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Risk-Sensitive Control of Vibratory Energy Harvesters

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***Abstract*— Linear-quadratic-Gaussian (LQG) optimal control theory can be used to maximize the average electrical power generated by a vibratory energy harvester subjected to random disturbances. However, feedback controllers designed using the LQG framework often require large peak power flows for their successful implementation, which may be undesirable for several reasons. In this paper, we propose using a risk-sensitive performance measure to synthesize control laws for stochastic vibratory energy harvesters. The proposed methodology is applied in two examples, in which we show how the risk-sensitive parameter can be systematically tuned to maximize power generation and mitigate excessive power flows. The first example involves a simple single-degree-of-freedom oscillator subjected to a bandpass filtered noise excitation, and the second pertains to ocean wave energy harvesting.**

I. Introduction

The successful extraction of energy from vibratory phenomena is a challenging problem, requiring efficient conversion of mechanical energy to electrical energy. Depending on the scale of the particular application, this harvested energy can be delivered to a grid or stored and used locally. For example, it is possible to generate utility-scale power from ocean waves. On the other hand, it is advantageous to scavenge energy directly from large civil infrastructure in order to power sensing/control systems. Both of these applications concern mechanical systems subjected to random disturbances, and necessitate the use of optimal control theory to maximize average power generation.

Over the past few decades, an immense amount of research has been conducted on technologies to harvest energy from mechanical vibrations. These efforts have mainly focused on small-scale technologies, with power levels less than 1mW and frequencies greater than 25Hz [1]. In this setting, several types of mechanical-to-electrical energy transducers have been successfully implemented, including piezoelectric [2], electromagnetic [3], and electrostatic [4] technologies. Typically, the transducer is embedded within a resonant mechanical system, which is designed such that its resonance frequency coincides with the dominant excitation frequency of the vibration energy to be harvested. The transducer is connected to an isolated power bus or rechargeable storage device, providing a conduit for energy conversion. One important application of low-level vibratory energy harvesting is to power sensors embedded within civil structures, which vibrate when subjected to vehicular and pedestrian traffic loads (see e.g., [2], [5]–[7]). This allows the sensing technologies to be operated in energy-autonomy, which is desirable if it is difficult or costly to physically access the sensors' batteries.

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Vibration energy harvesting is also useful at larger power scales, and at lower frequencies. A principal application in this regime is the generation of utility-scale power from ocean waves. Wave energy converters (WECs) are emerging as a promising alternative to other sources of renewable energy [8]. WECs are typically operated at average power levels greater than 50kW, and at frequencies between $0.05-0.2$Hz. As another example, large-scale transducers can be used to capture energy from the dynamic responses of wind or seismically excited buildings, at power scales above 1kW and frequencies below 1Hz [9]. This harvested energy can, in turn, be used to power feedback control systems for vibration suppression, resulting in closed-loop systems that operate in energy-autonomy [10], [11].

If the vibratory disturbance is a stationary stochastic process, the plant is linear, and the main dissipative losses are quadratic (i.e., I^2R) losses, it has been shown that the optimal energy-harvesting feedback law is the solution to a sign-indefinite Linear-Quadratic-Gaussian (LQG) control problem [12]. The theoretically-optimal feedback law typically requires that the transducer current be controlled continuously via a power-electronic drive using high-frequency pulse-width-modulation. Several subsequent studies have developed related optimal control techniques which account for nonlinearities in the harvester dynamics [13], non-quadratic loss models [14], and non-stationary disturbances [15].

Unfortunately, optimal feedback controllers designed using the LQG framework often require significant bidirectional power flows in order to provide the optimal mean generated power. This is undesirable, as over-designed transduction and power-electronic hardware becomes necessary for the controllers' successful implementation, leading to increased system costs. Reducing so-called peak-to-average-power ratios is a common goal in the wave-energy harvesting community (see e.g., [16]–[20]). In this paper, we consider the use of the more general Linear-Exponential-Quadratic-Gaussian (LEQG) [21], [22] or *risk-sensitive* [23], [24] performance index to design feedback controllers for vibratory energy harvesters. In this setting, one minimizes the expectation of the exponential of an integral-quadratic cost function multiplied by a risk-sensitivity parameter denoted ρ. The sign of ρ determines if the optimizer is *risk-seeking* or *risk-averse*. In the risk-averse case, statistical variation in the cost function is penalized more heavily than in the standard LQG formulation. In the context of energy harvesting, the cost function is the integral of generated power over a given time horizon (finite or infinite). Hence, we see risk-sensitive control as a potential means to improve the consistency of the quantity of harvested energy and consequently reduce large peak power flows.

The paper is organized as follows. In Section II, we provide the modeling assumptions for the plant and disturbance. In Section III, we introduce the sign-indefinite, risk-sensitive performance index and derive optimal full-state feedback control laws for the finite-time and infinite-time horizon cases. Importantly, we show that optimal controls always exist in the risk-averse case. This is quite different than in standard applications of risk-sensitive control, in which the optimization "breaks down" and produces infinite cost at some finite value of the risk-sensitive parameter ρ [23]. Section IV provides two examples of risk-sensitive energy harvesting: one involving a single-degree-of-freedom (SDOF) oscillator subjected to filtered noise excitation, and another related to ocean wave energy harvesting. Finally, Section V contains some brief conclusions and a discussion of future work.

II. Modeling and Assumptions

We assume that the vibratory energy harvester under consideration can be adequately modeled as a finite-dimensional, linear time-invariant (LTI) system $\mathcal{P}$, with state-space realization

$$\mathcal{P}:\begin{cases} \frac{d}{dt}x_p(t) = & A_p x_p(t) + B_{pu}u(t) + B_{pa}a(t) \\ \quad v(t) = & C_{pv}x_p(t) \end{cases} \tag{1}$$

where x_p is the state vector and a is the disturbance vector. In addition, $u(t)$ is the vector of "flow" variables (e.g., current, force, etc.) and $v(t)$ is the corresponding vector of colocated "effort" variables (e.g., voltage, velocity, etc.). As such, the inner product $u^T(t)v(t)$ has units of power. We assume that matrix A_p is Hurwitz and mapping $u \mapsto v$ is passive.

We model the exogenous disturbance a as stationary Gauss-Markov process, with stochastic state-space

$$\mathcal{A}:\begin{cases} dx_a(t) = & A_a x_a(t)dt + B_a dw(t) \\ \quad a(t) = & C_a x_a(t) \end{cases} \tag{2}$$

where x_a is the process state vector, w is a zero-mean Wiener process with $\frac{d}{dt}\mathcal{E}\{ww^T\} = I$, and matrix A_a is Hurwitz.

We augment $\mathcal{P}$ and $\mathcal{A}$ to obtain the composite stochastic state-space model

$$\mathcal{S}:\begin{cases} dx(t) = & (Ax(t) + B_u u(t))\,dt + B_w dw(t) \\ \quad v(t) = & C_v x(t) \end{cases} \tag{3}$$

where the state vector $x \triangleq \begin{bmatrix} x_p^T & x_a^T \end{bmatrix}^T$ and

$$A = \begin{bmatrix} A_p & B_{pa}C_a \\ 0 & A_a \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ B_a \end{bmatrix}, \quad B_u = \begin{bmatrix} B_{pu} \\ 0 \end{bmatrix} \tag{4}$$

$$C_v = \begin{bmatrix} C_{pv} & 0 \end{bmatrix}. \tag{5}$$

Since A_p and A_a are Hurwitz, it follows that A is Hurwitz. Finally, we assume that (A, C_v) is observable and $\left(A, \begin{bmatrix} B_u & B_w \end{bmatrix}\right)$ is controllable.

The dissipative losses in the transducers and power electronics can be approximately modeled as a quadratic form, i.e., $P_{loss}(t) = u^T(t)Ru(t)$ where matrix R is diagonal and positive definite. The instantaneous generated electrical power is then given by

$$P_{gen}(t) = -u^T(t)v(t) - u^T(t)Ru(t) \tag{6}$$

Over a finite-time horizon $[0, T]$, the harvested energy is equal to the integral of the generated power, i.e.,

$$E_h(T) = \int_0^T P_{gen}(t)dt. \tag{7}$$

We conclude this section with a variant of the Positive Real Lemma.

Lemma 1: If $u \mapsto v$ is passive, A is Hurwitz, and $R = R^T > 0$, it follows that the Riccati equation

$$0 = A^T W + WA + \left(B_u^T W - \tfrac{1}{2}C_v\right)^T R^{-1}\left(B_u^T W - \tfrac{1}{2}C_v\right) \tag{8}$$

has a unique solution $W = W^T > 0$, for which $A + B_u R^{-1}(B_u^T W - \frac{1}{2}C_v)$ is Hurwitz.

Proof: First note that if $u \mapsto v$ is passive, then its transfer function is positive-real. But if this is the case, A is Hurwitz, and $R > 0$, then it follows that the mapping $u \mapsto v + \frac{1}{2}Ru$ has a transfer function that is strictly positive-real. Next, assume the state space is partitioned so as to isolate the subspace that is controllable from u, i.e.,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad B_u = \begin{bmatrix} B_{u1} \\ 0 \end{bmatrix}, \tag{9}$$

$$C_v = \begin{bmatrix} C_{v1} & C_{v2} \end{bmatrix} \tag{10}$$

Assume W is similarly partitioned. Then parameters $\{A_{11}, B_{u1}, \frac{1}{2}C_{v1}, \frac{1}{2}R\}$ correspond to a minimal state space realization, which is known to be strictly positive-real. It is then a standard result of the Positive-Real Lemma that the Riccati equation

$$\begin{aligned} 0 = {} & A_{11}^T W_{11} + W_{11}A_{11} \\ & + \left(B_{u1}^T W_{11} - \tfrac{1}{2}C_{v1}\right)^T R^{-1}\left(B_{u1}^T W_{11} - \tfrac{1}{2}C_{v1}\right) \end{aligned} \tag{11}$$

has a unique solution $W_{11} = W_{11}^T > 0$, and that $A_{11} + B_{u1}R^{-1}\left(B_{u1}^T W_{11} - \frac{1}{2}C_{v1}\right)$ is Hurwitz. To show that terms W_{12} and W_{22} exist satisfying (8), we simply note that the resultant partitioned equation for W_{12} is a Sylvester equation which is guaranteed to have a unique solution because A_{22} and $A_{11} - B_{u1}R^{-1}\left(B_{u1}^T W_{11} - \frac{1}{2}C_{v1}\right)$ are both Hurwitz. The remaining equation for W_{22} is then a Lyapunov equation, which is guaranteed to have a unique solution because A_{22} is Hurwitz. This proves that $W = W^T$ exists. The fact that $A + B_u R^{-1}(B_u^T W - \frac{1}{2}C_v)$ is Hurwitz can be seen from the fact that it is block-upper-triangular, with diagonal blocks equal to $A_{11} + B_{u1}R^{-1}(B_{u1}^T W_{11} - \frac{1}{2}C_{v1})$ and A_{22}, both of which are Hurwitz. The fact that $W \geqslant 0$ comes from the fact that A is Hurwitz, and consequently all solutions to (8) must be positive-semidefinite. Positive-definiteness can be proved by contradiction. Let $W\eta = 0$ for $\eta \neq 0$. Then it follows via a quadratic form on (8) that $C_v\eta = 0$. If this is true then multiplying (8) from the right by η gives that $WA\eta = 0$ which implies that $A\eta$ is also in the null space of W. Consequently, we conclude that the null space of W must be A-invariant. But if this is the case, then it follows that there exists an eigenvector of A that is in the null space of W, and for which is also in the null space of C_v. This violates the assumption that (A, C_v) is observable, leading to a contradiction. ∎

III. Control Formulation

A. Optimal power generation

Let $\bar{P}_{gen} \triangleq \mathcal{E}\{P_{gen}\}$ denote the expected value of the generated power in stationarity. We will first consider the problem of synthesizing a full-state feedback controller such that $\bar{P}_{gen}$ is maximized in closed-loop. Equivalently, we seek to minimize the quantity

$$-\bar{P}_{gen} = \mathcal{E}\left\{x^T C_v^T u + u^T R u\right\} = \lim_{T\to\infty} \frac{1}{T}\,\mathcal{E}\left\{\int_0^T x^T(t) C_v^T u(t) + u^T(t) R u(t) dt\right\}. \tag{12}$$

This is an infinite-time horizon, LQ-optimal control problem, with a sign-indefinite cost function. The optimal controller in this case is given in the following theorem.

Theorem 1 ([12]): Given composite system $\mathcal{S}$, the optimal causal mapping $x \mapsto u$ (linear or nonlinear) that maximizes $\bar{P}_{gen}$ is feedback law

$$u(t) = Kx(t) = R^{-1}\left(B_u^T W - \tfrac{1}{2} C_v\right) x(t) \tag{13}$$

where W is as in (8). Furthermore, the matrix $(A + B_u K)$ is Hurwitz and the optimal mean generated power is

$$\bar{P}_{gen}^{*} = \mathrm{tr}\{B_w^T W B_w\} \tag{14}$$

We note that the controller given in (13) is agnostic to the magnitude of the disturbance, i.e., matrix B_w does not appear in (8). This is consistent with standard LQG theory.

B. Finite horizon, risk-sensitive energy harvesting

Next, we consider the minimization of the following risk-sensitive performance objective function:

$$J_\rho \triangleq \frac{1}{T}\frac{2}{\rho}\log\left\{\mathcal{E}\left\{\exp\left\{\frac{\rho}{2}\int_0^T -P_{gen}(t)dt\right\}\right\}\right\} \tag{15}$$

where constant scalar ρ is the risk-sensitivity parameter. Here we only consider the risk-averse case, i.e., when $\rho > 0$. In this case, the exponential more heavily penalizes realizations of P_{gen} that result in large energy *injection*.

To get a better sense of the meaning of metric J_ρ, recall that for an arbitrary random variable X, $f(\alpha) = \log\{\mathcal{E}\{\exp\{\alpha X\}\}$ is the cumulant-generating function of X [25]. As such, (15) can be rewritten as the Taylor series expansion

$$J_\rho = \frac{1}{T}\frac{2}{\rho}\sum_{n=1}^{\infty}\kappa_n\left\{\int_0^T P_{gen}(t)dt\right\}\frac{(-\rho)^n}{2^n n!} \tag{16}$$

$$= \frac{1}{T}\sum_{n=1}^{\infty}\kappa_n\left\{E_h(T)\right\}\frac{(-1)^n\rho^{n-1}}{2^{n-1}n!} \tag{17}$$

where $\kappa_n\{E_h(T)\}$ is the n^{th} cumulant of $E_h(T)$. Thus, J_ρ is a linear, weighted combination of the cumulants of $E_h(T)$, where the even cumulants have positive weights and the odd cumulants have negative weights.

Recalling that the first two cumulants are equal to mean and variance, it follows that when ρ is small

$$J_\rho \approx \frac{1}{T}\left(-\mathcal{E}\left\{E_h(T)\right\} + \frac{\rho}{4}\mathrm{Var}\left\{E_h(T)\right\}\right) \tag{18}$$

Consequently, by minimizing J_ρ we can approximately maximize the average energy harvested over the time horizon while also penalizing statistical variation in $E_h(T)$. In some sense, this allows us to achieve better consistency in the quantity of harvested energy. Note that when $\rho = 0$, the problem becomes equivalent to minimizing the finite-time horizon version of (12). When ρ is large, J_ρ becomes more difficult to interpret, as importance is shifted to higher-order cumulants of $E_h(T)$ in the optimization. However, as ρ is increased, minimization of J_ρ should yield feedback control laws that result in decreasing power injection.

Theorem 2: Given composite system $\mathcal{S}$ with initial condition $x(0) = x_0$, loss parameter R, risk-sensitive parameter $\rho > 0$, and time horizon T, J_ρ has a finite minimum. The optimal control that minimizes J_ρ is given by

$$u(t) = -R^{-1}\left(B_u^T S(t) + \tfrac{1}{2}C_v\right)x(t) \tag{19}$$

where $S(t)$ satisfies the Riccati equation

$$\tfrac{d}{dt}S(t) + A^T S(t) + S(t)A - \left(B_u^T S(t) + \tfrac{1}{2}C_v\right)^T R^{-1}\left(B_u^T S(t) + \tfrac{1}{2}C_v\right) + \rho S(t) B_w B_w^T S(t) = 0 \tag{20}$$

with final value $S(T) = 0$. The optimal performance is equal to

$$J_\rho^{*} = \frac{1}{T}\left(x_0^T S(0) x_0 + \int_0^T \mathrm{tr}\left\{B_w^T S(t) B_w\right\}dt\right) \tag{21}$$

Proof: The first part of this proof is a variant of a classical result. Similar derivations may be found in [21], [26], and [27].

Expressions (19) – (21) are derived by solving a stochastic Hamilton-Jacobi-Bellman (HJB) equation. First, define

$$V(\xi, t) \triangleq \mathcal{E}\left\{\exp\left\{\frac{\rho}{2}\int_t^T -P_{gen}(\tau)d\tau\right\}\;\middle|\; x(t) = \xi\right\} \tag{22}$$

and then let $V^*(\xi,t)$ denote the optimal value of (22), i.e., $V^*(\xi,t) \triangleq \min_u V(\xi,t)$. The pertinent HJB equation is

$$-\frac{\partial V^*(\xi,t)}{\partial t} = \min_u \Bigg\{ \frac{\rho}{2}\left(\xi^T C_v^T u + u^T R u\right) V^*(\xi,t) + \left(\frac{\partial V^*(\xi,t)}{\partial \xi}\right)^T (A\xi + B_u u) + \frac{1}{2}\operatorname{tr}\left\{\frac{\partial^2 V^*(\xi,t)}{\partial \xi^2} B_w B_w^T\right\} \Bigg\} \tag{23}$$

Minimization of the right-hand-side of (23) implies that the optimal control is given by

$$u^* = -\frac{1}{2}R^{-1}\left(\frac{2}{\rho V^*(\xi,t)} B_u^T \frac{\partial V^*(\xi,t)}{\partial \xi} + C_v \xi\right) \tag{24}$$

We assume the value function to be of the form

$$V^*(\xi,t) = \exp\left\{\frac{\rho}{2}\left(\xi^T S(t)\xi + c(t)\right)\right\} \tag{25}$$

The relevant partial derivatives of (25) are

$$\frac{\partial V^*(\xi,t)}{\partial \xi} = \rho \exp\left\{\tfrac{1}{2}\rho\left(\xi^T S(t)\xi + c(t)\right)\right\} S(t)\xi$$

$$\frac{\partial^2 V^*(\xi,t)}{\partial \xi^2} = \rho \exp\left\{\tfrac{1}{2}\rho\left(\xi^T S(t)\xi + c(t)\right)\right\} \times \left(\rho S(t)\xi\xi^T S(t) + S(t)\right)$$

$$\frac{\partial V^*(\xi,t)}{\partial t} = \rho \exp\left\{\frac{1}{2}\rho\left(\xi^T S(t)\xi + c(t)\right)\right\} \times \tfrac{1}{2}\left(\xi^T \tfrac{d}{dt}S(t)\xi + \tfrac{d}{dt}c(t)\right)$$

After making the appropriate substitutions, (24) simplifies to (19). In addition, (23) becomes

$$\begin{aligned}
&\tfrac{1}{2}\left(\xi^T \tfrac{d}{dt}S(t)\xi + \tfrac{d}{dt}c(t)\right) \\
&\quad - \tfrac{1}{2}\xi^T C_v^T R^{-1}\left(B_u^T S(t) + \tfrac{1}{2}C_v\right)\xi \\
&\quad + \tfrac{1}{2}\xi^T\left(B_u^T S(t) + \tfrac{1}{2}C_v\right)^T R^{-1}\left(B_u^T S(t) + \tfrac{1}{2}C_v\right)\xi \\
&\quad + \xi^T S(t)\left(A - B_u R^{-1}\left(B_u^T S(t) + \tfrac{1}{2}C_v\right)\right)\xi \\
&\quad + \tfrac{1}{2}\operatorname{tr}\left\{\rho S(t)\xi\xi^T S(t) B_w B_w^T + S(t) B_w B_w^T\right\} = 0
\end{aligned}$$

which may be simplified further to

$$\begin{aligned}
\xi^T\Big(&\tfrac{d}{dt}S(t) + S(t)A + A^T S(t) + \rho S(t) B_w B_w^T S(t) \\
&- \left(B_u^T S(t) + \tfrac{1}{2}C_v\right)^T R^{-1}\left(B_u^T S(t) + \tfrac{1}{2}C_v\right)\Big)\xi \\
&+ \tfrac{d}{dt}c(t) + \operatorname{tr}\{B_w^T S(t) B_w\} = 0
\end{aligned} \tag{26}$$

For (26) to hold for all ξ, it must be the case that (20) and

$$\tfrac{d}{dt}c(t) = -\operatorname{tr}\left\{B_w^T S(t) B_w\right\} \tag{27}$$

hold for all $t \in [0,T]$. The boundary conditions $S(T) = 0$ and $c(T) = 0$ are imposed from the fact that $V^*(\xi,T) = 1$. Finally, the optimal performance (21) is computed by substituting $V^*(x_0,0)$ into (15), with $S(0)$ found via backward integration of (20) and $c(0)$ found by subsequently integrating (27).

To show that (20) has a solution for all positive ρ, we exploit the passivity of the mapping $u \mapsto v$. First we show that $S(t)$ is monotonically decreasing in reverse-time. Next we show that $S(t)$ is bounded from below. These two facts together ensure the existence of a bounded solution $S(t)$ for all $t \in [0,T]$.

To show that $S(t)$ is monotonically decreasing in reverse-time, we note that from Lemma 1, that there exists a matrix $W = W^T > 0$ satisfying (8). Define $\Xi(t) = S(t) + W$, noting that $\Xi(T) = W > 0$. Then $\Xi(t)$ obeys

$$\begin{aligned}
-\tfrac{d}{dt}\Xi(t) = \tilde{A}^T\Xi(t) + \Xi(t)\tilde{A} \\
+ \Xi(t)\left[\rho B_w B_w^T - B_u R^{-1} B_u^T\right]\Xi(t) \\
+ \rho W B_w B_w^T W
\end{aligned} \tag{28}$$

where $\tilde{A} \triangleq A + B_u R^{-1}\left(B_u^T W - \frac{1}{2}C_v\right) - \rho B_w B_w^T W$. Define $\Theta(t) = \Xi^{-1}(t)$ and we have that

$$\begin{aligned}
\tfrac{d}{dt}\Theta(t) = \Theta(t)\tilde{A}^T + \tilde{A}\Theta(t) + \rho B_w B_w^T - B_u R^{-1} B_u^T \\
+ \Theta(t)\left[\rho W B_w B_w^T W\right]\Theta(t)
\end{aligned} \tag{29}$$

Take a second derivative, to get that

$$\begin{aligned}
\tfrac{d^2}{dt^2}\Theta(t) = \tfrac{d}{dt}\Theta(t)\left[\tilde{A} + \Theta(t)\rho W B_w B_w^T W\right]^T \\
+ \left[\tilde{A} + \Theta(t)\rho W B_w B_w^T W\right]\tfrac{d}{dt}\Theta(t)
\end{aligned} \tag{30}$$

with final-value condition $\frac{d}{dt}\Theta(T) = -\frac{1}{4}W^{-1}C_v^T R^{-1} C_v W^{-1}$. Let interval $t \in (t_1, T]$ be the largest interval over which $\Xi(t)$ is bounded. Then over this interval we have that

$$\tfrac{d}{dt}\Theta(t) = -\Phi(t,T)W^{-1}C_v^T R^{-1} C_v W^{-1}\Phi^T(t,T) \tag{31}$$

where $\Phi(t,T)$ is the state transition matrix satisfying

$$\tfrac{d}{dt}\Phi(t,T) = \left[\tilde{A} + \Theta(t)\rho W B_w B_w^T W\right]\Phi(t,T) \tag{32}$$

with boundary condition $\Phi(T,T) = I$. We conclude that $\frac{d}{dt}\Theta(t) \leqslant 0$ on $t \in (t_1,T]$, and consequently, that $\frac{d}{dt}\Xi(t) \geqslant 0$ on the same interval. As such, $\Xi(t)$ is monotonically decreasing in reverse-time. But $\Xi(t) = S(t) + W$, so we conclude the same for $S(t)$.

To show that $S(t)$ is bounded from below, we first decompose $v(\tau)$ for $\tau \in [t,T]$ into components due to $u(\tau)$, $w(\tau)$, and $x(t) = \xi$, as

$$v_u(\tau) = \int_t^\tau C_v \exp\{A(\tau-\theta)\} B_u u(\theta) d\theta \tag{33}$$

$$v_w(\tau) = \int_t^\tau C_v \exp\{A(\tau-\theta)\} B_w w(\theta) d\theta \tag{34}$$

$$v_\xi(\tau) = C_v \exp\{A(\tau-t)\}\xi \tag{35}$$

Then we have that because $u \mapsto v$ is passive, it is the case that

$$\int_t^T u^T(\tau) v_u(\tau) d\tau \geqslant 0 \tag{36}$$

and consequently

$$\int_t^T -P_{gen}(\tau)d\tau \geqslant \int_t^T \big[u^T(\tau)Ru(\tau) + u^T(\tau)v_w(\tau) + u^T(\tau)v_\xi(\tau)\big]d\tau \quad (37)$$

Completing the square,

$$\int_t^T -P_{gen}(\tau)d\tau \geqslant \int_t^T \|u(\tau) + \tfrac{1}{2}R^{-1}(v_w(\tau) + v_\xi(\tau))\|_R^2 d\tau - \int_t^T \tfrac{1}{4}\|v_w(\tau) + v_\xi(\tau)\|_{R^{-1}}^2 d\tau \quad (38)$$

$$\geqslant - \int_t^T \tfrac{1}{4}\left(\|v_w(\tau)\|_{R^{-1}}^2 + \|v_\xi(\tau)\|_{R^{-1}}^2\right) \quad (39)$$

Because A is Hurwitz, there exists constant χ such that

$$\int_0^T \|v_\xi(\tau)\|_{R^{-1}}^2 d\tau \leqslant \chi\|\xi\|^2 \quad (40)$$

Consequently we have that

$$V(\xi,t) \geqslant \mathcal{E}\left\{\exp\left\{-\frac{\rho}{8}\left(\int_0^T \|v_w(\tau)\|_{R^{-1}}^2 d\tau + \chi\|\xi\|^2\right)\right\}\right\} \quad (41)$$

$$= \delta(\rho)\exp\left\{-\frac{\rho\chi}{8}\|\xi\|^2\right\} \quad (42)$$

where we have that

$$\delta(\rho) = \mathcal{E}\left\{\exp\left\{-\frac{\rho}{8}\int_0^T \|v_w(\tau)\|_{R^{-1}}^2\right\}\right\}. \quad (43)$$

Because v_w is Gaussian distributed with finite variance, $\delta(\rho) > 0$ for all $\rho > 0$. Consequently we have that

$$\xi^T S(t)\xi + c(t) \geqslant \frac{2}{\rho}\log\delta(\rho) - \frac{\chi}{4}\|\xi\|^2 \quad (44)$$

That this must hold for all ξ assures that $S(t)$ has a finite lower bound, thus completing the proof. ■

C. *Infinite horizon, risk-sensitive energy harvesting*

Next, we consider the case in which the time horizon is taken to be $[0,\infty)$, and the performance index is defined as

$$J_{\rho,\infty} \triangleq \lim_{T\to\infty} J_\rho \quad (45)$$

Theorem 3: Given composite system $\mathcal{S}$, loss parameter R, and risk-sensitive parameter $\rho > 0$, the optimal control that minimizes $J_{\rho,\infty}$ is given by

$$u(t) = \bar{K}x(t) = -R^{-1}\left(B_u^T\bar{S} + \tfrac{1}{2}C_v\right)x(t) \quad (46)$$

where $\bar{S} = \bar{S}^T$ is the unique stabilizing solution to algebraic Riccati equation

$$A^T\bar{S} + \bar{S}A - \left(B_u^T\bar{S} + \tfrac{1}{2}C_v\right)^T R^{-1}\left(B_u^T\bar{S} + \tfrac{1}{2}C_v\right) + \rho\bar{S}B_wB_w^T\bar{S} = 0. \quad (47)$$

In addition, the optimal performance is

$$J_{\rho,\infty}^* = \operatorname{tr}\left\{B_w^T\bar{S}B_w\right\}. \quad (48)$$

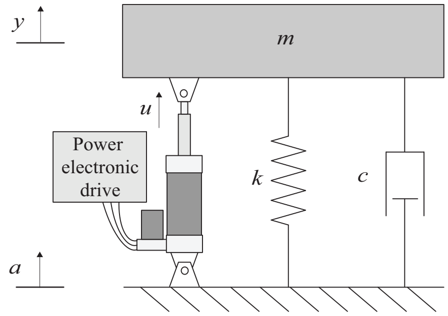

Fig. 1. SDOF energy harvester

and $\left(A + B_u\bar{K}\right)$ is Hurwitz.

Proof: In the proof to Theorem 2 it is shown that the solution $S(t)$ to (20) is monotonically decreasing in reverse-time, starting from a final value of $S(T) = 0$, and that it is bounded from below. It follows from these two facts that $S(t)$ has a well-defined limit $\bar{S}$ as $t \to -\infty$. Equivalently, $S(0)$ has a well-defined limit $\bar{S}$ as $T \to \infty$. It remains only to show that $\bar{S}$ satisfies algebraic Riccati equation (47), or equivalently, that $\frac{d}{dt}S(t) \to 0$ as $t \to -\infty$ for the solution to (20). This can be proven through a continuity argument identical to that used in [28] (Lemma 3.7.7). ■

IV. Examples

In this section, we explore the application of infinite-time horizon, risk-sensitive control in two examples.

A. *SDOF oscillator*

Consider the SDOF vibratory energy harvester portrayed in Figure 1. It consists of a mechanical oscillator, with mass m, stiffness k, and viscous damping coefficient c, coupled with an electromechanical transducer. The dynamics of the oscillator are governed by the following differential equation

$$\frac{d^2}{dt^2}y(t) + 2\zeta\omega_0\frac{d}{dt}y(t) + \omega_0^2 y(t) = -a(t) + \frac{1}{m}u(t) \quad (49)$$

where $y(t)$ is the relative displacement of the mass, $a(t)$ is the stochastic base acceleration, $\omega_0 \triangleq \sqrt{k/m}$ is the oscillator's natural frequency, and $\zeta \triangleq c/(2m\omega_0)$ is the damping ratio. The electromechanical force $u(t)$ and relative velocity $v(t) = \frac{d}{dt}y(t)$ are assumed to be proportional to the transducer current and back-EMF voltage, respectively. The transducer is connected to a power-electronic drive, which is used regulate the current and track desired control forces at high bandwidth. We note that the parasitic loss parameter R has units of power/force^2.

For this example, the plant state-space matrices are

$$A_p = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}, \quad B_{pu} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \quad (50)$$

$$B_{pa} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C_{pv} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (51)$$

We assume disturbance $a(t)$ has a second-order bandpass spectrum, i.e., $\mathcal{A}$ has state-space matrices

$$A_a = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta_a\omega_a \end{bmatrix},\ B_a = \begin{bmatrix} 0 \\ 2\sigma_a\sqrt{\zeta_a\omega_a} \end{bmatrix},\ C_a = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{52}$$

where ω_a is the passband frequency of the spectrum, σ_a is the disturbance intensity, and ζ_a is the damping ratio.

We consider the design of an infinite horizon, risk-sensitive controller for this system, and study the effect of ρ on the resulting feedback control law and the closed-loop performance characteristics. We take the system parameters to be $m = 1$, $\omega_0 = \omega_a = 2\pi$, $\zeta = 0.01$, $\zeta_a = 0.1$, $\sigma_a = 1$, and $R = \frac{1}{4}$. Since the closed-loop dynamics are LTI, we can directly compute $\bar{P}_{gen}$ as follows

$$\bar{P}_{gen} = -\operatorname{tr}\{\tilde{C}X\} \tag{53}$$

where $\tilde{C} = C_v^T K + K^T R K$ and state covariance matrix $X = X^T > 0$ is the solution to Lyapunov equation

$$(A + B_u\bar{K})X + X(A + B_u\bar{K})^T + B_w B_w^T = 0 \tag{54}$$

In addition, we define the root-mean-square (RMS) generated power as

$$\mathrm{RMS}\{P_{gen}\} \triangleq \sqrt{\mathcal{E}\{P_{gen}^2\}} \tag{55}$$

This metric can be used to assess the magnitude of fluctuation in P_{gen}. Ideally, we would like the ratio $\mathrm{RMS}\{P_{gen}\}/\bar{P}_{gen}$ to be small, as that reduces the need for overrated hardware. It is straightforward to show, via some matrix algebra, that

$$\mathrm{RMS}\{P_{gen}\} = \sqrt{\operatorname{tr}\left\{\tilde{C}X\right\}^2 + \operatorname{tr}\left\{\tilde{C}X\left(\tilde{C} + \tilde{C}^T\right)X\right\}} \tag{56}$$

We computed $J_{\rho,\infty}^*$, $\bar{P}_{gen}$, and $\mathrm{RMS}\{P_{gen}\}$ for $\rho \in [0, 10^{20}]$, and report these results in Figure 2. There are some obvious trends. First, note that for $\rho = 0$, we have $-J_{\rho,\infty}^* = \bar{P}_{gen} = \bar{P}_{gen}^*$, as expected. As ρ is increased $-J_{\rho,\infty}^*$ and $\bar{P}_{gen}$ diverge, with $J_{\rho,\infty}^* \to 0$ and $\bar{P}_{gen}$ approaching a constant value, which corresponds to the control law given in the Theorem 4 below. Furthermore, in the bottom plot in Figure 2 we see that the ratio of RMS to mean generated power decreases as ρ is increased, illustrating how tuning ρ can be used to reduce large power fluctuations.

Theorem 4: Let $u^*(t)$ denote the optimal feedback law given in (46) for composite system defined by plant matrices (50) – (51) and disturbance matrices (52) with $\omega_0 = \omega_a$ and loss parameter R. Then

$$\lim_{\rho\to\infty} u^*(t) = \begin{bmatrix} 0 & -\frac{1}{2R} & 0 & 0 \end{bmatrix} x(t) = -\frac{1}{2R}v(t). \tag{57}$$

Proof: This result can be obtained by solving (47) symbolically, taking the limit as $\rho \to \infty$ which yields $\bar{S} = 0$, and substituting this into (46). These calculations result in intermediate expressions that are rather lengthy and thus are omitted for the sake of the reader. ■

This result can be interpreted as follows: the most risk-averse control law for this particular energy-harvesting system is synthetic viscous damping with a coefficient which only depends on the loss parameter R, and does not take into consideration the disturbance or plant dynamics. Furthermore, control law (57) does not result in bidirectional power flows, i.e., power is only absorbed, not injected.

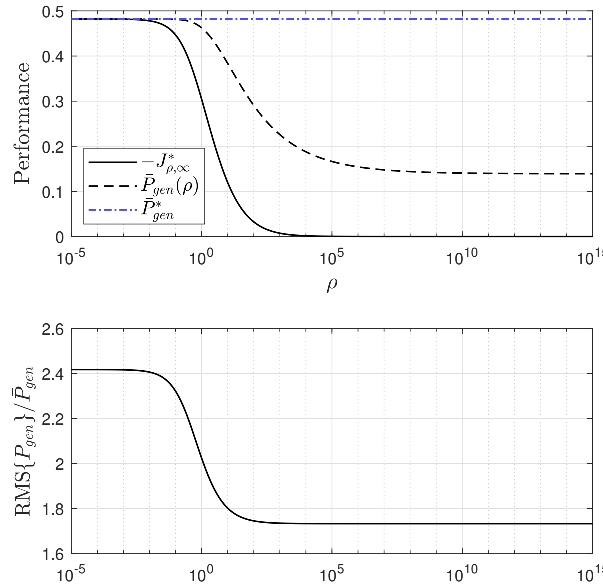

Fig. 2. Effect of risk-sensitive parameter ρ on performance index $J_{\rho,\infty}$ and mean generated power (top); and effect of ρ on ratio of root-mean-square (RMS) generated power to mean generated power for SDOF energy harvester (with $m = 1$, $\omega_0 = \omega_a = 2\pi$, $\zeta = 0.01$, $\zeta_a = 0.1$, $\sigma_a = 1$, and $R = \frac{1}{4}$)

Interestingly, it can also be shown for this particular system that if $\omega_0 = \omega_a$, then for any $\rho \geq 0$ the state feedback gain has the form

$$\bar{K} = \begin{bmatrix} 0 & \bar{k}_{12} & 0 & \bar{k}_{14} \end{bmatrix} \tag{58}$$

as illustrated in Figure 3. This implies that the only measurements needed to implement the controller are the transducer velocity $v(t)$ and the base acceleration $a(t)$. Due to space constraints, we do not include a proof of this statement but note that this same result holds in the LQG case and direct the reader to [29] for an analogous proof of that result. We stress that $\bar{K}$ does not generally have this special form if $\omega_0 \neq \omega_a$.

B. Wave energy harvesting

To demonstrate another application of risk-sensitive energy harvesting, consider the wave energy converter (WEC) system illustrated in Figure 4. The WEC consists of a floating, slack-moored buoy, in which a tuned vibration absorber (TVA) is housed. It is presumed that the mooring system restrains the buoy motion to heave. The TVA is comprised of a mass-spring-dashpot assembly, with the transducer (usually referred to as the power take-off or PTO in the wave-energy literature) located between the mass and the buoy. The dynamics of the mass along the buoy's vertical axis, together with the heave motion of the buoy, constitute a 2DOF vibratory system.

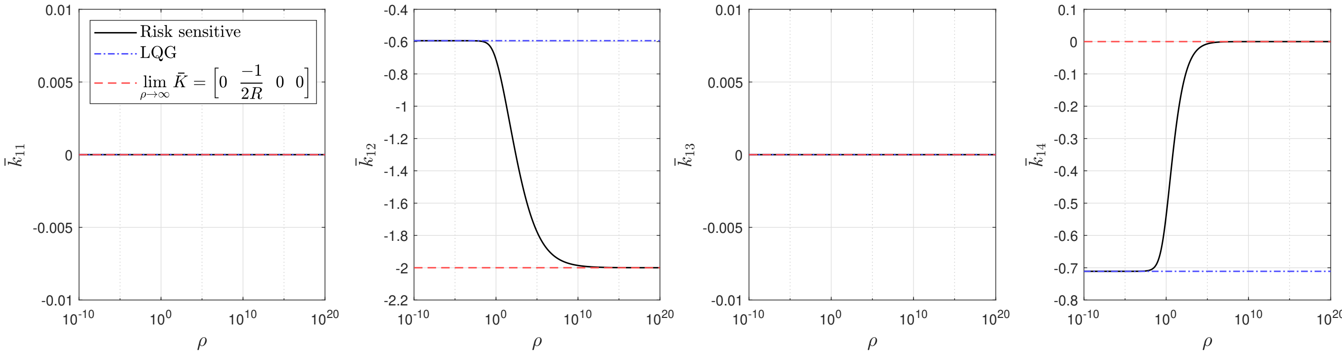

Fig. 3. Effect of risk-sensitive parameter ρ on infinite-time horizon feedback gain $\bar{K}$ for SDOF energy harvester (with $m = 1$, $\omega_0 = \omega_a = 2\pi$, $\zeta = 0.01$, $\zeta_a = 0.1$, $\sigma_a = 1$, and $R = \frac{1}{4}$)

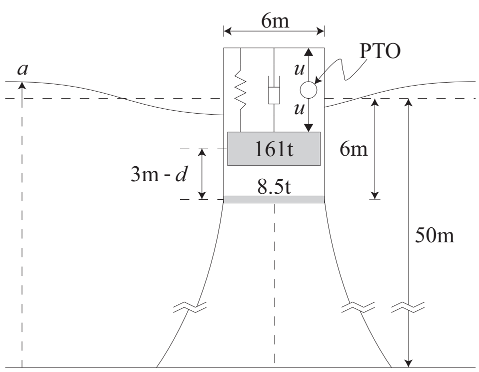

Fig. 4. Example WEC system

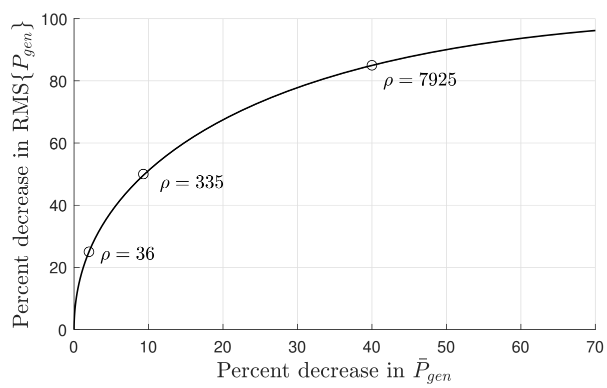

Fig. 5. Percent reduction in root-mean-square (RMS) generated power compared to percent reduction in mean generated power parametrized by increasing ρ values for WEC example

The masses of the buoy and TVA are such that the system is in hydrostatic equilibrium in the configuration shown in Figure 4 (with displacement $d = 0$). The fundamental vibratory mode of the system (including the added mass of the displaced fluid) has a natural period of approximately 9s, and a damping ratio of 0.5%. The loss parameter of the PTO was chosen as $R = 50\text{kW/MN}^2$. This value is reasonable if the PTO was a permanent-magnet synchronous machine.

We assume the free surface wave elevation $a(t)$ is characterized by a Pierson-Moskowitz spectrum [30], i.e.,

$$S_a(\omega) = c_a \left|\frac{\omega_p}{\omega}\right|^5 \exp\left\{-\frac{5}{4}\left(\frac{\omega_p}{\omega}\right)^4\right\} \tag{59}$$

which is parametrized by its significant wave height H_s and peak wave period T_p, with these parameters determining c_a and ω_p in (59). We have that $T_p = \frac{2\pi}{\omega_p}$ and $H_s = 4\sigma_a$, with $\sigma_a^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_a(\omega)d\omega$. We presume that $T_p = 9$s and $H_s = 1$m. The reader is directed to [31] for additional information regarding the modeling of this system.

We conducted a similar sensitivity analysis for this system. Figure 5 illustrates how ρ can be tuned to obtain controllers that result in smaller peak power flows. For example, with $\rho = 335$, we achieve more than a 50% reduction in RMS generated power while only reducing the mean generated power less than 10%. Simulated realizations of P_{gen} corresponding to $\rho = 0$ and $\rho = 335$, over the span of one hour, are shown in Figure 6. There is a clear reduction in power fluctuation as compared to the optimal LQG controller.

V. Conclusions

In this paper, we have examined the application of risk-sensitive control theory to the problem of vibratory energy harvesting. We showed that the corresponding risk-sensitive Riccati equation always has a solution, and hence optimal feedback controllers always exist, in the risk-averse case. We provided two examples which demonstrated how the risk-sensitive parameter ρ can be tuned to reduce the variance of the generated power. A straightforward extension of this work is to consider the output feedback case. In addition, more extensive numerical studies should be performed to better quantify the effect of ρ on other statistical quantities related to power and energy beyond those discussed herein.

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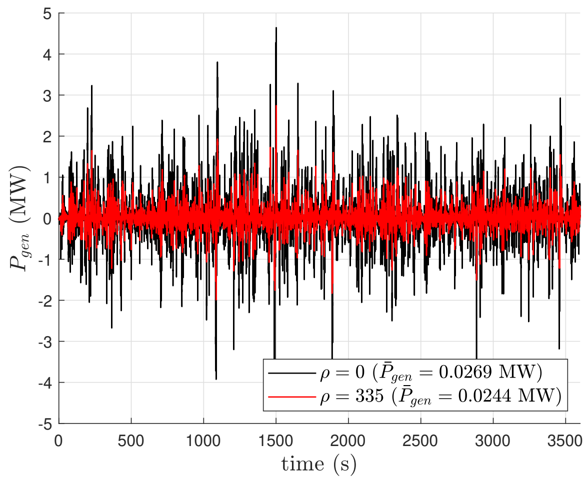

Fig. 6. Comparison of simulated P_{gen} time histories corresponding to LQ-optimal controller ($\rho = 0$) and risk-sensitive controller ($\rho = 335$) for WEC example

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