \ge t_0 \ge 0$ the solution of system (6) is given by

$$z(t) = e^{\mathcal{A}_0(t-t_0)}z(t_0) + \int_{t_0}^t e^{\mathcal{A}_0(t-\tau)}\mathcal{B}u(\tau)d\tau + \int_{t_0}^t e^{\mathcal{A}_0(t-\tau)}g(z(\tau))d\tau, \quad (7)$$

where $g(z(t)) = \mathcal{A}_1 z^+(t) + \mathcal{A}_2 z^-(t)$. The expression (7), with a sampled control $u(t) = u(t_k)$, for $t \in [t_k, t_{k+1})$, and $k \in \mathbb{N}_+$, and $t_0 = 0$, generates a discrete–time sequence:

$$z(t_{k+1}) = e^{\mathcal{A}_0 h}z(t_k) + \int_0^h e^{\mathcal{A}_0(h-s)}\mathcal{B}ds \cdot u(t_k) + \int_{t_k}^{t_{k+1}} e^{\mathcal{A}_0(t_k-s)}g(z(s))ds,$$

where $h = t_{k+1} - t_k$ is the sampling interval, which we select to be constant for brevity, and t_k is the sampling time instant. Then, adding and subtracting the term $\int_0^h e^{\mathcal{A}_0(h-s)}ds \cdot g(z_k)$, it follows that

$$z(t_{k+1}) = e^{\mathcal{A}_0 h}z(t_k) + \int_0^h e^{\mathcal{A}_0(h-s)}\mathcal{B}ds \cdot u(t_k) + \int_0^h e^{\mathcal{A}_0(h-s)}ds \cdot g(z(t_k)) + \varphi_k, \quad (8)$$

with $\varphi_k = \int_{t_k}^{t_{k+1}} e^{\mathcal{A}_0(t_{k+1}-s)}[g(z(s)) - g(z(t_k))]ds$. Let us consider the notation $z(t_k) = z_k$, $u(t_k) = u_k$, and $\mathcal{G} = \int_0^h e^{\mathcal{A}_0(h-s)}ds \in \mathbb{R}^{2n\times 2n}$. Thus, (8) can be rewritten as

$$z_{k+1} = \overline{\mathcal{A}}_0 z_k + \overline{\mathcal{B}}u_k + \mathcal{G}g(z_k) + \varphi_k, \quad (9)$$

with $\overline{\mathcal{A}}_0 = e^{\mathcal{A}_0 h} \in \mathbb{R}^{2n\times 2n}$ and $\overline{\mathcal{B}} = \mathcal{G}\mathcal{B} \in \mathbb{R}^{2n\times p}$.

However, note that the interval predictor (9) cannot be implemented since φ_k, which characterizes the discretization accuracy and that is dependent on $z(t)$ with $t \in [t_k, t_{k+1}]$, cannot be computed. Thus, consider the following predictor

$$\hat{z}_{k+1} = \overline{\mathcal{A}}_0 \hat{z}_k + \overline{\mathcal{B}}u_k + \mathcal{G}g(\hat{z}_k). \quad (10)$$

Then, in order to stabilize the nonlinear dynamics (1), we need to design u_k to take the trajectories of the system (10) to zero dealing with the state and input constraints. Let us propose the following control law for the system (9)

$$u_k = \begin{cases} \bar{u}_k & \text{if} \quad \hat{z}_k \notin \mathscr{R}, \\ \hat{u}_k & \text{if} \quad \hat{z}_k \in \mathscr{R}, \end{cases} \quad (11)$$

where $\bar{u}_k$ is the control signal computed by the MPC algorithm and $\hat{u}_k$ is the state–feedback control law. Be aware that $\hat{u}_k$ is applied only inside the safe set $\mathscr{R}$, which is defined further on, where the states constraints are not transgressed.

In the following sections we describe the design of each part of the controller.

B. State–Feedback Control

Let us propose the following state–feedback control

$$\hat{u}_k = K_0\hat{z}_k + K_1\hat{z}_k^+ + K_2\hat{z}_k^-. \tag{12}$$

Be aware that (12) is a nonlinear state–feedback control law due to the terms $\hat{z}_k^-$ and $\hat{z}_k^+$. The next theorem provides a way to compute the controller gains $K_0, K_1, K_2 \in \mathbb{R}^{p\times 2n}$.

Lemma 3. *Let Assumptions 1 and 2 be satisfied, and the control law* (12) *be applied to the system* (9)*. Suppose that there exist some symmetric matrices* Q_0*,* Q_1*,* $Q_2 \in \mathbb{R}^{2n\times 2n}$*, a diagonal matrix* $R_3 \in \mathbb{R}^{2n\times 2n}$*, some matrices* $0 \prec X_1^\top = X_1 \in \mathbb{R}^{2n\times 2n}$*,* Y_1*,* $Y_2 \in \mathbb{R}^{2n\times 2n}$ $X_2 \in \mathbb{R}^{2n\times p}$*,* $K_1 \in \mathbb{R}^{p\times 2n}$*, and* $K_2 \in \mathbb{R}^{p\times 2n}$*, such that the following set of LMIs*

$$\begin{pmatrix} -X_1 & Y_2 & Y_3 & 0 & \Sigma_{16} & \Sigma_{17} & X_1 \\ \star & \Sigma_{22} & \Sigma_{23} & 0 & \Sigma_{26} & 0 & 0 \\ \star & \star & \Sigma_{33} & 0 & \Sigma_{36} & 0 & 0 \\ \star & \star & \star & -\gamma I & I & 0 & 0 \\ \star & \star & \star & \star & -X_1 & 0 & 0 \\ \star & \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & \star & -X_2 \end{pmatrix} \preceq 0, \tag{13}$$

$$\Sigma_{16} = X_1\overline{\mathcal{A}}_0^\top + Y_1\overline{\mathcal{B}}^\top,\ \Sigma_{17} = 2\gamma k_a X_1,$$
$$\Sigma_{22} = 4\gamma k_{e_2}^2\mathcal{A}_1^\top\mathcal{A}_1 + Q_1,\ \Sigma_{23} = 4\gamma k_{e_2}^2\mathcal{A}_1^\top\mathcal{A}_2 + R_3,$$
$$\Sigma_{26} = \mathcal{A}_1^\top\mathcal{G}^\top + K_1^\top\overline{\mathcal{B}}^\top,\ \Sigma_{33} = 4\gamma k_{e_2}^2\mathcal{A}_2^\top\mathcal{A}_2 + Q_2,$$
$$\Sigma_{36} = \mathcal{A}_2^\top\mathcal{G}^\top + K_2^\top\overline{\mathcal{B}}^\top,$$

is feasible for some fixed constants

$$k_{e_1} = 1 + e^{k_\xi}k_\xi,\ k_{e_2} = k_{e_1}k_\xi,$$
$$k_\xi = \xi\nu^{-1}(1 - e^{-\nu h}),\ k_a = k_{e_1}\xi - ||\overline{\mathcal{A}}_0|| - L||\mathcal{G}||,$$

with $\gamma, \xi, \alpha > 0$*,* $L = 2\max\{||\mathcal{A}_1||, ||\mathcal{A}_2||\}$*, and* $\nu = |\mathrm{Re}\{\lambda_{\max}\{\mathcal{A}_0\}\}|$*. If the following inequality*

$$\bar{X} + \min\{Q_1, Q_2\} + 2\min\{R_1, R_2\} \succeq 0, \tag{14}$$

holds with $\bar{X} = -\alpha X_1^{-1} + X_2^{-1}$*,* $R_1 = X_1^{-1}Y_1$*, and* $R_2 = X_1^{-1}Y_2$*; the IP* (10) *is initialized as* $\hat{z}_0 = (\underline{x}_0^\top, \overline{x}_0^\top)^\top$*, and the controller gains are selected as* $K_0 = X_2^\top X_1^{-1}$*,* K_1*,* K_2*, and* S_1*; then, the origin of the system dynamics* (9) *is practically Uniformly Asymptotically Stable (practically UAS)*[2]*. Moreover, the safe set is given as*

$$\mathscr{R} = \{z_k \in \mathbb{R}^{2n} : \hat{z}_k^\top X_1^{-1}\hat{z}_k \le \varepsilon\}, \tag{15}$$

with $\varepsilon = \alpha^{-1}q_1$ *and*

$$q_1 = 8(k_{e_1} - ||\overline{\mathcal{A}}_0|| - L||\mathcal{G}||)^2\tilde{z}^2 + 8k_{e_2}^2||\mathcal{B}||^2\overline{u}_{\max}^2 + (2\tilde{e}\tilde{z} + \tilde{e}^2)\lambda_{\max}\{P\} + 2[\tilde{z}c_1 + \tilde{e}c_2 + c_4\tilde{\varphi}]\tilde{e},$$

$$\tilde{z} = \sqrt{2k_x\lambda_{\max}^{-1}\{\mathbf{b}^\top\mathbf{b}\}},\ \overline{u}_{\max} = ||u_{\max}||,$$
$$\tilde{e} = [k_{e_1}(\xi + 1) + k_\xi(||\mathcal{A}_1|| + ||\mathcal{A}_2||)]\tilde{z} + k_\xi[||\mathcal{B}||\overline{u}_{\max} + \lambda_{\max}^{-1/2}\{R_0\}],$$
$$\tilde{\varphi} = \tilde{e} - [2||\overline{\mathcal{A}}_0|| + 2L||\mathcal{G}||]\tilde{z}.$$

for a given $\mathbf{b}$*,* $u_{\max}$*, and*

$$c_1 = (||\tilde{\mathcal{A}}_0|| + ||\tilde{\mathcal{A}}_1|| + ||\tilde{\mathcal{A}}_2||)||P\overline{\mathcal{B}}||\overline{K}, \tag{17a}$$
$$c_2 = (0.5||K_0|| + ||K_1|| + ||K_2||)||P\overline{\mathcal{B}}K_0|| + (||K_0|| + 0.5||K_1|| + ||K_2||)||P\overline{\mathcal{B}}K_1|| + (||K_0|| + ||K_1|| + 0.5||K_2||)||P\overline{\mathcal{B}}K_2||, \tag{17b}$$
$$c_3 = ||\tilde{S}_1^\top P\overline{\mathcal{B}}||\overline{K}, \tag{17c}$$
$$c_4 = ||P\overline{\mathcal{B}}||\overline{K}, \tag{17d}$$

where $\tilde{\mathcal{A}}_0 = \overline{\mathcal{A}}_0 + \overline{\mathcal{B}}K_0$*,* $\tilde{\mathcal{A}}_1 = \mathcal{G}\mathcal{A}_1 + \overline{\mathcal{B}}K_1$*,* $\tilde{\mathcal{A}}_2 = \mathcal{G}\mathcal{A}_2 + \overline{\mathcal{B}}K_2$*, and* $\overline{K} = ||K_0|| + ||K_1|| + ||K_2||$*. Additionally, for* $\overline{P} = \alpha P/q_1 \in \mathbb{R}^{2n\times 2n}$*, with* $P = X_1^{-1}$*, if the following inequalities*

$$b_i^\top P_{\underline{x}}^{-1}b_i \le 1,\ b_i^\top P_{\overline{x}}^{-1}b_i \le 1, \tag{18}$$

hold for $P_{\underline{x}} = \overline{P}_{11} - \overline{P}_{21}^\top\overline{P}_{22}^{-1}\overline{P}_{21}$ *and* $P_{\overline{x}} = \overline{P}_{22} - \overline{P}_{21}^\top\overline{P}_{11}^{-1}\overline{P}_{21}$*, with* $\overline{P}_{11} \in \mathbb{R}^{n\times n}$*,* $\overline{P}_{21} \in \mathbb{R}^{n\times n}$*,* $\overline{P}_{22} \in \mathbb{R}^{n\times n}$*, and*

$$\overline{P} = \begin{pmatrix} \overline{P}_{11} & \overline{P}_{21}^\top \\ \overline{P}_{21} & \overline{P}_{22} \end{pmatrix},$$

then, the trajectories of the system (1) *fulfill the state constraints, i.e.,* $x \in \mathcal{X}$*.*

Due to space limitation, the proofs of all the results are omitted.

C. Design of the Model Predictive Control

For the design of the MPC the following assumption is considered.

Assumption 3. *There exist some gains* $K_0, K_1, K_2 \in \mathbb{R}^{p\times 2n}$*, satisfying the conditions of Lemma 3, and* $\hat{u}_k(t) = K_0\hat{z}_k(t) + K_1\hat{z}_k^+(t) + K_2\hat{z}_k^-(t) \in \mathscr{U}$*, for all* $z_k \in \mathscr{R}$ *and* $t \ge 0$*.*

Assumption 3 implies that the invariant set $\mathscr{R}$, which is the safe set of system (1), is completely contained inside the polytope (2), *i.e.*, the constrained region $\mathcal{X}$.

Let us consider that the prediction can be performed in a receding horizon fashion with a window length $N > 1$. The interval predictor (10) has a sequence of inputs $\mathcal{S}_N = \{s_0, \ldots, s_{N-1}\}$, with $s_m \in \mathscr{U}$ for all $m = \overline{0, N-1}$, the values $\hat{z}_{k,m+1}$ will be calculated for $m = \overline{0, N-1}$ under substitution $u_{k+m} = s_m$. Let us define $\hat{z}_{k+j} = \hat{z}_{k,j}$, for $j = \overline{0, N}$. Thus, the optimal control problem to be solved by the MPC is given as follows:

Problem 1. *For given matrices* $0 \prec W = W^\top \in \mathbb{R}^{2n\times 2n}$*,* $0 \prec H = H^\top \in \mathbb{R}^{2n\times 2n}$*, and* $M = M^\top \in \mathbb{R}^{p\times p}$*, to find the control signals*

$$\mathcal{S}_N^k = \mathrm{argmin} V_N(\hat{z}_{k,0}, \ldots, \hat{z}_{k,N}, \mathcal{S}_N), \tag{19}$$

[2]For more details of this type of convergence rate, please see [20].

with the cost function V_N

$$V_N(\hat{z}_{k,0},\ldots,\hat{z}_{k,N},\mathcal{S}_N) = V_f(\hat{z}_{k,N}) + \sum_{m=0}^{N-1}\ell(\hat{z}_{k,m},s_m),$$

where $V_f(\hat{z}) = \hat{z}^\top W\hat{z}$, $\ell(\hat{z},s) = \hat{z}^\top H\hat{z} + s^\top Ms$, such that the following constraints are satisfied:

a) $\hat{z}_{k+j}$, *for* $j = \overline{0,N}$ *are computed by* (10).
b) $u_k \in \mathscr{U}$.
c) $\hat{z}_{k,N} \in \mathscr{R}$.

Finally, considering the results of Lemma 3 and the solution of the Problem 1 the following theorem is presented.

Theorem 1. *Let the Assumptions 1–3 be satisfied; then, the closed–loop system composed by* (1), (10) *and* (11) *has the following features: i) Recursive feasibility to reach $\mathscr{R}$ in N steps; ii) The origin of the system dynamics* (9) *is practically UAS, in the safe set $\mathscr{R}$; iii) The state and input constraints are not transgressed.*

Note that the proposed MPC is based on the conventional MPC scheme; thus, the proof of Theorem 1 is based on classic results for MPC schemes [4].

V. Simulation Results

Consider the following nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 6+2\sin(x_1) \\ 2\sin(x_2) & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u.$$

It is possible to show that this system satisfies Assumption 1. The matrices A_0 and A_j can be obtained by a convex polythopic method, *e.g.*, that one given in [21]. Let us consider that $u_{\max} = 5$, $x_1 \in (-12,12)$ and $x_2 \in (-5,5)$, and hence the state constraints are $b_1 = (1/12,0)^\top$, $b_2 = -b_1$, $b_3 = (0,1/5)^\top$, and $b_4 = -b_3$. The initial conditions for the system are $x(0) = (2,-1)^\top$ and for the IP are $\hat{z}_k(0) = (-2.5,-1.6,2.5,1.6)^\top$.

All the simulations are done in MATLAB with Euler discretization method, integration step equal to 0.0001 $[s]$ and sampling time $h = 0.1$ $[s]$. The solution for the LMI's are found by means of SDPT3 solver among YALMIP while the MPC is implemented using the nlmpc MATLAB toolbox.

For the matrix inequalities of Lemma 3, selecting $\gamma = 0.1$, and fixing $Q_1 = Q_2 = -0.05I$, the following feasible solution is obtained:

$$P = \begin{pmatrix} 0.6158 & 0.1853 & 0.0375 & -0.1304 \\ 0.1853 & 0.8670 & -0.1304 & -0.4782 \\ 0.0375 & -0.1304 & 0.6158 & 0.1853 \\ -0.1304 & -0.4782 & 0.1853 & 0.8670 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 0.0498 & 0.0089 & 0.0481 & 0.0094 \\ 0.0070 & 0.0392 & 0.0056 & 0.0430 \\ 0.0478 & 0.0054 & 0.0483 & 0.0015 \\ 0.0053 & 0.0363 & 0.0010 & 0.0286 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 0.0499 & 0.0157 & 0.0482 & 0.0073 \\ 0.0129 & 0.0512 & 0.0023 & 0.0337 \\ 0.0480 & 0.0033 & 0.0485 & 0.0083 \\ 0.0020 & 0.0270 & 0.0070 & 0.0406 \end{pmatrix},$$

$$K_0 = \begin{pmatrix} -2.8684 & -5.6117 & -2.8689 & -5.6113 \end{pmatrix},$$
$$K_1 = \begin{pmatrix} -0.1033 & -0.0700 & -0.1061 & -0.0471 \end{pmatrix},$$
$$K_2 = \begin{pmatrix} -0.0039 & 0.0525 & -0.0980 & -0.0724 \end{pmatrix}.$$

Then, with $\alpha = 0.1$ it is possible to verify that condition (14) is fulfilled. For the MPC, let us consider $W = I$, $H = 1000I$, $M = 0.0001$, and $N = 10$.

The system trajectories, the IP (10) trajectories and the corresponding state constraints are depicted in Fig. 1. Note that both the IP and the system trajectories never transgress the system constraints. On the other hand, in Fig. 2 it is noticed that the initial conditions of the system are outside the safe set $\mathscr{R}$, but inside of the state constraints. However, the MPC takes the trajectories of the system towards the safe set $\mathscr{R}$ and they remain inside of it. Additionally, in Fig. 3 it can be seen that the control signal do not transgress the input constraints.

VI. Conclusions

This paper deals with the design of a robust sampled–time controller for the stabilization of continuous–time nonlinear systems, taking into account state and input constraints. The proposed controller comprises the design of a robust control law, which is based on an interval predictor–based state–feedback controller and an MPC approach, that deals with the state and input constraints. A safe set is provided where the state constraints are not violated. The safe set characterizes the region where the MPC is activated, *i.e.*, out of this set, guaranteeing the fulfillment of the state and input constraints. The proposed switched control strategy guarantees the practical Uniform Asymptotic Stability of the considered nonlinear systems without using the system states.

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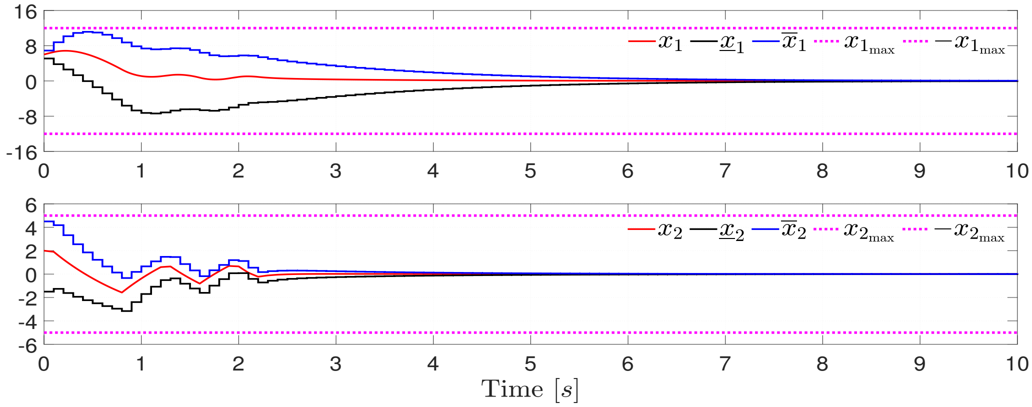

Figure 1. System trajectories and state constraints

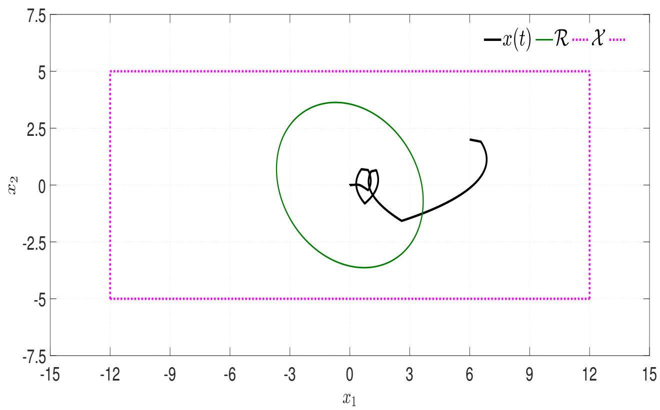

Figure 2. System trajectories, state constraints, and safe set $\mathscr{R}$

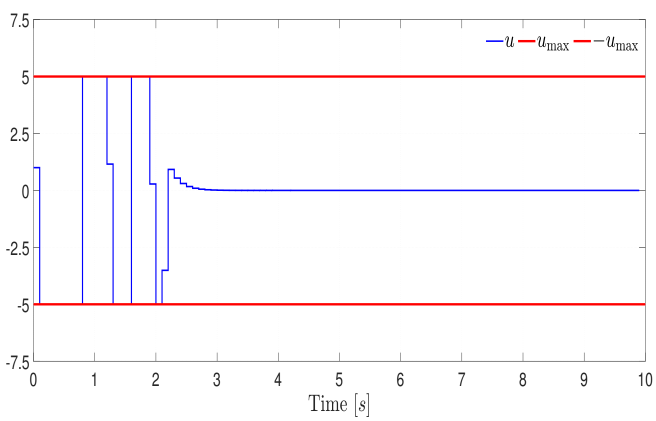

Figure 3. Control signal

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