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Consistent event-triggered consensus on complete graphs

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Abstract— **This paper starts by considering an optimal control formulation of the consensus problem on complete graphs with a cost capturing disagreement and agents modeled by integrators. An optimal control policy for this problem is shown to be the well-known consensus algorithm by which each agent resets its state to the average of its and other agents' state values received at every time step. The framework is extended to the case where agents can only exchange information periodically, with a period larger than one. Then an event-triggered control strategy is proposed that results in a better cost than that of the optimal periodic one with the same average transmission rate, that is, it is consistent. According to this strategy, each agent distributedly transmits its state if the error between its current state and a common consensus estimate based on previously transmitted agents' data exceeds a threshold. Simulation results are presented to illustrate the proposed strategy.**

I. Introduction

In many applications, a group of distributed agents needs to agree upon certain quantities of interest or states [1]–[3]. Under mild conditions on the communication graph, this can be achieved by having agents exchange and update their states based on weighted averages of neighbouring agent states [2]. The exchange of information is typically assumed to be time-triggered, particularly periodic.

However, applications where this information exchange is expensive and thus the communication load needs to be reduced, need more efficient communication protocols, such as event-triggered protocols. In event-triggered connumication and/or control, information exchange is based on the state rather than time [4]–[9]. Many consensus strategies that borrow ideas from event-triggered control (ETC) have been proposed [10].

One of the underlying principles of ETC is that it should outperform periodic control. However, this is not always the case [5], [11], and it is thus a desired property rather than a fact. Therefore, it is reasonable to pose the following question: given a performance index and a periodic control strategy that is optimal with respect to this index, is there an ETC strategy that improves this performance index while using the same communication resources as periodic control? This question was addressed in a pioneering paper on ETC [12]. Considering an integrator system and the output variance as a performance metric, [12] proves that this property holds for a threshold ETC strategy. ETC strategies that achieve this property, referred to as consistency [5], were later proposed for general linear systems with quadratic performance indices [5]–[9].

This question has been recently addressed in the context of consensus in [11]. Considering a complete communication graph, a continuous-time framework with agents described by integrators driven by white noise, and a natural performance index measuring disagreement between agents, [11] compares periodic and event-triggered implementations that are reasonable extensions of [12] to the consensus case. Surprisingly, [11] shows that periodic control yields a better index for the same average transmission rate when the number of agents is large. However, note that the event-triggered policy in [11] is not optimal for the performance index considered. Thus, it remains open whether there exists an ETC policy that can outperform an optimal periodic control policy for the performance index proposed in [11] using the same communication resources.

This paper considers a discrete-time version of the optimal control problem proposed in [11]. An optimal control policy is shown to be the natural consensus policy by which each agent resets its state to the average of its and other agents' state values received at every time step. Since there are other optimal control policies, including fully decentralized ones, the choice for the natural consensus strategy is motivated by a related stochastic reformulation of the problem. The framework is extended to the case where agents can only exchange information periodically, with a period larger than one. This is how agents can reduce the communication load in a periodic communication setting. A class of ETC policies is proposed for which the initial disagreement cost can be expressed as an output variance of an ETC policy for a single integrator. Building upon this fact, an ETC strategy is proposed that results in a better cost than that of the optimal periodic one with the same average transmission rate, that is, it is consistent. According to this strategy, each agent transmits if the error between its current state and a common estimate based on previously transmitted agents' data exceeds a threshold. This strategy can be described as a simple distributed algorithm (see Algorithm 2 below).

The paper is organized as follows. Section II formulates the optimal control problem, gives an optimal periodic control solution and states the problem. Section III provides the proposed ETC policies and the main results. Section IV discusses a numerical example and Section V gives concluding remarks. The proofs of the results are omitted for brevity.

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II. Problem formulation, Periodic Consensus, and Problem Statement

Section II-A formulates an optimal control problem with no communication constraints for which an optimal solution is a natural consensus policy. The optimal periodic solution with period larger than one is given in Section II-B and the problem considered in this paper is stated in Section II-C.

A. Problem Formulation

Consider a set of n agents indexed by $i \in \mathsf{N} := \{1, 2, \dots, n\}$ and each storing state $x_i[t] \in \mathbb{R}$ at time $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The agents are described by integrators

$$x_i[t+1] = x_i[t] + u_i[t] + w_i[t], \quad i \in \mathsf{N}, \quad t \in \mathbb{N}_0, \tag{1}$$

driven by zero-mean independent and identically Gaussian distributed disturbances $\{w_i[t] | t \in \mathbb{N}_0\}$ with $\mathbb{E}[w_i^2[t]] = \sigma_w^2$, for every $i \in \mathsf{N}$, where $u_i[t]$ is the control input of agent i at time t that allows that agent to change its state. The disturbances sequences are mutually independent.

The communication graph, determining which agents received information from which agents is assumed to be complete. This means that all the agents can communicate with all others at each time t although they might refrain from doing so to save communication resources.

Assumption 1: Each agent can communicate with all the other agents, i.e., the communication graph is complete. □

A natural consensus law in this case is

$$u[t] = -x[t] + \frac{1}{n}1_n 1_n^\intercal x[t], \quad t \in \mathbb{N}_0, \tag{2}$$

with $x[t] = \begin{bmatrix} x_1[t] & \dots & x_n[t] \end{bmatrix}^\intercal$, $u[t] = \begin{bmatrix} u_1[t] & \dots & u_n[t] \end{bmatrix}^\intercal$, and 1_n a column vector with n entries equal to one. This law can be written in the equivalent form $u_i[t] = \frac{1}{n}\sum_{j \in \mathsf{N}}(x_j[t] - x_i[t])$; it is clear that it depends on the relative distances of the agents' states.

A binary variable $\sigma_i[t]$ indicates if, at time t, agent $i \in \mathsf{N}$ decides to transmit to all the other agents its state ($\sigma_i[t] = 1$) or not ($\sigma_i[t] = 0$). Based on the past transmitted states, each agent has access to a common information set at time t,

$$\mathcal{I}_t = \cup_{i=1}^n \{x_i[\ell] \,| \sigma_i[\ell] = 1, 0 \le \ell \le t\}.$$

Each agent has access to its own state. At time t, each agent: 1) decides based on shared information up to time $t-1$, $\mathcal{I}_{t-1}$ and private state information up to time t either to transmit or not; 2) receives information from other agents at time t; 3) compute its control input based on shared and state information up to time t, $\mathcal{I}_t$. Thus, $\sigma_i[t] = \nu_{t,i}(\mathcal{J}_{t,i})$ and $u_i[t] = \mu_{t,i}(\mathcal{K}_{t,i})$ for some functions $\nu_{t,i}$, $\mu_{t,i}$, where $\mathcal{J}_{t,i} := \mathcal{I}_{t-1} \cup \{x_i[\ell] \,|\, 0 \le \ell \le t\}$ and $\mathcal{K}_{t,i} := \mathcal{I}_t \cup \{x_i[\ell] \,|\, 0 \le \ell \le t\}$.

Performance of a given control and transmission policy is measured by the average cost

$$J = \limsup_{T \to \infty} \frac{1}{T}\mathbb{E}[\sum_{t=0}^{T-1} g(x[t])], \; g(x[t]) = \frac{1}{2}\sum_{1 \le \ell, j \le n}(x_\ell[t] - x_j[t])^2. \tag{3}$$

Let I_n be the n-dimensional identity matrix. Defining the incident matrix $\Gamma = I_n \otimes 1_n - 1_n \otimes I_n$ and Laplacian $L = \frac{1}{2}\Gamma^\intercal \Gamma = nI_n - 1_n 1_n^\intercal$ of this fully connected graph, we can write $g(x[t]) = x[t]^\intercal L x[t]$.;T g measures the disagreement between states and is zero if all the agents' states are equal. It is a discrete-time version of the cost proposed in [11].

The next proposition provides an optimal policy when all the agents are allowed to transmit all the time.

Proposition 1: Suppose that $\sigma_i[t] = 1$ for every $i \in \mathsf{N}$ and $t \in \mathbb{N}_0$. Then, any policy of the form

$$u_i[t] = -x_i[t] + a(x[t], t), \quad i \in \mathsf{N}, \quad t \in \mathbb{N}_0,$$

for arbitrary $a(\cdot, \cdot)$ minimizes (3) for (1). The resulting optimal cost is

$$J = J_1 = (n-1)n\sigma_w^2. \qquad \square$$

While $a(\cdot, \cdot)$ could depend on other variables (e.g. previous states), it is well known in the context of Markov decision processes that it is enough to search for policies that depend on the state [13], which justifies this choice.

Interestingly, the natural consensus policy (2) is an optimal solution with $a(x[t], t) = \frac{1}{n}1_n^\intercal x[t]$ for every $t \in \mathbb{N}_0$. However, making $a(x[t], t) = 0$ for every $t \in \mathbb{N}_0$ results in a control policy for which each agent tries to reset its state to the origin, a simpler but non-cooperative way of achieving consensus. The choice of the natural consensus policy (2) is justified in the appendix. It discusses an alternative problem formulation that makes (2) unique in a given sense.

B. Periodic control solution with period larger than one

We now turn to the case where the communication protocol is still periodic but with general period $h \in \mathbb{N}$, i.e.,

$$\sigma_i[t] = \begin{cases} 1 \text{ if } t \in \{0, h, 2h, \dots\}, \\ 0 \text{ otherwise}, \qquad \forall i \in \mathsf{N}, t \in \mathbb{N}_0. \end{cases} \tag{4}$$

This is the way in a periodic setting one can reduce communication. As stated next, the optimal control policy and the optimal cost can still be computed, thus providing a benchmark for comparison with other (ETC) strategies.

Lemma 1: Suppose that the transmission protocol is periodic with period $h \in \mathbb{N}$, that is, it follows (4). Then, any policy of the form

$$u_i[t] = \begin{cases} -x_i[t] + b(\mathcal{I}_t, t) \text{ if } t \in \{0, h, 2h, \dots\}, \\ b(\mathcal{I}_t, t) \text{ otherwise}, \end{cases} \tag{5}$$

for arbitrary $b(\cdot, \cdot)$ minimizes (3) for (1). The resulting optimal cost is

$$J_h = J_1 + \phi_h(n-1)n\sigma_w^2, \quad \phi_h = \frac{(h-1)}{2}. \tag{6}$$

□

We chose here to make $b(\cdot, \cdot)$ $\mathcal{I}_t$ dependent, since $\mathcal{I}_t$ plays the role of state. The reformulation in the appendix leads to an optimal policy that is unique in a given sense and sets

$$b(\mathcal{I}_t, t) = \begin{cases} \frac{1}{n}1_n^\intercal x[t] \text{ if } t \in \{0, h, 2h, \dots\}, \\ 0, \text{ otherwise}. \end{cases}$$

This is a natural choice and leads to a very simple algorithm to be implemented by each agent distributedly. This is

Algorithm 1: Distributed periodic consensus algorithm run by each agent $i \in \mathsf{N}$

Set $\hat{x}_j[0] = x_j[0]$, for all $j \in \mathsf{N}$, $\alpha[0] = \frac{1}{n}\sum_{j=1}^{n} x_j[0]$
At each time $t \in \{1, 2, \dots\}$:
if $t \in \{h, 2h, \dots\}$ **then**
  Send $x_i[t]$ to all agents
  Receive messages from other agents
  Set $\alpha[t] = \frac{1}{n}\sum_{j=1}^{n} x_j[t]$, $u_i[t] = -x_i[t] + \alpha[t]$
else
  No action needed ($u_i[t] = 0$)

Algorithm 2: Distributed ETC consensus algorithm run by each agent $i \in \mathsf{N}$

Set $\hat{x}_j[0] = x_j[0]$, for all $j \in \mathbb{N}$, $\alpha[0] = \frac{1}{n}\sum_{j=1}^{n} x_j[0]$
At each time $t \in \{1, 2, \dots\}$:
if $|x_i[t] - \alpha[t-1]| > \theta$ **then**
  Send $x_i[t]$ to all agents and update $\hat{x}_i[t] = x_i[t]$
Receive possible messages and update $\hat{x}_j[t] = x_j[t]$ for nodes $j \neq i$ from which a message is received
if *either a message was sent or received* **then**
  Set $\alpha[t] = \frac{1}{n}\sum_{j=1}^{n} \hat{x}_j[t]$, $u_i[t] = -\hat{x}_i[t] + \alpha[t]$ and $\hat{x}_j[t+1] = \alpha[t]$ for all $j \in \mathsf{N}$.
else
  No action needed ($u_i[t] = 0$, $\alpha[t] = \alpha[t-1]$)

summarized in Algorithm 1 for later comparison with the ETC case. Note also that Lemma 1 provides the optimal cost when transmissions are periodic with period larger than one, which is crucial for comparison with ETC.

C. Problem Statement

Let us define the average transmission rate

$$r = \frac{1}{n}\sum_{i=1}^{n} r_i, \quad r_i = \limsup_{T \to \infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[\sigma^i[t]]$$

of a given control and communication policy and the average inter-transmission time $\bar{\tau} = \frac{1}{r}$. From Lemma 1, we know that the optimal cost for a fixed communication policy (4) with period h is $\bar{J}(h) = J_h$. Suppose that we linearly interpolate the points (h, J_h), i.e., consider the curve $(d, \bar{J}(d))$, for $d \in \mathbb{R}_{\geq 1}$,

$$\bar{J}(d) = J_h + (J_{h+1} - J_h)(d - h), \qquad d \in [h, h+1).$$

We are interested in finding a policy $\pi = \{\nu_{t,i}, \mu_{t,i} | t \in \mathbb{N}_0, i \in \mathsf{N}\}$ for communication and control for which the pair $(\bar{\tau}_\pi, J_\pi)$, where $\bar{\tau}_\pi$ is the average inter-sampling time of π and J_π is the cost of π, is below the curve $(d, \bar{J}(d))$, i.e.,

$$J_\pi \leq \bar{J}(\bar{\tau}_\pi). \tag{7}$$

Following the nomenclature in [5] and the discrete-time adaptation proposed in [6], we say that a policy π that satisfies (7) is consistent (although in [5] the definition is different, namely by considering a second desired property). We say that it is strictly consistent if (7) holds with strict inequality. Typically in ETC, there is parameter (e.g. θ in (11) below) that allows to tune $\bar{\tau}_\pi$. Also here we are interested in finding a parameterized class of consistent policies.

III. Consensus ETC policy and main results

The proposed ETC policy is given in Section III-A leading to Algorithm 2. Section III-B establishes consistency.

A. Consensus ETC policy

By convention, let us assume that each agent shares its state at time $t = 0$, i.e., $\sigma_i[0] = 1$ for all $i \in \mathsf{N}$. Suppose now that each agent can transmit asynchronously and independently from the other agents. For all $i \in \mathsf{N}$ let

$$\hat{x}_i[t] := \begin{cases} x_i[t], & \text{if } \sigma_i[t] = 1, \\ \alpha[t-1], & \text{otherwise}, \end{cases} \quad \text{for all } t \in \mathbb{N}_0,$$

and

$$\alpha[t] := \frac{1}{n}\sum_{j=1}^{n} \hat{x}_j[t], \text{ for all } t \in \mathbb{N}_0.$$

By convention $\alpha[-1] = 0$. Note that $\alpha[t+1] = \alpha[t]$ if $\sigma_i[t] = 0$ for all $i \in \mathsf{N}$. Note also that each agent can compute and keep track of $\hat{x}_j[t]$, and $\alpha[t]$, *for all* $j \in \mathsf{N}$, *which depend only on the shared information* $\mathcal{I}_t$; $\alpha[t]$ can be interpreted as an estimated consensus value, since it relies on state estimates rather than on states. The proposed control policy is

$$u_i[t] = \begin{cases} -\hat{x}_i[t] + \alpha[t] & \text{if } \sigma_j[t] = 1 \text{ for some } j \in \mathsf{N}, \\ 0 & \text{if } \sigma_j[t] = 0 \text{ for all } j \in \mathsf{N}. \end{cases} \tag{8}$$

Note that it takes the form $u_i[t] = \mu_{t,i}(\mathcal{K}_{t,i})$. Let $e_i[t] := x_i[t] - \hat{x}_i[t]$, $\tilde{e}_i[t] := x_i[t] - \alpha[t-1]$ which evolve as

$$\begin{aligned} e_i[t] &= \begin{cases} \tilde{e}_i[t] \text{ if } \sigma_i[t] = 0 \\ 0 \text{ if } \sigma_i[t] = 1 \end{cases} \\ \tilde{e}_i[t+1] &= e_i[t] + w_i[t]. \end{aligned} \tag{9}$$

Note that $\tilde{e}_i[t]$ and $e_i[t]$ are *local* errors that can be computed only by each agent i by measuring its own state $x_i[t]$ and keeping track of the shared $\hat{x}_i[t]$, $\alpha[t-1]$.

We define and restrict ourselves to scheduling policies which are functions of $\tilde{e}_i[t]$, i.e.,

$$\sigma_i[t] = \eta(\tilde{e}_i[t]). \tag{10}$$

Note that indeed these take the form $\sigma_i[t] = \nu_{t,i}(\mathcal{J}_{t,i})$. An important special case is the following threshold policy

$$\sigma_i[t] = \begin{cases} 1 \text{ if } |\tilde{e}_i[t]| > \theta \\ 0 \text{ otherwise} \end{cases} \tag{11}$$

where the threshold θ is a positive constant.

This proposed consensus ETC policy can be summarized as in Algorithm 2. We can interpret this algorithm as follows. All the agents keep track of the same value $\alpha[t]$ which is equal to the average of the last transmitted state values; if an agents' state $x[t]$ deviates from $\alpha[t-1]$ by more than θ it communicates it to other agents. When this happens, all the agents need to update $\alpha[t]$ and simultaneously try to make their states close to $\alpha[t]$ by setting $u_i[t] = -\hat{x}_i[t] + \alpha[t]$.

Let us now compare the communication and computation requirements of ETC and periodic control. The transmission rate of agent i in the ETC case is only determined by Step 1 in Algorithm 2. However, this ETC algorithm will execute steps 2 and 3 at a typically much higher rate than this transmission rate, whereas in the periodic case steps 1,2,3 in Algorithm 1 are executed at the same rate $\frac{1}{h}$. This means the agents can be put to sleep in between steps $t \in \{h, 2h, \dots\}$ in the periodic case, but must perform internal computations in the ETC case. Besides in the ETC case, agents need to update the input at a typically much higher rate. The way we define consistency only takes into account the communication rate. Still, we acknowledge this is a disadvantage of the ETC algorithm when computation and control updates and not only communication are expensive.

B. Main result

For every $i \in \mathsf{N}$, let the times t at which $\sigma_i[t] = 1$ be

$$s_i[\ell+1] = s_i[\ell] + \tau_i[\ell], \quad \ell \in \mathbb{N}_0,$$

where $s_i[0] = 0$ and $\tau_i[\ell+1] = \inf\{t \in \mathbb{N} | \sigma_i[s_i[\ell]+t] = 1\}$. Note that $\tau_i[\ell]$ are stopping times, with the same statistical properties, for every $i \in \mathsf{N}$ and every ℓ. Let us denote by τ one such stopping time, e.g,. $\tau = \tau_1[0]$. Note that

$$\tau = \inf\{t \in \mathbb{N} | \ |\tilde{e}_1[t]| > \theta\} \tag{12}$$

with

$$\tilde{e}_1[t+1] = \tilde{e}_1[t] + w_i[t], t \in \mathbb{N}, \tilde{e}_1[0] = 0. \tag{13}$$

Moreover, $r_i = \frac{1}{\mathbb{E}[\tau]}$, for every $i \in \mathsf{N}$, and $r = \frac{1}{\mathbb{E}[\tau]}$ if $\mathbb{E}[\tau] < \infty$.

Theorem 1: Consider (1) and suppose the control policy is given by (8) and the scheduling policy takes the form (10) for some function η such that $\mathbb{E}[\tau] = \bar{\tau} < \infty$. Then the average cost (3) is

$$J = J_1 + \psi_\tau (n-1) n \sigma_w^2 \tag{14}$$

with

$$\psi_\tau = \frac{1}{\sigma_w^2 \mathbb{E}[\tau]} \mathbb{E}[\sum_{t=1}^{\tau-1} \tilde{e}_1[t]^2]. \qquad \square$$

This theorem is the main result of the paper. It reveals that the cost of (8), (10) is obtained by picking a scheduling policy (or stopping time τ) for a scalar integrator system (13). Thus, consider the curve $(d, \bar{\phi}(d))$, for $d \in \mathbb{R}_{\geq 1}$,

$$\bar{\phi}(d) = \phi_h + (\phi_{h+1} - \phi_h)(d-h), \qquad d \in [h, h+1).$$

which, due to $\phi_h = \frac{h-1}{2}$, boils down to $\bar{\phi}(d) = \frac{d-1}{2}, d \in \mathbb{R}_{\geq 1}$. If we find a stopping time τ that can ensure that $(\mathbb{E}[\tau], \psi_\tau)$ is below this curve, i.e., $\psi_\tau < \bar{\phi}(\tau)$ then due to Theorem 1 it immediately follows that (7) holds with strict inequality, *irrespective* of n. While there are multiples other ways of achieving this, see [5], [5]–[8], we show next that (11) guarantees $\psi_\tau < \bar{\phi}(\tau)$. Note that for (11), ψ_τ only depends on the ratio $\frac{\theta}{\sigma_w}$. In fact, we can write

$$\psi_\tau = \frac{1}{\mathbb{E}[\tau]} \mathbb{E}[\sum_{t=1}^{\tau-1} \tilde{e}_s[t]^2], \quad \tilde{e}_s[t] = \frac{\tilde{e}_1[t]}{\sigma_w} \tag{15}$$

where for unitary variance random variables $w_s[t] = \frac{w_1[t]}{\sigma_w}$, $\tilde{e}_s[t+1] = \tilde{e}_s[t] + w_s[t], t \in \mathbb{N}, \tilde{e}_s[0] = 0$, and the triggering condition in (11) is now $|\tilde{e}_s| > \frac{\theta}{\sigma_w}$. The next proposition provides a method to compute $\mathbb{E}[\tau]$ and ψ_τ; it relies on computing the conditioned probability distribution of the error given that the triggering condition has not been met up to time t, which is denoted by $p_t(\cdot)$ using the Bayes filter. Let $a(s) * b(s) = \int_{\infty}^{\infty} a(r) b(s-r) dr$ denotes convolution of two real functions in $\mathbb{R}$, $\mathbf{1}_{[c,d]}(s) := 1$ if $s \in [c,d]$ and $\mathbf{1}_{[c,d]}(s) := 0$ if $s \notin [c,d]$, and $p_w(s) = \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{s^2}{2\sigma_w^2}}$ be the probability distribution of $w_i[t]$ for any i, t.

Proposition 2: Consider (13) and the stopping time (12) for a positive constant θ. Then,

$$\mathbb{E}[\tau] = \sum_{t=1}^{\infty} \beta_t, \quad \beta_t = \Pi_{\ell=0}^{t-1} \lambda_\ell$$

$$\mathbb{E}[\sum_{t=1}^{\tau-1} \tilde{e}_1[t]^2] = \sum_{t=1}^{\infty} v_t \beta_{t+1}, \quad v_t = \int_{-\infty}^{\infty} s^2 p_t(s) ds$$

where λ_t, $p_t(.)$, can be computed by taking initially $\tilde{p}_1(s) = p_w(s)$ and $\lambda_0 = 1$ and then recursively computing for $t \in \mathbb{N}$

$$p_t(s) = \frac{1}{\lambda_t} \tilde{p}_t(s) \mathbf{1}_{[-\theta,\theta]}(s), \quad \lambda_t = \int_{-\theta}^{\theta} \tilde{p}_t(s) ds,$$

$$\tilde{p}_{t+1}(s) = p_t(s) * p_w(s). \qquad \square$$

An outcome of this proposition is that $\mathbb{E}[\tau] < \infty$ for (11).

The result of applying this procedure is shown in Figure 1 for various values of θ/σ_w; this plot can be used to predict the performance for any value of n and σ_w. It is then clear that the proposed ETC policy is consistent for any n and σ_w.

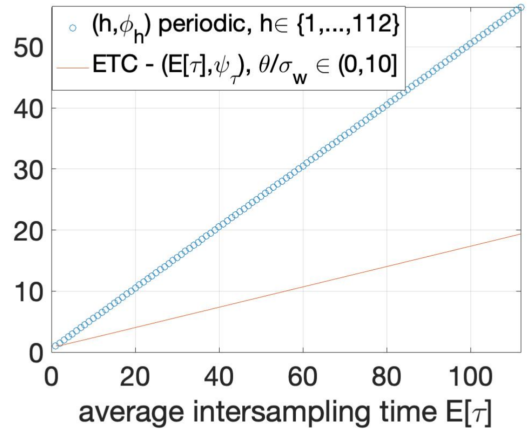

Fig. 1: Components ψ_τ and ϕ_τ of average costs (3) versus average inter-sampling time $\mathbb{E}[\tau]$ for single integrator described by (9) for threshold ETC (see (11)) and periodic control. From (14), (6) we conclude the ETC is consistent.

IV. Numerical Example

Figure 2 plots the results of a numerical simulation with $n = 3$ for periodic control with $h = 5$ and Figure 3 the results for ETC with $\theta = 0.15$, $\sigma_w = 0.1$ which leads to approximately the same average inter-transmission time $r = 0.218$. As expected, after transmission times the states move closer to $\alpha[t]$. For the values plotted, periodic control has overall $5\times3 = 15$ transmissions, whereas ETC $5+7+5 = 17$ transmissions and the cost $\sum_{k=1}^{20} x^\intercal Lx$ of periodic control is 8.2041 and for ETC 7.7134. Over a large time horizon of 10000, Monte-carlo simulation confirms the value for the average costs and average intersampling times. These are compared to the theoretical values obtained from the plot in Figure 1 for some values of n and σ_w in Table I.

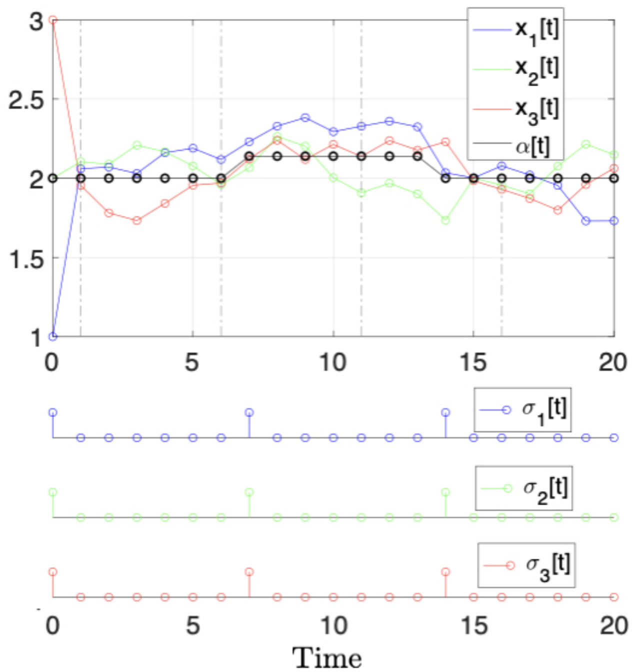

Fig. 2: Periodic

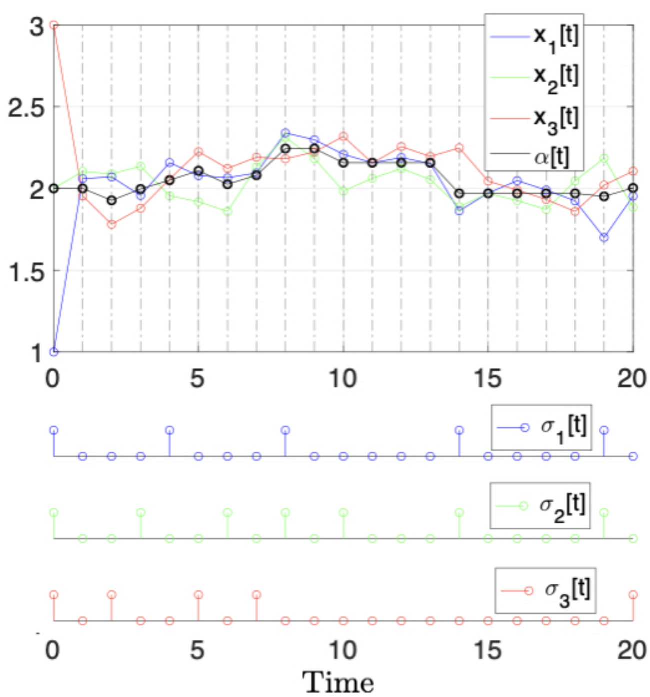

Fig. 3: ETC for $\theta = 0.15$

Periodic Control	MC J	Th. J	MC $1/r$	Th. $1/r$
$n = 3$, $\sigma_w = 1$, $h = 7$	23.973	24	7	7
$n = 5$, $\sigma_w = 2$, $h = 13$	34.832	35	13	13
ETC	MC J	Th. J	MC $1/r$	Th. $1/r$
$n = 3$, $\sigma_w = 1$, $\theta = 2$	11.182	11.159	7.037	6.913
$n = 5, \sigma_w = 0.5, \theta = 1.5$	14.324	14.389	13.147	13.085

TABLE I: Comparison of Monte-Carlo (MC) and Theoretical (Th.) values of the average cost J and average intersampling time $1/r$

V. Conclusions

Recently, [11] has shown that an ETC consensus law can perform worse than periodic control with respect to a cost measuring disagreement for the same average communication rate, as the number of agents becomes large. Such an ETC consensus law is a natural extension of [12] to multi-agents systems with a complete communication graph structure. Here, we consider a discrete-time version of the problem considered in [11]. We provide a different ETC policy that guarantees a better performance/cost for the same average communication rate irrespective of the number of agents. Such an ETC policy, can be described by a simple distributed algorithm. The connection between the present work and [11] as well as considering general communication graphs, not necessarily complete, are topics for future work.

Appendix

This appendix proposes a different problem formulation that leads to an optimal policy coinciding with the natural consensus policy (2) that is unique in a given sense. Suppose that possibly p, $0 \leq p < n$, of the agents are leaders, and the leaders might change in time. If i is a leader at time t, $i \in \mathsf{L}_t \subseteq \mathsf{N}$; when there are no leaders at time t, $\mathsf{L}_t = \emptyset$. The case of no-leaders is the most interesting scenario, but accounting for possible leaders plays an important role. The non-leader agents are described as before $x_i[t+1] = x_i[t] + u_i[t] + w_i[t], \quad i \in \mathsf{N} \setminus \mathsf{L}_t, \quad t \in \mathbb{N}_0$. A leader state is described by $x_i[t+1] = x_i[t], \quad i \in \mathsf{L}_t, \quad t \in \mathbb{N}_0$, meaning that it cannot change its state; disturbances are not included for leaders since (3) would not be finite otherwise.

The shared information set is modified to include information about the leader

$$\mathcal{I}_t = \cup_{i=1}^n \{x_i[\ell]\,|\sigma_i[\ell] = 1, 0 \leq \ell \leq t\} \cup \{\mathsf{L}_t | 0 \leq \ell \leq t\}.$$

The next proposition generalizes Proposition 1 to the case where the set of leaders is fixed throughout time. Since only the asymptotic behavior is important for cost (3), optimal control policies are not unique. For instance given any optimal policy $\mu_{t,i}$, $u_i[t] = \mu_{t,i}(\mathcal{I}_t, t) + \kappa_t$ is also optimal, where κ_t is arbitrary in $t \in \{0, 1, \ldots, M-1\}$ and $\kappa_t = 0$ for $t \geq M$, for an arbitrary $M \in \mathbb{N}$. Therefore, we consider the following finite-horizon version of cost (3)

$$\mathbb{E}[\sum_{t=0}^{T} g(x[t])] \tag{16}$$

As proved next, the optimal policy considering (16) is unique and time-invariant *for any given* horizon $T \in \mathbb{N}$. If we extend such a unique optimal policy for $t > T$, it is also an optimal policy when (3) is considered.

Proposition 3: Suppose that $\sigma_i[t] = 1$ for every $i \in \mathsf{N}$ and $t \in \mathbb{N}_0$, that $\mathsf{L}_t = \mathsf{L} \subset \mathsf{N}$ for every $t \in \mathbb{N}_0$, and $p > 0$. Then, an optimal control policy for (1) that minimizes (3) is

$$u_i[t] = -x_i[t] + \frac{1}{p}(\sum_{j \in \mathsf{L}} x_j[t]), \quad i \in \mathsf{N} \setminus \mathsf{L}. \tag{17}$$

Moreover, the resulting optimal cost (3) is

$$J = (n-1)(n-p)\sigma_w^2 + \frac{n}{2p} \sum_{\ell, j \in \mathsf{L}, \ell \neq j} (x_\ell[t] - x_j[t])^2.$$

Furthermore, (17) is the unique optimal policy for (1) that minimizes (16) for any $T \in \mathbb{N}$. □

Policy (17) imposes collaboration with the leaders since each agent needs to average the leaders' states. but not with the remaining agents. We thus proceed to a formulation with stochastic leaders, to arrive at the desired form (2).

Consider that at each time t there is either one randomly picked agent with probability ϵ or no leader is selected with probability $1 - \epsilon$; all agents have the same probability of being the leader at time t. We can model this as

$$x[t+1] = x[t] + \Omega_{\delta[t]}(u[t] + w[t]) \tag{18}$$

where, $\Omega_i := I_n - \mathsf{e}_i \mathsf{e}_i^\mathsf{T}$, for $i \in \mathsf{N}$, $\Omega_{n+1} = I_n$ with $\{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$ the canonical basis vectors in $\mathbb{R}^n$, and

$$\text{Prob}[\delta[t] = i] = \frac{\epsilon}{n}, \text{ for } i \in \mathsf{N}, \quad \text{Prob}[\delta[t] = n+1] = 1 - \epsilon,$$

where $\mathsf{e}_i \in \mathbb{R}^n$, $i \in \{1, \ldots, n\}$ is a column vector with entry i equal to one and all other entries equal to zero. Note that implicitly $\delta[t]$ through L_t is part of the information set $\mathcal{I}_t$.

Proposition 4: Suppose that $\sigma_i[t] = 1$ for every $i \in \mathsf{N}$ and $t \in \mathbb{N}_0$ and consider the problem of finding a control policy to minimize (3) for system (18). Then, an optimal policy is

$$u_i[t] = \mathsf{e}_i^\mathsf{T} \tilde{u}[t], \quad i \in \mathsf{N} \setminus \mathsf{L}_t, \quad \tilde{u}[t] = K_\epsilon x[t], \tag{19}$$

where $K_\epsilon = -\frac{(1-\frac{\epsilon}{n})}{(n-\epsilon-\frac{\epsilon}{n})} L$. Moreover, the resulting cost is

$$J = J_{\epsilon,1} = \frac{n(n-\epsilon-\frac{\epsilon}{n})}{(n-\epsilon)^2} n(n-(1+\epsilon)+\frac{\epsilon}{n})\sigma_\omega^2.$$

Thus, as $\epsilon \searrow 0$, $K_\epsilon \to -I + \frac{1}{n}1_n 1_n^\mathsf{T}$, $J_{\epsilon,1} \to n(n-1)\sigma_\omega^2$. Furthermore, (19) is the unique optimal policy for (1) that minimizes (16) for any $T \in \mathbb{N}$. □

This proposition states that the parameters K_ϵ optimal control policy as $\epsilon \searrow 0$ converge to parameters that recovers the natural consensus policy (2). Note then that from all the policies taking the form $u_i[t] = -x_i[t] + a(x[t], t)$ in Proposition 1, in the case of no leaders, we now have a good reason to pick the natural consensus policy (2), and it is the (unique) limit of a sequence of policies parameterized by ϵ as ϵ converges to zero, themselves unique is we consider (16).

Lemma 2: Suppose that the transmission protocol is periodic with period h, that it follows (4), and consider the problem of finding a control policy to minimize (3) for system (18). Then, an optimal policy is given by

$$u_i[t] = \mathsf{e}_i^\mathsf{T} \tilde{u}[t], \quad i \in \mathsf{N} \setminus \mathsf{L}_t, \quad \tilde{u}[t] = K_\epsilon \hat{x}[t], \tag{20}$$

where $\hat{x}[k] = \begin{cases} x[k] & \text{if } k \text{ is a multiple of } h \\ \tilde{x}[k] & \text{otherwise,} \end{cases}$,' and $\tilde{x}[t+1] = (I + \Omega_{\delta[t]} K_\epsilon)\hat{x}[t]$, Moreover, the cost is

$$J_{\epsilon,h} = J_{\epsilon,1} + \phi_h \frac{(n-\epsilon)^2}{(n-\epsilon-\frac{\epsilon}{n})n}(n-1)n, \; \phi_h := \frac{(h-1)}{2}(1-\frac{\epsilon}{n}).$$

Furthermore, this same optimal control policy is the unique optimal policy for (1) that minimizes (16) for any $T \in \mathbb{N}$. □

Again, $\epsilon \searrow 0$ recovers the policy and cost of Lemma 1.

We can provide consistent ETC policies for the case $\epsilon > 0$ and recover the one proposed in Section III when $\epsilon \searrow 0$. While we do not pursue this here, we note that we can only easily implement this policy in a distributed way when $\epsilon = 0$ both in the ETC and periodic case. For example to compute (20) $\delta[t]$ labeling the stochastically chosen leader would have to be communicated to all agents.

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