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Differential Game with Mixed Strategies: A Weak Approximation Approach

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Abstract— **This paper utilizes the weak approximation method to analyze differential games that involve mixed strategies. Mixed strategies have the potential to produce unique strategic behaviors, whereas traditional models and tools in pure strategy games cannot be directly applied. Based on the stochastic processes with independent increments, we define the mixed strategy without assuming the knowledge of the opponents' strategy and system state. However, this general mixed strategy poses challenges in evaluating game payoff and game value. To overcome these challenges, we utilize the weak approximation method to employ a stochastic differential game to characterize the dynamics of the mixed strategy game. We demonstrate that the game's payoff function can be precisely approximated with an error of the same scale as the step size. Furthermore, we estimate the upper and lower value functions of the weak approximated game to analyze the existence of game value. Finally, we provide numerical examples to illustrate and elaborate on our findings.**

I. Introduction

Differential game theory is a branch of game theory that deals with dynamic, continuous-time interactions between multiple decision-makers or agents. In contrast to the classical game theory, which assumes that players choose their strategies simultaneously, differential games account for the fact that players may choose their strategies over time, and the evolution of the game depends on the states of the system.

One important subclass of differential games is differential games with mixed strategies. In such games, players select their strategies based on probability distributions over a set of possible pure strategies. This allows for a richer representation of the player's behavior, as they can exhibit varying degrees of randomness or unpredictability in their decision-making.

The use of mixed strategies in differential games can capture important features of real-world problems, such as incomplete information, imperfect competition, and stochastic dynamics. Mixed strategies can also generate novel strategic behavior that may not emerge in pure strategy games. In addition, the use of mixed strategies can be a useful approach for addressing the challenge of determining a value for differential games in cases where the Isaacs condition is not satisfied.

The study of differential games has a rich history, with numerous contributions from researchers in game theory, control theory, and applied mathematics. One of the earliest works in this area was the pioneering work by Isaacs [1]. Since then, a large body of literature has emerged on various aspects of differential games, including their existence and uniqueness of equilibrium solutions, numerical methods for solving them, and applications in diverse fields such as economics [2], robotics [3], and biology [4].

The analysis of differential games with mixed strategies presents a significant challenge in recent studies. To define the mixed strategy, two primary models are used: the nonanticipative strategy with delay (NAD strategy) and the Markov strategy. Both models use time discretization to approach the final definition. [5] introduce the concept of mixed strategies along a partition of the time interval associated with classical NAD strategies, which helps to address the existence of the game value for differential games without the Isaacs condition. This work is further extended to stochastic differential games in [6], and cases in asymmetric information are also investigated in [7], [8]. The first model assumes the knowledge of the opponents' strategy, while the second model assumes the knowledge of the system state. The Markov strategy, which resembles the closed-loop control in the optimal control theory, is utilized to define mixed strategies and design nearly optimal game strategies [9], [10]. However, for the condition where state information is not available, both two models become inappropriate. To tackle this open problem, we use a general stochastic process with independent increments to define the mixed strategy, so that the knowledge of opponents' strategies and system state is not needed.

The evaluation of the game payoff and the analysis of the existence of game value are always of great challenge. Even for the differential game with pure strategies, the game value generically has no explicit expression or just does not exist [11]. Traditionally, upper and lower value functions and viscosity solutions are introduced to describe the game value. However, these methods cannot be directly applied to the differential games with mixed strategies. To deal with this challenge, we introduce a novel weak approximation method [12]–[14]. It manages to approximate the game by a standard stochastic differential game driven by Brownian motion.

The contributions of this paper are three-fold:

- Based on the stochastic processes with independent increments, a general model of mixed strategy is defined, which does not assume the knowledge of the opponents' strategy and system state.
- A novel weak approximation method is introduced to estimate the payoff function, whose approximation

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error is provably guaranteed of the same scale as the discretized step size.
- To study the existence of game value, the challenging problem of optimizing over stochastic process is tackled by estimating the upper and lower value functions of the weak approximated game. Additionally, some sufficient conditions for the existence are provided.

The remainder of this paper is organized as follows. Section II defines the differential game with mixed strategies and describes the problem of interest. Sec. III expounds on the weak approximation method and proved the main theorem on the approximation performance. Sec. IV provides some sufficient conditions for the existence of game value. Sec. V shows simulation results and analysis. Sec. VI presents concluding remarks and further research issues.

II. Problem Formulation

In this section, we expound on the dynamics of the differential game and introduce the concept of δ-game, which serves as a discrete approximation to the original differential game. We then define the mixed strategy within the differential game framework by utilizing δ-game and outline the primary issues that are of concern in our study.

A. Differential Game

The presence of mixed strategies in zero-sum differential games, such as the classical pursuit-evasion game, presents a formidable challenge for the pursuer. Despite possessing the ability to capture the evader in the sense of expectation, the task of maintaining a constant distance from the evader becomes exceedingly difficult due to the stochastic nature of mixed strategies.

To study the differential games with mixed strategies, we generally consider a linear time-invariant zero-sum differential game with mixed strategies,

$$\mathcal{G}^0:\begin{cases} J^0(t_0,x_0,u_1,u_2)=\mathbb{E}\left\{g(x(T))+\int_{t_0}^{T} h(t,u_1,u_2,x)\mathrm{d}t\right\} \\ \dot{x}(t)=Ax(t)+B_1u_1(t)+B_2u_2(t),\ x(t_0)=x_0 \end{cases}$$

where $x(t)\in\mathbb{R}^n$ is the game state, the matrices A, B_1 and B_2 are the system and input matrices with appropriate dimensions. $g(x_T)$ represents the terminal reward of the game, and $h(t,u_1,u_2,x)$ describes the energy cost during the game.

For each $t\in[t_0,T]$ and $i\in\{1,2\}$, $u_i(t)\in U$ is a random variable where U is a compact set. $\{u_i(t)\}_{t=t_0}^{T}$ is a stochastic process with independent increments, and it is controlled by the player i. For ease of notation, we say $u_i\in U_{[t_0,T]}$. $J^0(t_0,x_0,u_1,u_2)$ is the payoff that the player 1 wants to minimize and the player 2 wants to maximize.

The subsequent assumption constrains the scope of the payoff function J to those exhibiting polynomial growth. Specifically, the definition of polynomial growth functions is provided as follows.

Definition 1 (Polynomial growth functions [14]): Let G denote the set of continuous functions $\mathbb{R}^d\to\mathbb{R}$ of at most polynomial growth, i.e., $g(\cdot)\in G$ if there exists positive integers $\kappa_1,\kappa_2>0$ such that

$$|g(x)|\le\kappa_1\left(1+|x|^{2\kappa_2}\right)$$

for all $x\in\mathbb{R}^d$. Moreover, for each integer $\alpha\ge 1$ we denote by G^α the set of α-times continuously differentiable functions $\mathbb{R}^d\to\mathbb{R}$ which, together with its partial derivatives up to and including order α, belong to G.

Definition 2: The upper and lower value functions of the game $\mathcal{G}^0$ are given by

$$\mathbf{V}_-^0(t_0,x_0)=\sup_{u_2\in U_{[t_0,T]}}\inf_{u_1\in U_{[t_0,T]}} J^0(t_0,x_0,u_1,u_2),$$

and

$$\mathbf{V}_+^0(t_0,x_0)=\inf_{u_1\in U_{[t_0,T]}}\sup_{u_2\in U_{[t_0,T]}} J^0(t_0,x_0,u_1,u_2).$$

The primary issue to be considered for a game is whether the game value exists, i.e., $\mathbf{V}_+(t_0,x_0)=\mathbf{V}_-(t_0,x_0)$.

B. δ-Game

The formulation of mixed strategies hinges on the utilization of stochastic processes, which pose significant challenges both in terms of optimization and practical implementation for the players involved [5], [7], [8]. The δ-game is a useful approach for investigating mixed strategies in differential games, as it involves utilizing a discretized approximation of the continuous stochastic process inputs utilized in the original game [15].

We divide the interval $[t_0,T]$ into n intervals I_k of equal length δ :

$$I_k=\{t;t_{k-1}<t\le t_k\}$$

where $t_k=t_0+k\delta, 0\le k\le n$, and $n=\lfloor\frac{T}{\delta}\rfloor$. On each interval I_k, the control input for player i is a random variable u_{ik}. We can replace the stochastic process $\{u_i(t)\}_{t=t_0}^{T}$ in the game $\mathcal{G}^0$ by this independent random variables sequence $\{u_{ik}\}_{k=0}^{n-1}$, i.e.,

$$u_i(t)=\sum_{k=0}^{n-1}\left(u_{ik}\cdot\mathbb{I}_{I_k}(t)\right),$$

where $\mathbb{I}_{I_k}(t)=1$ if $t\in I_k$, and $\mathbb{I}_{I_k}(t)=0$ otherwise. For these discrete type of controls, we say $u_i\in U_{[t_0,T]}^{\delta}\subseteq U_{[t_0,T]}$. Let $x_k=x(k\delta)$ and discretize the game $\mathcal{G}^0$, we have the following δ-game.

Definition 3 (δ-game): Given a constant $\delta\in(0,1)$, the δ-game of the differential game $\mathcal{G}^0$ is

$$\mathcal{G}^1:\begin{cases} J^1(t_0,x_0,u_1,u_2)=\mathbb{E}\left\{g(x_n)+\sum_{k=0}^{n-1}h(t,u_{1k},u_{2k},x_k)\delta\right\} \\ x_{k+1}=e^{A\delta}x_k+(B_1u_{1k}+B_2u_{2k})\int_0^{\delta}e^{A\tau}\mathrm{d}\tau, x_0=x_0 \end{cases}$$

From another perspective, the mixed strategy for a δ-game is the random variable pairs (u_{1k},u_{2k}) for $k=0,1,\cdots,n-1$. Let $\delta\to 0$, a more tractable definition of the mixed strategy is available as follows.

Definition 4 (Mixed strategy): The mixed strategy for the differential game $\mathcal{G}^0$ is a pair of stochastic process $\{(u_1, u_2)\}_{t=t_0}^{T}$ such that,

$$u_i(t) = \lim_{\delta \to 0} \sum_{k=0}^{\lfloor \frac{T}{\delta} \rfloor - 1} \left(u_{ik} \cdot \mathbb{I}_{I_k}(t) \right),$$

where $\{(u_{1k}, u_{2k})\}_{k=0}^{\lfloor \frac{T}{\delta} \rfloor - 1}$ is a mixed strategy for the δ-game $\mathcal{G}^1$.
Similarly, we study the game value of this δ-game by defining the upper and lower value functions.

Definition 5: The upper and lower value functions of the δ-game $\mathcal{G}^1$ are given by

$$\mathbf{V}_-^1(t_0, x_0) = \sup_{u_2 \in U_{[t_0,T]}^{\delta}} \inf_{u_1 \in U_{[t_0,T]}^{\delta}} J^1(t_0, x_0, u_1, u_2),$$

and

$$\mathbf{V}_+^1(t_0, x_0) = \inf_{u_1 \in U_{[t_0,T]}^{\delta}} \sup_{u_2 \in U_{[t_0,T]}^{\delta}} J^1(t_0, x_0, u_1, u_2).$$

C. Problem in Interest

- Given the mixed strategies u_1 and u_2, our primary goal is to estimate the payoff function $J^0(t_0, x_0, u_1, u_2)$ accurately. Our focus is on estimating the δ-games initially, followed by the original game by gradually reducing δ to zero. Specifically, we design a stochastic differential game to approximate the δ-game, and the weak approximation theory guarantees the estimation accuracy.
- Our second objective is to analyze the existence of game value and investigate the sufficient conditions that ensure its existence. To deal with the challenging problem of optimizing over stochastic processes, we study the existence problem of the approximated stochastic differential game and then prove the relationship to the original game.

III. Weak Approximation of the Game

In this section, we introduce the weak approximation method as a tool for studying the δ-game with mixed strategies. We provide preliminaries on the weak approximation, highlighting its significance and indispensability in the analysis of mixed strategy.

A. Preliminaries on Weak Approximation

Let's delve into the concept of weak approximation - a method of approximating the distribution of sample paths instead of the paths themselves. By comparing the expectations of two processes over a broad range of test functions, we can determine their closeness. Our definition involves a massive test function class, including all polynomials, ensuring that all moments become close at an impressive rate. As a result, the distributions of both processes must also be equally close.

Definition 6 (Weak convergence [14]): For $T > 0$, $\delta \in (0, 1 \wedge T)$, and $\alpha \geq 1$ as an integer, let $N = \lfloor T/\delta \rfloor$. We define a continuous-time stochastic process $\{X_t : t \in [0,T]\}$ to be an order α weak approximation of a discrete stochastic process $\{x_k : k = 0, \ldots, N\}$ if for every $g \in G^{\alpha+1}$, there is a positive constant C that does not depend on δ such that

$$\max_{k=0,\ldots,N} |\mathbb{E}g(x_k) - \mathbb{E}g(X_{k\delta})| \leq C\delta^\alpha.$$

B. Heuristic Design of Weak Approximation

Rewrite the δ-game $\mathcal{G}^1$ as

$$x_{k+1} - x_k = (e^{A\delta} - I)x_k + (B_1 u_{1k} + B_2 u_{2k}) \int_0^\delta e^{A\tau} \mathrm{d}\tau, \quad (1)$$

where I is an identity matrix. Let γ_k and σ_k be the expectation and covariance of $(B_1 u_{1k} + B_2 u_{2k})$ respectively, i.e.,

$$\gamma_k := \mathbb{E}\{B_1 u_{1k} + B_2 u_{2k}\},$$

and

$$\sigma_k := \mathrm{Cov}\{B_1 u_{1k} + B_2 u_{2k}\}.$$

Heuristically, $(B_1 u_{1k} + B_2 u_{2k})$ can be approximated by a random variable subjected to the Gaussian distribution $\mathcal{N}(\gamma_k, \sigma_k)$ because of their matchings on the first two orders of moments. Motivated by this, we approximately rewrite the right-hand side as

$$(e^{A\delta} - I)x_k + \left[\delta^{-\frac{1}{2}} \sigma_k^{\frac{1}{2}} \left(W_{(k+1)\delta} - W_{k\delta}\right) + \gamma_k\right] \int_0^\delta e^{A\tau} \mathrm{d}\tau,$$

where $\{W_t\}_{t=0}^{\infty}$ is a standard winner process. When δ is small, there is $(e^{A\delta} - I) \approx Ae^{A\delta}\delta$, $\int_0^\delta e^{A\tau} \mathrm{d}\tau \approx \delta$, then it follows approximately

$$(Ae^{A\delta} x_k + \gamma_k)\delta + \sqrt{\delta}\sigma_k^{\frac{1}{2}} \left(W_{(k+1)\delta} - W_{k\delta}\right). \quad (2)$$

Substituting $x_{k+1} - x_k$, δ and $W_{(k+1)\delta} - W_{k\delta}$ with $\mathrm{d}X_t$, $\mathrm{d}t$ and $\mathrm{d}W_t$ respectively, we have the following stochastic differential equation:

$$\mathrm{d}X_t = (Ae^{A\delta} X_t + \Gamma_t)\mathrm{d}t + \sqrt{\delta}\Sigma_t^{\frac{1}{2}} \mathrm{d}W_t, \quad (3)$$

where Γ_t and Σ_t are the expectation and covariance of $B_1 u_{1,\lfloor \frac{t}{\delta} \rfloor} + B_2 u_{2,\lfloor \frac{t}{\delta} \rfloor}$ respectively, i.e.,

$$\Gamma_t := \mathbb{E}\left\{(B_1 u_{1,\lfloor \frac{t}{\delta} \rfloor} + B_2 u_{2,\lfloor \frac{t}{\delta} \rfloor}\right\},$$

and

$$\Sigma_t := \mathrm{Cov}\left\{(B_1 u_{1,\lfloor \frac{t}{\delta} \rfloor} + B_2 u_{2,\lfloor \frac{t}{\delta} \rfloor}\right\}.$$

Therefore, we have a stochastic differential game (SDG) as a weak approximation to the δ-game $\mathcal{G}^1$.

Definition 7 (SDG): Given a constant $\delta \in (0, 1)$, a zero-sum stochastic differential game with payoff function J^2 is

$$\mathcal{G}^2 : \begin{cases} J^2(t_0, x_0, u_1, u_2) = \mathbb{E}\left\{g(X(T)) + \int_{t_0}^{T} h(t, u_1, u_2, X)\mathrm{d}t\right\} \\ \mathrm{d}X_t = (Ae^{A\delta} X_t + \Gamma_t)\mathrm{d}t + \sqrt{\delta}\Sigma_t^{\frac{1}{2}} \mathrm{d}W_t, X_{t_0} = x_0 \end{cases}$$

Similarly, we consider the value of this stochastic game by studying the upper and lower value functions.

Definition 8: The upper and lower value functions of the stochastic differential game $\mathcal{G}^2$ are given by

$$\mathbf{V}_-^2(t_0, x_0) = \sup_{u_2 \in U_{[t_0,T]}^{\delta}} \inf_{u_1 \in U_{[t_0,T]}^{\delta}} J^2(t_0, x_0, u_1, u_2),$$

and

$$\mathbf{V}_{+}^{2}(t_0,x_0)=\inf_{u_1\in U^{\delta}_{[t_0,T]}}\sup_{u_2\in U^{\delta}_{[t_0,T]}} J^2(t_0,x_0,u_1,u_2).$$

If the SDG $\mathcal{G}^2$ indeed approximates the δ-game $\mathcal{G}^1$ in a weak sense such that the approximation error is of scale $O(\delta)$, then the original game $\mathcal{G}^0$ with mixed strategies can be accurately described by the dynamic of SDG $\mathcal{G}^2$ when $\delta \to 0$. In the rest of this section, it is proved that this heuristic design is an order 1 weak approximation.

C. Proof of Weak Approximation

This section presents the primary approximation theorem, which is obtained via a two-step process. Initially, we demonstrate that the one-step approximations for both games $\mathcal{G}^2$ and game $\mathcal{G}^1$ exhibit second-order accuracy. Subsequently, we establish a connection between one-step approximation and approximation on a finite time interval, guaranteeing that the approximation on a finite interval is of order 1.

To begin with, we study the one-step approximation errors of the stochastic game $\mathcal{G}^2$, and compare the errors to δ.

Lemma 1: Let $0<\eta<1$. Consider the SDG $\mathcal{G}^2$ and define the one-step difference $\Delta = X_\delta - x$. Let $\Delta_{(i)}$ be the ith element of Δ, then we have

1) $\mathbb{E}\Delta_{(i)} = (Ae^{A\delta}x_0+\Gamma_0)_{(i)}\delta + \mathcal{O}\left(\delta^2\right)$
2) $\mathbb{E}\Delta_{(i)}\Delta_{(j)} = O(\delta^2)$

Proof: This lemma is a direct application of the Lemma 4 in [14] by letting $b_0(x,\epsilon)=b_0(x)=Ae^{A\delta}x+\Gamma_0$ without relying on the auxiliary variable ϵ. ■

Then, we study the one-step approximation errors of the δ-game $\mathcal{G}^1$, and compare the errors to δ.

Lemma 2: Let $0<\eta<1$. Consider δ-game $\mathcal{G}^1$ and define the one-step difference $\bar{\Delta} = x_1 - x_0$. Let $\bar{\Delta}_{(i)}$ be the ith element of $\bar{\Delta}$, then we have

1) $\mathbb{E}\bar{\Delta}_{(i)} = \left[Ae^{A\delta}x_0+\gamma_0\right]_{(i)}\delta + O(\delta^2)$
2) $\mathbb{E}\bar{\Delta}_{(i)}\bar{\Delta}_{(j)} = O(\delta^2)$

Proof: According to the definition, there is

$$\bar{\Delta} = (e^{A\delta}-I)x_0 + (B_1u_{1,0}+B_2u_{2,0})\int_0^\delta e^{A\tau}\mathrm{d}\tau, \tag{4}$$

then

$$\begin{aligned}
&\mathbb{E}\bar{\Delta} - \left[Ae^{A\delta}x_0+\gamma_0\right]\delta \\
=&(e^{A\delta}-I-\delta Ae^{A\delta})x_0 + \left(\int_0^\delta e^{A\tau}\mathrm{d}\tau - \delta\right)\gamma_0 \\
=&(e^{A\delta}-I-\delta Ae^{A\delta})x_0 + \left[\int_0^\delta (e^{A\tau}-I)\mathrm{d}\tau\right]\gamma_0.
\end{aligned} \tag{5}$$

Notice that $e^{A\delta}-I=\delta A+\frac{\delta^2}{2}A^2+o(\delta^2)$, the above equation follows that

$$\begin{aligned}
&(\delta A+\frac{\delta^2}{2}A^2+o(\delta^2)-\delta Ae^{A\delta})x_0 \\
&+\left[\int_0^\delta (\delta A+\frac{\delta^2}{2}A^2+o(\delta^2))\mathrm{d}\tau\right]\gamma_0 \\
=&\left[\delta A(I-e^{\delta A})+O(\delta^2)\right]x_0 + \left(\delta^2 A+\frac{\delta^3}{2}A^2\right)\gamma_0 + o(\delta^3) \\
=&\left[\delta A(-\delta A+O(\delta^2))+O(\delta^2)\right]x_0 + O(\delta^2) \\
=&O(\delta^2),
\end{aligned} \tag{6}$$

Hence, the first conclusion is proved. Notice that $e^{A\delta}-I=O(\delta)$ and $\int_0^\delta e^{A\tau}\mathrm{d}\tau = O(\delta)$, we have

$$\bar{\Delta}_{(i)} = O(\delta), \quad \forall i.$$

Therefore, the second conclusion can be easily obtained. ■

Finally, we combine the above two lemmas to derive the main theorem.

Theorem 1 (Approximation accuracy): Stochastic differential equation $\mathcal{G}^2$ is an order 1 weak approximation to the δ-game $\mathcal{G}^1$ such that for every $g\in G$, there exists $C>0$, independent of δ, such that for all $k=0,1,\ldots,n$,

$$\left|\mathbb{E}g\left(x_{k\delta}\right)-\mathbb{E}g\left(x_k\right)\right| < C\delta.$$

Proof: First, according to the above two lemmas, there exists $K_1\in G$ independent of δ such that

$$\left|\mathbb{E}\prod_{j=1}^{s}\Delta_{(i_j)}(x) - \mathbb{E}\prod_{j=1}^{s}\bar{\Delta}_{(i_j)}(x)\right| \le K_1(x)\delta^2, \tag{7}$$

for $s=1$ and

$$\mathbb{E}\prod_{j=1}^{2}\left|\Delta_{(i_j)}(x)\right| \le K_1(x)\delta^2 \tag{8}$$

for all $i_j\in\{1,\ldots,d\}$.

Second, according to the compactness of U, for each $m\ge 1$, the $2m$-moment of x_k is uniformly bounded with respect to k and δ, i.e., there exists a $K_2\in G$, independent of δ, k, such that

$$\mathbb{E}\left|x_k\right|^{2m} \le K_2(x) \tag{9}$$

for all $k=0,\ldots,\lfloor T/\eta\rfloor$.

Finally, according to Theorem 3 in [14], we have the weak approximation conclusion proved. ■

IV. Approximation Performance and Existence of the Game Value

In this section, our focus is on examining the accuracy of the value functions of the continuous SDG $\mathcal{G}^2$ and the δ-game $\mathcal{G}^1$, and demonstrating that the approximation error is of the order of $O(\delta)$. We also investigate the correlation between the value functions of the δ-game $\mathcal{G}^1$ and the original game $\mathcal{G}^0$, and suggest a general sufficient condition for the existence of the game value. Additionally, we prove that the game value exists when the function $h(t,x,u_1,u_2)$ can be separated with respect to u_1 and u_2.

To begin with, we apply Theorem 1 to the payoff functions, and the following lemma shows that the approximation error of payoff functions is also of scale $O(\delta)$.

Lemma 3 (Payoff approximation): Given u_1 and u_2, there exists a constant $C_1(u_1, u_2) > 0$ such that

$$|J^1(t_0, x_0, u_1, u_2) - J^2(t_0, x_0, u_1, u_2)| \leq C_1(u_1, u_2)\delta.$$

Proof: According to the definitions of J^1 and J^2, we have

$$\begin{aligned}
&|J^1(t_0, x_0, u_1, u_2) - J^2(t_0, x_0, u_1, u_2)| \\
=&|\mathbb{E}\{g(x_n) + \sum_{k=0}^{n-1} h(t, u_{1k}, u_{2k}, x_k)\delta \\
&- g(X(T)) - \int_{t_0}^{T} h(t, u_1, u_2, X)\mathrm{d}t\}| \\
\leq&|\mathbb{E}\{\sum_{k=0}^{n-1} h(t, u_{1k}, u_{2k}, x_k)\delta - \int_{t_0}^{T} h(t, u_1, u_2, X)\mathrm{d}t\}| \\
&+ |\mathbb{E}\{g(x_n) - g(X(T))\}| \\
\leq& \int_{t_0}^{T} |\mathbb{E}\{h(t, u_1, u_2, x) - h(t, u_1, u_2, X)\}|\mathrm{d}t \\
&+ |\mathbb{E}\{g(x_n) - g(X(T))\}|.
\end{aligned} \tag{10}$$

By the weak approximation theorem, there are $C_2(u_1, u_2), C_3(u_1, u_2)$ such that

$$|\mathbb{E}\{h(t, u_1, u_2, x) - h(t, u_1, u_2, X)\}| \leq C_2(u_1, u_2)\delta,$$

and

$$|\mathbb{E}\{g(x_n) - g(X(T))\}| \leq C_3(u_1, u_2).$$

Let $C_1(u_1, u_2) = (T - t_0)C_2(u_1, u_2) + C_3(u_1, u_2)$, the proof is completed. ■

The payoff approximation lemma helps to prove that the upper and lower value functions of δ-game $\mathcal{G}^1$ and SDG $\mathcal{G}^2$ are also well approximated with the scale of $O(\delta)$.

Theorem 2 (Value approximation): There exists $C > 0$, independent of δ, such that

$$|\mathbf{V}_-^1(t_0, x_0) - \mathbf{V}_-^2(t_0, x_0)| \leq C\delta,$$

and

$$|\mathbf{V}_+^1(t_0, x_0) - \mathbf{V}_+^2(t_0, x_0)| \leq C\delta.$$

Proof: Without loss of generality, we only need to prove the approximation for the lower value function. Let $\mathbf{V}_-^1(t_0, x_0) = J^1(t_0, x_0, u_1^{(1)}, u_2^{(1)})$ and $\mathbf{V}_-^2(t_0, x_0) = J^2(t_0, x_0, u_1^{(2)}, u_2^{(2)})$. For ease of notation, we omit t_0, x_0 in J^1 and J^2 in this proof without causing any confusion.

According to the definition of the lower value function, it follows that

$$\begin{aligned}
J^1(u_1^{(1)}, u_2^{(2)}) \leq J^1(u_1^{(1)}, u_2^{(1)}) \leq J^1(u_1^{(2)}, u_2^{(1)}), \\
J^2(u_1^{(2)}, u_2^{(1)}) \leq J^2(u_1^{(2)}, u_2^{(2)}) \leq J^2(u_1^{(1)}, u_2^{(2)}).
\end{aligned} \tag{11}$$

Furthermore, Lemma 3 reveals that

$$\begin{aligned}
|J^1(u_1^{(1)}, u_2^{(2)}) - J^2(u_1^{(1)}, u_2^{(2)})| \leq C_1(u_1^{(1)}, u_2^{(2)})\delta, \\
|J^1(u_1^{(2)}, u_2^{(1)}) - J^2(u_1^{(2)}, u_2^{(1)})| \leq C_1(u_1^{(2)}, u_2^{(1)})\delta.
\end{aligned} \tag{12}$$

Let $C = \max\{C_1(u_1^{(1)}, u_2^{(2)}), C_1(u_1^{(2)}, u_2^{(1)})\}$, we have

$$\begin{aligned}
&|J^1(u_1^{(1)}, u_2^{(1)}) - J^2(u_1^{(2)}, u_2^{(2)})| \\
\leq&|J^1(u_1^{(1)}, u_2^{(2)}) - J^2(u_1^{(1)}, u_2^{(2)})| \\
\leq&C\delta.
\end{aligned} \tag{13}$$

■

Now that the SDG $\mathcal{G}^2$ approximates well with respect to the upper and lower functions of δ-game, it helps to develop a sufficient condition to ensure the existence of the game value of the original game $\mathcal{G}^0$.

Theorem 3: A sufficient condition for the existence of a game value for the game $\mathcal{G}^0$, i.e., $\mathbf{V}_-^0(t_0, x_0) = \mathbf{V}_+^0(t_0, x_0)$, is that Isaacs' condition holds for any $\delta \in (0, 1)$: for all $(t, x, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d$, there is

$$H^-(t, x, p, A) = H^+(t, x, p, A) := H(t, x, p, A),$$

where

$$\begin{aligned}
H^-(t, x, p, A) = \sup_{u_2(t) \in U} \inf_{u_1(t) \in U} \{\langle p, Ae^{A\delta}X_t + \Gamma_t\rangle \\
+ \frac{1}{2}\operatorname{tr}[\delta\Sigma_t A] + h(t, x, u_1(t), u_2(t))\Big\},
\end{aligned}$$

and

$$\begin{aligned}
H^+(t, x, p, A) = \inf_{u_1(t) \in U} \sup_{u_2(t) \in U} \{\langle p, Ae^{A\delta}X_t + \Gamma_t\rangle \\
+ \frac{1}{2}\operatorname{tr}[\delta\Sigma_t A] + h(t, x, u_1(t), u_2(t))\Big\}.
\end{aligned}$$

Proof: Given $\delta \in (0, 1)$, if the Isaacs' condition holds, we can conclude that the value of the SDG $\mathcal{G}^2$ exists, i.e., $\bar{\mathbf{V}}_\delta^-(t_0, x_0) = \bar{\mathbf{V}}_\delta^+(t_0, x_0)$. For the classical theories on the Isaacs' condition, please refer to some surveys, e.g., [16].

According to the definition of mixed strategy and Theorem 2, it follows that

$$\begin{aligned}
&|\mathbf{V}_-(t_0, x_0) - \mathbf{V}_+(t_0, x_0)| \\
=&\lim_{\delta \to 0} |\mathbf{V}_-^1(t_0, x_0) - \mathbf{V}_+^1(t_0, x_0)| \\
=&\lim_{\delta \to 0} |\mathbf{V}_-^1(t_0, x_0) - \bar{\mathbf{V}}_\delta^-(t_0, x_0) + \bar{\mathbf{V}}_\delta^-(t_0, x_0) \\
&- \bar{\mathbf{V}}_\delta^+(t_0, x_0) + \bar{\mathbf{V}}_\delta^+(t_0, x_0) - \mathbf{V}_+^1(t_0, x_0)| \\
\leq&\lim_{\delta \to 0} |\mathbf{V}_-^1(t_0, x_0) - \bar{\mathbf{V}}_\delta^-(t_0, x_0)| + |\bar{\mathbf{V}}_\delta^-(t_0, x_0) \\
&- \bar{\mathbf{V}}_\delta^+(t_0, x_0)| + |\bar{\mathbf{V}}_\delta^+(t_0, x_0) - \mathbf{V}_+^1(t_0, x_0)| \\
\leq&\lim_{\delta \to 0} |\bar{\mathbf{V}}_\delta^-(t_0, x_0) - \bar{\mathbf{V}}_\delta^+(t_0, x_0)| + 2C\delta \\
=&0.
\end{aligned} \tag{14}$$

■

Corollary 1: If there exist functions h_1 and h_2 such that $h(t, x, u_1(t), u_2(t)) = h_1(t, x, u_1(t)) + h_2(t, x, u_2(t))$, the game value exists for the game $\mathcal{G}^0$.

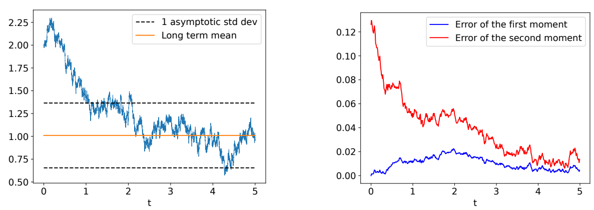
Remark 1: The proposed sufficient condition for the existence of game value is inclusive, as it encompasses a vast majority of the commonly employed payoff functions, including but not limited to the linear quadratic game and a variety of other games.

V. Simulation

A. Simulation Setup

For a one-dimensional differential game with mixed strategy $\mathcal{G}^0$, suppose $\Gamma_t = \gamma \in \mathbb{R}$, constant $\Sigma_t = \sigma \in \mathbb{R}$ and $A \in \mathbb{R} < 0$. In this section, we let $A = -1$, $B_1 = B_2 = 1$, $\gamma = 1$, $\sigma = 25$ and $x_0 = 2$. We numerically simulate the dynamics of the δ-game $\mathcal{G}^1$ and the weak approximated game $\mathcal{G}^2$, where δ is set as 0.01. For simplicity, we suppose that $B_1 u_{1k} + B_2 u_{2k}$ is a uniformly distributed random variable. Since its expectation and variance are γ and σ respectively, we have $B_1 u_{1k} + B_2 u_{2k} \sim \mathrm{Uni}(\gamma - \sqrt{3\sigma}, \gamma - \sqrt{3\sigma})$.

B. Results and Analysis



(a) A trajectory of the approximated stochastic differential equation (b) Approximation errors of the first two moments

Fig. 1. Simulation of a one-dimensional differential game with mixed strategies. (a) presents the dynamics of the weak approximated game $\mathcal{G}^2$; (b) shows the approximation errors between $\mathcal{G}^1$ and $\mathcal{G}^2$ on the first two moments

Fig. 1(b) verifies the approximation error between δ-game $\mathcal{G}^1$ and the weak approximated game $\mathcal{G}^2$ is indeed of the scale $O(\delta)$. Therefore, it is reasonable to use the dynamic of game $\mathcal{G}^2$ to analyze the game $\mathcal{G}^1$. $\mathcal{G}^2$ has a huge advantage for analyzing the game dynamic in this simulation setting, because it shows explicitly the distribution of state X_t, which asymptotically converges to an equilibrium Gaussian distribution $\mathcal{N}\left(-\frac{\gamma}{Ae^{A\delta}}, -\frac{\delta\sigma}{2Ae^{A\delta}}\right)$. When δ is fixed, the expectation and variance are determined by γ and σ respectively.

The dynamic of $\mathcal{G}^2$ is presented in Fig. 1(a). The expected value of X_t given by $x_0 e^{-\theta t} + \xi\left(1 - e^{-\theta t}\right)$ approaches ξ exponentially with a decay rate of $-\theta$. However, the variance of X_t given by $\frac{\delta\sigma}{2\theta}\left(1 - e^{-2\theta t}\right)$ increases from zero to a limiting value of $\frac{\delta\sigma}{2\theta}$. The transition time point between the descent phase and the fluctuation phase is determined by equating the expected value of X_{t^*} to the square root of its variance. When $t < t^*$, descent dominates, and when $t > t^*$, fluctuation dominates.

In a pursuit-evasion game setting, γ represents the purser's prediction error on the expected strategy of the evader at each δ step, and σ quantifies how random the mixed strategies are. Intuitively, the pursuer desires to minimize γ and σ so that the expected distance is 0 and the fluctuation is so small that with a high probability, the distance is maintained in an acceptable scale.

VI. Conclusions

In conclusion, this paper contributes to the differential games with mixed strategies by demonstrating the effectiveness of the weak approximation method in analysis. By defining mixed strategy in terms of stochastic processes with independent increments, the study provides a comprehensive framework for exploring the novel strategic behavior generated by mixed strategies. Moreover, the establishment of sufficient conditions for ensuring the existence of game value enhances the practical applicability of the approach. The precise approximation of the payoff function achieved through the proposed weak approximation method also increases the accuracy and reliability of the results. Future researches include designing optimal game strategies and generalizing the weak approximation method to the time-variant systems.

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