2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Robustness of Log-Linear Learning in Network Coordination Games with Stubborn Players

Aqsa Shahzadi Akber and Hassan Jaleel

***Abstract*— We analyze Log-Linear Learning (LLL) in a networked multi-agent system with stubborn players that can influence other players but do not update their actions. We are interested in the robustness of LLL against stubborn players in a coordination game setup in which the players have to decide between a status quo and an innovative practice that is inherently superior to the status quo. We investigate the impact of interaction network topology and the payoff gain offered by the innovation on the steady state behavior of the population in the presence of stubborn agent/s. We present conditions for the robustness of various networks, namely a class of 3-regular networks, $n \times n$ grid networks, and Erdős-Rényi (ER) random networks. For these networks, we derive the threshold values of the payoff gain for which the system behavior is robust to the presence of stubborn players under LLL.**

I. Introduction

In the field of population dynamics, a key research challenge involves analyzing the influence of the actions taken by a small subset of the population on the global behavior of the network. For instance, the fields of sociology and epidemiology investigate the spread of ideas, decisions, and diseases through a population based on the interactions among individuals. This problem of diffusion of behavior is typically investigated under the setup of network coordination games (see for instance [1], [2], [3], [4], [5], [6], and [7]).

An underlying assumption in the current population dynamics literature is that the agents within the studied population are homogeneous, meaning that all players' decision strategies are identical, and the agents follow some form of noisy best/better response dynamics (see for instance the setups in the references [1]–[7]). This model oversimplifies the behavior of large populations of players since individuals can have diverse decision strategies. As a result, it is worth exploring whether a few players with notably different decision strategies can influence the long-term behavior of the population.

In this paper, we focus on a scenario in which a few stubborn individuals are introduced into the system. We define stubborn agents as those that can influence the decisions of other players but they do not update their own actions. Our objective is to establish the implication of having stubborn player/s on the long-run behavior of the overall population in network coordination games. In particular, we deal with the problem of finding conditions based on game parameters and network structures that assist in changing the otherwise stochastically stable states. This paper is based on our earlier work [8], in which we introduced a novel notion of robustness to quantify the impact of heterogeneous players. Here, we extend our analysis of stubborn players for some important classes of networks such as 3-regular networks, 3×3 grid networks, and Erdos-Renyi networks.

A. S. Akber and H. Jaleel are with the Intelligent Machines & Sociotechnical Systems (iMaSS) Lab, Department of Electrical Engineering, Syed Babar Ali School of Science & Engineering at LUMS, Lahore, Pakistan. H. Jaleel is also affiliated with NCRA National Agricultural Robotics Lab at LUMS. Email: 19060048@lums.edu.pk and hassan.jaleel@lums.edu.pk

In the recent literature on evolutionary game theory, the robustness of stochastic learning dynamics towards player heterogeneity has been examined in various aspects. In [9], authors investigated the robustness of behavioral rules when 1) a single agent takes different actions at different times 2) different agents follow different behavioral rules 3) agents update their actions synchronously. Their analysis was based on an asymmetry property of the learning dynamics. The authors in [10] examined the robustness of stochastically stable states to simultaneous action updates.

The impact of interaction structure and behavioral rule on equilibrium behavior is widely studied in the literature on games in networks. In [11], authors showed that best response dynamics lead to the spread of risk-dominant equilibrium in coordination games over a network of players. They presented conditions on the payoff parameter for which an action becomes the best response of all the players in the network. The impact of heterogeneous player preferences on the heterogeneity in steady state behavior is investigated in [12] and a class of networks is identified that exhibit this behavior. In [13], [14], and [15], authors evaluated the influence of a static adversary on the performance of the system evolving under graphical coordination games. In particular, they determine the relation between the complexity of the adversary's strategy, the level of the adversary's knowledge about the system, and the degradation of efficiency in the overall system. In [16], authors studied the effect of the degree of rationality of a player on the risk-dominant Nash equilibrium [17] under best response dynamics.

Motivated by the analysis mentioned above, we are interested in the scenarios under which a network becomes robust to the presence of stubborn agents for classes of some important networks such as 3-regular networks, $m \times n$ grid networks, and Erdos-Renyi networks.Our objective is to understand how can a stubborn agent exploit the interaction structure, and which factors are responsible for assisting this behavior.

II. Setup

We consider a 2×2 symmetric coordination game played over a network in which each player i interacts with a

979-8-3503-0123-6/23/$31.00 ©2023 IEEE

subset of players. In a coordination game, each player has two action choices, namely A, which typically represents an innovation, and B, which represents the status quo. Let $A_i = \{A, B\}$ be the set of actions of player i. The payoff matrix for a two player coordination game is shown in Fig. 1, where α is called the payoff gain of innovation A over B.

	A	B
A	$1+\alpha, 1+\alpha$	$0, 0$
B	$0, 0$	$1, 1$

Fig. 1. Payoff matrix of a 2×2 coordination game.

The interaction network is represented by a connected and undirected graph $\Gamma(V, E)$ in which the vertex set corresponds to the set of players $N = \{1, 2, ..., n\}$, and the edge set corresponds to the interactions between the players. If an edge $i \sim j$ between players i and j exists in the edge set E, then we say that i and j are neighbors. Let $\mathcal{N}_i$ represents the set of neighbors of player or node i i.e. $\mathcal{N}_i = \{j \in N \mid i \sim j \in E\}$. The total payoff for a player is obtained by summing the payoffs obtained from each individual interaction with all of its neighbors in the set $\mathcal{N}_i$.

Action profile at any time step t is an $n-$dimensional vector $\sigma \in \{A, B\}^n$. We will also represent σ as $\sigma = (\sigma_i, \sigma_{-i})$, where σ_i is the action of player i and σ_{-i} is the vector of actions of all the player other than i. Given an action profile σ, let $\eta_A^\sigma(i)$ and $\eta_B^\sigma(i)$ be the fraction of the neighbors of i playing actions A and B, respectively, i.e., $\eta_A^\sigma(i) + \eta_B^\sigma(i) = 1$. For notational convenience, we will suppress σ in $\eta_A^\sigma(i)$ and $\eta_B^\sigma(i)$.

The payoffs of player i for actions A and B given σ_{-i} are $U_i(A, \sigma_{-i}) = \eta_A(i)(1 + \alpha)$ and $U_i(B, \sigma_{-i}) = \eta_B(i)$, respectively. Player i's best response is to select an action that maximizes its payoff from its interaction with the neighboring players, i.e., action A belongs to $B_i(\sigma_{-i})$, which is the best response set of player i, if $U_i(A, \sigma_{-i}) \geq U_i(B, \sigma_{-i})$. Let 1_A and 1_B be the n-dimensional vectors that represent the action profiles where all the players are playing actions A and B, respectively. Furthermore, $1_A^k/1_B^k$ represents the state where k players have switched to action B/A and the remaining $N - k$ players continue to play action A/B, respectively.

A. Learning Dynamics

We consider an update rule in which the players update their strategies based on noisy best response dynamics. One such dynamics is Log-Linear-Learning (LLL), which was originally proposed in [3]. In LLL, players update their actions at discrete time steps. At each time t, one player, say player i, is selected uniformly at random. All the other players repeat their actions from previous step $t - 1$. The selected player updates its action from $\sigma = (\sigma_i, \sigma_{-i})$ to $\sigma' = (\sigma_i', \sigma_{-i})$ with probability

$$p_i(\sigma_i^{'}, \sigma_{-i}) = \frac{e^{-\frac{1}{\tau}[U_i(\sigma_i^*, \sigma_{-i}) - U_i(\sigma_i', \sigma_{-i})]}}{\sum\limits_{\bar{\sigma}_i \in A_i} e^{-\frac{1}{\tau}[U_i(\sigma_i^*, \sigma_{-i}) - U_i(\bar{\sigma}_i, \sigma_{-i})]}}, \quad (1)$$

where σ_i^* is the best response of player i to σ_{-i} at time t and τ is the noise in decision making. The probability of selecting an action σ_i decreases as the utility of that action relative to the best action decreases. This relative utility is called the resistance of the transition from an action profile σ to σ', and is defined as

$$R(\sigma, \sigma') = V_i(\sigma_{-i}) - U_i(\sigma', \sigma_{-i}), \quad (2)$$

where $V_i(\sigma_{-i}) := \max_{\sigma_i \in A_i} U_i(\sigma_i, \sigma_{-i})$. A path $P = \{\sigma^0 \to \sigma^1 \to \sigma^2 \to \cdots \to \sigma^k\}$ is a sequence of joint action profiles such that each subsequent action profile is a result of a single player update. The resistance of the path is the sum of resistances of each transition, i.e. $R(P) = \sum_{i=1}^k R(\sigma^{i-1}, \sigma^i)$. Given any two states σ and σ', $R(\sigma, \sigma')$ is the resistance of the minimum resistance path from σ to σ'.

Log-linear learning induces a regularly perturbed Markov chain with perturbation parameter τ over the set of joint action profiles with a unique stationary distribution $\mu^\tau(\sigma)$ [1]. *A state is said to be stochastically stable if and only if* $\lim_{\tau \to 0} \mu^\tau(\sigma) > 0$. *A state is stochastically stable if it has a positive probability in the stationary distribution as noise vanishes.*

B. Analysis Approach

To determine stochastically stable states, we will employ Radius(Rd) and Coradius(CR) based analysis, which was originally proposed in [18] and was later extended for LLL in [19]. The Radius and Coradius of a state σ are defined as

$$Rd(\sigma) = \min\{R(\sigma, \sigma') \mid R(\sigma', \sigma) \neq 0\}, \text{ and} \quad (3)$$

$$CR(\sigma) = \max\{R(\sigma', \sigma) \mid R(\sigma', \sigma) \neq 0\}. \quad (4)$$

Here, Radius is a measure of how easy it is to leave a state and Coradius is a measure of how difficult it is to reach a particular state from any other state. Note that state and action profile are used interchangeably throughout the paper. *According to Prop.2 of [19], given an action profile* σ*, if* $Rd(\sigma) > CR(\sigma)$ *then* σ *is stochastically stable state.* Thus, the *Rd-CR* criteria of Prop. 2 is typically a *sufficient condition* to characterize the stochastic stability of a given state.

Next, we establish that for a two-action coordination game setup, *Rd-CR* criteria is a necessary condition as well.

Proposition 1: For a network coordination game with the pairwise payoff matrix given in Fig. 1, an action profile σ is stochastically stable if and only if $Rd(\sigma) > CR(\sigma)$.

Proof: In the coordination game of Fig. 1, there are two candidates for stochastically stable profiles $\mathbf{1}_A$ and $\mathbf{1}_B$. Let $P_{A,B}$ and $P_{B,A}$ be the minimum resistance path from $\mathbf{1}_A$ to $\mathbf{1}_B$ and from $\mathbf{1}_B$ to $\mathbf{1}_A$, respectively. Then, the Radius and Coradius of the two profiles are

$$Rd(\mathbf{1}_A) = R(P_{A,B}) \text{ and } CR(\mathbf{1}_A) = R(P_{B,A}).$$
$$Rd(\mathbf{1}_B) = R(P_{B,A}) \text{ and } CR(\mathbf{1}_B) = R(P_{A,B}).$$

Since $Rd(\mathbf{1}_A) = CR(\mathbf{1}_B)$ and $CR(\mathbf{1}_A) = Rd(\mathbf{1}_B)$, it is obvious that either $Rd(\mathbf{1}_A) > CR(\mathbf{1}_A)$ or $Rd(\mathbf{1}_B) > CR(\mathbf{1}_B)$. Which ever profile satisfies this criteria will be the unique stochastically stable profile. ■

III. Robustness Against Single Stubborn Player

Consider a coordination game with the payoff matrix given in Fig. 1 with $\alpha \in (0, 1)$. The game is played over a network in which all the players are updating their actions using LLL. We refer to this framework with homogeneous players as the standard setup. If we incorporate a few stubborn players in the standard setup, we will characterize the resulting setup as a *heterogeneous setup with players that never update their actions and always play action* B. Our goal is to assess the influence of stubborn players on the population's behavior, assuming that the other players update their strategies using LLL. To measure the impact of the stubborn players, we employ the concept of robustness, which was first introduced in [8].

Definition 3.1: [8] Let S be the set of stochastically stable action profiles for stochastic learning dynamics in the standard setup, and let σ be an element in S. Suppose all the players in a subset $H \subset N$ are replaced with stubborn players, and let S_H be the set of stochastically stable action profiles in the heterogeneous setup. Then, $\sigma = (\sigma_H, \sigma_{-H})$ is robust to stubborn players in H if there exists a σ' in S_H such that $\sigma'_{-H} = \sigma_{-H}$.

Thus, a stochastically stable action profile is robust to stubborn behavior if the behavior of the remaining population remains unchanged when any subset H of players is replaced with stubborn players.

In the following section, we will derive the conditions under which a class of 3-regular networks and $n \times n$ grid networks are robust against the addition of a single stubborn player. These networks are considered because of their significance in the population dynamics literature [11]. An important characteristic of these networks is that there does not exist any central node with an overwhelming influence on the rest of the network like the central node in the wheel network. Even within these networks, we demonstrate that a single player has the potential to alter the long-run behavior of the entire population.

A. 3-Regular networks

A 3-regular network, also known as a cubic network, is a graph in which every node has exactly three neighbors. We consider a subset of 3-regular networks that have non-overlapping neighborhood sets.

Definition 3.2: A 3-regular network has non-overlapping neighborhood sets if for any node pair i and j such that $j \in \mathcal{N}_i$ and $i \in \mathcal{N}_j$, $\mathcal{N}_i \cap \mathcal{N}_j = \phi$.

We first present an algorithm for computing the minimum resistance paths for 3-regular networks with non-overlapping neighborhood sets.

Definition 3.3: Given an undirected network Γ in which $N = \{1, ...n\}$ is the node set, let C be a subset of N. A node i in the set N is covered by C if either i belongs to C or $\mathcal{N}_i \cap C \neq \phi$, where $\mathcal{N}_i$ is the neighborhood set of i. Furthermore, if $|\mathcal{N}_i \cap C| = k$, we say that i is *k-covered* by the set C.

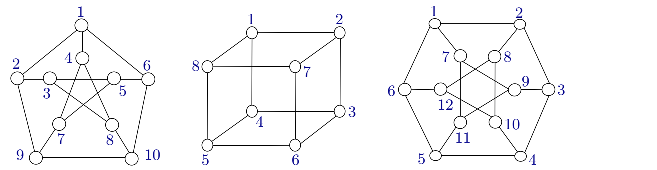
Minimum Resistance Path Algorithm: We consider a 3-regular network with N nodes and non-overlapping neighborhood sets. Given a covering set $C \subset N$, let C_N be the set of nodes in N that are covered by C. Let C^k by the covering set after k iterations of the algorithm presented below.

1) [Step: 0 Initialize] $C^0 = \phi$, and $C^0_N = \phi$.
2) [Step: 1 Add first node to the covering set] $C^1 = \{i_1\}$, where i_1 is a randomly selected node from N and $C^1_N = \{i_1\} \cup \mathcal{N}_{i_1}$. Since each node is connected with 3 other nodes, $|C_{N,1}| = 4$.
3) [Step 2] Select a node i_2 from the set $\mathcal{N}_{i_1}$ and add it to C^1, which results in $C^2 = C^1 \cup \{i_2\}$ and $C^2_N = C^1_N \cup \mathcal{N}_{i_2}$. Since i_2 and i_1 are already in C^1_N, we get $|C^2_N| = |C^1_N| + (3-1)$.
4) [Step: k : Add subsequent nodes to the covering set] Select a node i_k such that $i_k \in C^{k-1}_N \backslash C^{k-1}$ and i_k has two uncovered neighbors. Add i_k to C^{k-1}, which results in $C^k = C^{k-1} \cup \{i_k\}$ and $C^k_N = C^{k-1}_N \cup \mathcal{N}_{i_k}$. Repeat the above step until all the nodes in the set N are at least 1-covered. After k steps of the algorithm, we will have $|C^k_N| = 4 + 2(k-1)$. Let k' be the number of steps after which all the nodes are at least one covered. Then, $4 + 2(k'-1) = N \implies k' = \frac{N-2}{2}$.
5) [Add node for 2-coverage] After all the nodes are 1-covered, add one more element $i_F \in C^{k'}_N$ to $C^{k'}$ and let the final set be denoted as C^F. As a result of this final node, at least one of the nodes in C^F_N will be two covered by C^F.

As the algorithm terminates, sequence in the set C^F gives us a path from an initial node i_1 to some terminal node. To better explain the steps of the minimum resistance path algorithm, we demonstrate its steps for Peterson network. A Peterson graph is a 3 regular graph with 10 nodes and 15 edges as shown in Fig. 2(a).

1) Lets initialize C^1 with node 1 i.e. $C^1 = \{1\}$, $C^1_N = \{1, 2, 4, 6\}$, $|C^1_N| = 3 + 1 = 4$
2) For $i_2 = 6$, $C^2 = \{1, 6\}$, $C^2_N = \{1, 2, 4, 6, 5, 10\}$, $|C^2_N| = |C^1_N| + 2 = 6$
3) Let $i_3 = 5$, $C^3 = \{1, 6, 5\}$, $C^3_N = \{1, 2, 4, 6, 5, 10, 3, 7\}$, $|C^3_N| = |C^2_N| + 2 = 8$
4) Let $i_4 = 10$, $C^4 = \{1, 6, 5, 10\}$, $C^4_N = \{1, 2, 4, 6, 5, 10, 3, 7, 9, 8\}$, $|C^4_N| = |C^3_N| + 2 = 10$
5) Since all the nodes are converted to $1-$covered nodes in the previous step, we add an additional node to C^4 to make a node $2-$covered i.e. $C^F = \{1, 6, 5, 10, 4\}$.

The question that arises is how this algorithm is relevant to our robustness analysis. In the $RD-CR$ analysis, we have to compute minimum resistance paths between states 1_A and 1_B. For a 3-regular network with non-overlapping neighborhood sets, this algorithm gives us one such path without explicitly computing and comparing the resistances of all possible paths. In particular, starting from $\mathbf{1}_B$ and $\mathbf{1}_A$, the sum of the resistances of the path $\{i_1, i_2, \ldots, i_F\}$, which are the nodes in the set C^F, gives us the Radius and the Coradius of the action profile 1_B, respectively, in the absence



(a) Peterson graph (b) Cube graph (c) Generalized Peterson graph $G(6,2)$

Fig. 2. Examples of 3-regular network with non-overlapping neighborhood

of any stubborn player.

The inclusion of a stubborn player that always plays action B only introduces a change in the choice of initial node i_1. When computing the Coradius of $\mathbf{1}_B$, we start with the configuration $\mathbf{1}_A^1$ (stubborn player always plays B). Then, the stubborn node is selected as i_1 because we are interested in computing a minimum resistance path that leads to state 1_B. However, the stubborn node does not incur any resistance since it always plays action B. Therefore, the resistance path will be $P = \{1_A^1, 1_A^2, \dots, 1_B\}$. Switching a node that is a neighbor of a stubborn node gives us the minimum resistance of the switch from 1_A^1 to 1_A^2. We get a minimum resistance path if we follow our minimum resistance path algorithm. While computing the radius of $\mathbf{1}_B$, a node that is not covered by a stubborn node is selected as i_1 to maximize its impact on the network. Selecting a node from the neighborhood set of the stubborn player will not have the maximum impact since the stubborn player cannot be influenced.

Theorem 3.1: Given a 3-regular network with non-overlapping neighborhood sets such that any neighboring node pair (i,j) satisfies $\mathcal{N}_i \cap \mathcal{N}_j = \phi$, action profile 1_A is robust against a single stubborn player if $\alpha > 2/(N-2)$.

Proof: Suppose we start with configuration 1_A. Then, applying the minimum resistance path algorithm, the resistance of the first noisy action is $R(1_A, 1_A^1) = 3(1+\alpha)$. Afterward, each of the remaining noisy actions will have a resistance of $R(\mathbf{1}_A^k, \mathbf{1}_A^{k+1}) = 2(1+\alpha) - 1 = 1 + 2\alpha$. Minimum resistance from 1_A to 1_B is $R(1_A, 1_B) = 3(1+\alpha) + \frac{N-2}{2}(1+2\alpha)$. Starting from the other extreme of $\mathbf{1}_B$ and applying the minimum resistance path algorithm, the resistance of the first noisy action will be $R(\mathbf{1}_B, \mathbf{1}_B^1) = 3$. Each of the remaining noisy actions will have a resistance of $R(\mathbf{1}_B^k, \mathbf{1}_B^{k+1}) = 2 - (1+\alpha) = 1 - \alpha$. Thus, the minimum resistance from 1_B to 1_A is $R(1_B, 1_A) = 3 + \frac{N-2}{2}(1-\alpha)$. To check for the robustness of 1_A against a single stubborn player, we place a single stubborn player in the network. Then, the effective resistance from 1_A^1 to 1_B is $R(1_A^1, 1_B) = \frac{N-2}{2}(1+2\alpha)$. And the effective resistance from 1_B to 1_A^1 is same as $R(1_B, 1_A)$. For 1_A to be stochastically stable, we should have $R(1_A^1, 1_B) > R(1_B, 1_A^1)$, which results in the desired condition $\alpha > 2/(N-2)$. ■

Illustrative examples:

1) A *cube network* is also a type of 3-regular network consisting of 8 nodes and 12 edges as displayed in Fig. 2(b). Node 1 is stubborn at B. The outcome of the minimum resistance path algorithm is the path $C = \{1,2,3,5\}$ for Radius and $C = \{8,7,4,6\}$ for Coradius. Radius and Coradius of these paths are $Rd(1_B) = 3 + 3(1-\alpha)$, and $CR(1_B) = 3(1+2\alpha)$. For $\alpha > 1/3$, the cube network is robust to the placement of a single stubborn agent.
2) *Generalized Petersen graphs* $G(n,k)$, as shown in Fig. 2(c) for $n = 6$ and $k = 2$, belong to the family of 3-regular graphs constructed by joining the nodes of an outer polygon comprising n nodes and inner star polygon of same size, inner polygon is formed such that each of its node is connected to the node at k^{th} distance. For this particular network, given that node 1 is made stubborn, a minimum resistance path from 1_A^1 to 1_B and 1_B to 1_A^1, as computed using the minimum resistance path algorithm, are $\{1,2,3,4,5,6\}$ and $\{n, n+1, \dots, 2n-1\}$, respectively, where nodes are numbered as shown in Fig. 2(c). The corresponding resistances of these paths are $R(1_A^1 \rightarrow 1_B) = 5(1+2\alpha)$, and $R(1_B \rightarrow 1_A^1) = 3 + 5(1-\alpha)$. Thus, $\alpha > 1/5$ results in the robustness of $\mathbf{1}_A$.

We have seen examples of 3-regular networks where a single stubborn player can alter the stochastically stable action profile if the payoff parameter satisfy certain condition. In general, increasing the number of stubborn agents relaxes the condition on the payoff parameter. Now the question arises, can there be a scenario when addition of a set of stubborn agents do not have any impact on the population behavior at all, no matter how large that set is. To illustrate this point concretely, we present a condition for $d-$regular network under which no number of stubborn agents can alter the equilibrium behavior.

Proposition 2: Any $d \geq 3$ regular network is robust to addition of any number of stubborn players if $\alpha > d-2$.

Proof: Consider a finite $d \geq 3$ regular network with N total players out of which $N-2$ nodes are stubborn. Radius of action profile 1_B is r and $CR(1_B) = 0$ for $\alpha \geq d-2$ and $CR(1_B) = \alpha - (d-2)$ for $\alpha < d-2$. It is apparent that for $\alpha > d-2$, network is robust to placement of $N-2$ stubborn agents. ■

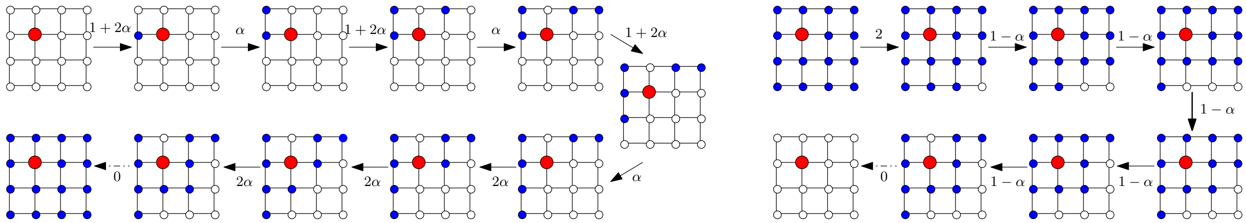
This result is helpful from system design perspective and provides a sufficient condition for robustness.

B. 2D Grid Network

Next, we consider a family of $m \times n$ grid networks. These networks are important because they are studied extensively in the context of network coordination games. We first present the robustness result for the square grid network and later extend it for any $m \times n$ grid network.

Theorem 3.2: An $n \times n$ square grid network with n^2 agents is not robust to the addition of a single stubborn player if $\alpha < 2/(n^2 - n - 2)$.

Proof: We begin the proof with an example of 4×4 grid network and then extend the analysis to a general $n \times n$ grid network. Since the internal nodes have a maximum degree, one of the internal nodes is made stubborn at action B to analyze the worst-case impact of a single stubborn



(a) Minimum resistance path from 1_A to 1_B (b) Minimum resistance path from 1_B to 1_A

Fig. 3. Minimum resistance paths between the states 1_A^1 and 1_B in the grid network. Blue nodes are the players with action B, white nodes are the players with action A, and stubborn players are shaded with red.

agent. The minimum resistance paths of transitioning from action profile 1_A^1 to 1_B and 1_B to 1_A^1 are shown in Fig. 3. These paths are obtained by sequentially selecting nodes that can switch to noisy action with minimum resistance. Based on this approach, there exist multiple minimum resistance paths and Fig. 3 presents one particular instance. Radius and Coradius of action profile 1_B are $Rd(1_B) = 2+5(1-\alpha)$, and $CR(1_B) = 3(1+2\alpha)+3\alpha+3(2\alpha)$. Comparison of Rd and CR gives us the condition that for $\alpha < 1/5$ action profile 1_B is stochastically stable. An $n\times n$ grid consists of 4 corner nodes with degree 2, $4(n-2)$ boundary nodes with degree 3 and $(n-2)^2$ internal nodes with degree 4. While computing the Coradius of 1_B, it is evident that corner nodes always switch with the resistance of α, boundary nodes switch with the resistance of $1+2\alpha$, and internal nodes switch with the resistance of 2α. Radius and Coradius of the action profile 1_B are $Rd(1_B) = 2 + 2(n-2)(1-\alpha) + (1-\alpha)$, and $CR(1_B) = (n-2)(1+2\alpha) + (n-3)(1+2\alpha) + 3\alpha + ((n-2)^2-1)(2\alpha)$. Comparing the above two, Rd is greater than CR if $\alpha < 2/(n^2-n-2)$. ■

Corollary 3.3: An $m \times n$ rectangular grid network with $m \times n$ agents is not robust to addition of single stubborn player if $\alpha < 2/(mn-2n+m-2)$.

The above result is a direct extension of the the result proved in Thm. 3.2.

IV. Robustness Results for Multiple Stubborn Players

This section is concerned with the problem that for a given value of α, what is the minimum number of players, such that if they are fixed at playing action B, LLL will ensure that action B is eventually played everywhere. In other words, what is the minimum number/set of stubborn agents such that 1_B becomes the stochastically stable action profile? We consider Erdős-Rényi (ER) random networks for the analysis of multiple stubborn players.

An Erdos-Renyi(ER) graph, denoted by $G(N,p)$, is a random graph with N vertices and an edge is formed between any two vertices with probability p. ER graphs are important as they serve as baseline models for statistical comparisons. In this network model, the parameter p can be interpreted as the degree of influence that the players have on each other. For ER networks, we presented results with a single stubborn player in [20]. Now we are extending the analysis for multiple stubborn players and providing a sufficient condition on the threshold of α such that the network is no longer robust to k stubborn players.

Theorem 4.1: An ER graph with N nodes and connectivity p is robust to the presence of k stubborn players if $\alpha > 2k/(N-k-1)$.

Proof: Let k be the number of stubborn players and N be the total number of players in the network. Here we assume that the stubborn players have the same level of influence on the rest of the population as that of any other players. In the random network setup, the utility function of any player, say player i, is the expected utility. Let n_A and n_B be the number of homogeneous players other than i playing actions A and B, respectively such that $n_A + n_B = N-1-k$. Then, the utility function of player i is $U_i(A, a_{-i}) = p\frac{n_A}{N}(1+\alpha)$, and $U_i(B, a_{-i}) = p\frac{n_B}{N} + p\frac{k}{N}$. Consider the case when all players are initially playing action B. Then, the resistance of going from 1_B to 1_B^1 is

$$R(1_B, 1_B^1) = \frac{p(N-1-k)+pk}{N} = \frac{p(N-1)}{N}. \quad (5)$$

Also, the resistance of going from 1_A^k to 1_A^{k+1}, where $k \geq 0$, is

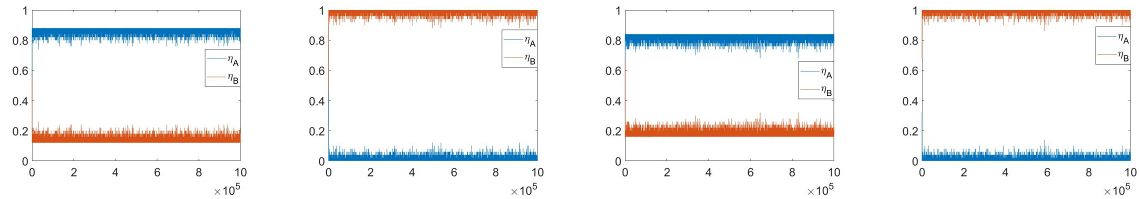
$$R(1_A^k, 1_A^{k+1}) = \frac{p(N-k-1)(1+\alpha)}{N} - \frac{pk}{N}. \quad (6)$$

We are interested to find out the condition for which 1_A remains stochastically stable, i.e. $R(1_A^k \to 1_A^{k+1}) > R(1_B \to 1_B^1)$. A simple comparison gives the desired condition on the value of α above which the network is robust to the addition of k stubborn players, i.e. $\alpha > 2k/(N-k-1)$.

Similarly, based on the proof of Proposition 1, which establishes $Rd(\mathbf{1}_A) = CR(\mathbf{1}_B)$ and $CR(\mathbf{1}_A) = Rd(\mathbf{1}_B)$, it becomes clear that the network loses its robustness against the addition of k stubborn players when α is less than $2k/(N-k-1)$. ■

If the strategic player has the same level of influence as the other players in the population, then we need k stubborn players, as specified by Thm. 4.1, to alter the behavior of the entire population. Whereas, if we assume that the strategic player can have a higher level of influence than the rest of the population, then we would require less stubborn players to cause contagion to 1_B. This higher influence is modeled by p_h, where p_h is the probability that the strategic player has an any edge with any other player in the network.

Corollary 4.2: In a random ER graph of connectivity p, and let p_h be the connectivity of strategic players, the



(a) $k = 6$, $\alpha = 0.329$, $p_h = p = 0.3$ (b) $k = 6$, $\alpha = 0.229$, $p_h = p = 0.3$ (c) $\alpha = 0.9$, $k = 8$, $p_h = 0.7$, $p = 0.3$ (d) $\alpha = 0.9$, $k = 10$, $p_h = 0.7$, $p = 0.3$

Fig. 4. Evolution of population in ER network in the presence of k stubborn players with $N = 50$,noise $\tau = 0.2$, and 1×10^6 iterations.

minimum number of stubborn players required to make it non-robust are $\lceil (pN\alpha)/(p\alpha + 2p_h) \rceil$.

Simulation results for ER network: To verify the result in the Thm. 4.1 and Cor. 4.2, we simulated a population with $N = 50$ players. In Fig. 4(a) and 4(b), random network is generated with parameter $p = p_h = 0.3$, Payoff gain is $\alpha + \epsilon$ in Fig. 4(a) and $\alpha - \epsilon$ in Fig. 4(b), with $\epsilon = 0.05$ and α is computed from the threshold given in Thm. 4.1. The players update their actions using LLL with $\tau = 0.2$. All players are initialized with random actions and simulated for 10^6 iterations. Blue graph represents the fraction of players playing action A and red graph represents fraction of players playing action B. For Fig. 4(c) and 4(d), a stubborn player is included in the population with $p_h = 0.7$. Parameter p_h indicates that the stubborn player has more influence over other players compared to the rest of the nodes in the network. Also, k is $\lceil (pN\alpha)/(p\alpha + 2p_h) \rceil - 1$ for Fig. 4(c) and $\lceil (pN\alpha)/(p\alpha + 2p_h) \rceil + 1$ for Fig. 4(d). The results show that k stubborn players were not successful in changing the behavior of the population from 1_A to 1_B for the cases when number of stubborn players are less than threshold given in Cor. 4.2, whereas when number of stubborn players exceeds this threshold, 1_B becomes the equilibrium action.

V. Conclusion

In this paper we have analyzed the impact of stubborn player/s in coordination games played over network of players under log linear learning dynamics. We have provided sufficient conditions for the robustness of stochastically stable states for a class of networks including 3-regular network, $n \times n$ grid network and ER network. These networks have uniform degree distribution, which means no single player have an advantage to influence the nodes more than any other node. Our results highlight that robustness property of log linear learning are function of payoff parameter α, for a certain value of α same network is robust and for other it becomes non-robust. For ER network, we have established results both for uniform and variable degree distribution. Furthermore, we have provided a graph theoretic algorithm for the computation of minimum resistance paths in 3-regular networks which are required for the computation of radius and coradius in stochastic stability analysis.

References

[1] H. P. Young, "The evolution of conventions," *Econometrica: Journal of the Econometric Society*, pp. 57–84, 1993.

[2] M. Kandori, G. J. Mailath, and R. Rob, "Learning, mutation, and long run equilibria in games," *Econometrica: Journal of the Econometric Society*, pp. 29–56, 1993.

[3] L. E. Blume, "The statistical mechanics of strategic interaction," *Games and economic behavior*, vol. 5, no. 3, pp. 387–424, 1993.

[4] G. E. Kreindler and H. P. Young, "Rapid innovation diffusion in social networks," *Proceedings of the National Academy of Sciences*, vol. 111, no. Supplement 3, pp. 10 881–10 888, 2014.

[5] J. Newton and D. Sercombe, "Agency, potential and contagion," *Games and Economic Behavior*, vol. 119, pp. 79–97, 2020.

[6] A. Montanari and A. Saberi, "The spread of innovations in social networks," *Proceedings of the National Academy of Sciences*, vol. 107, no. 47, pp. 20 196–20 201, 2010.

[7] H. P. Young, "The dynamics of social innovation," *Proceedings of the National Academy of Sciences*, vol. 108, no. supplement_4, pp. 21 285–21 291, 2011.

[8] H. Jaleel, W. Abbas, and J. S. Shamma, "Robustness of stochastic learning dynamics to player heterogeneity in games," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 5002–5007.

[9] J. Newton, "Conventions under Heterogeneous Behavioural Rules," *The Review of Economic Studies*, vol. 88, no. 4, pp. 2094–2118, 10 2020.

[10] C. Alós-Ferrer and N. Netzer, "Robust stochastic stability," *Economic Theory*, vol. 58, no. 1, pp. 31–57, 2015.

[11] S. Morris, "Contagion," *The Review of Economic Studies*, vol. 67, no. 1, pp. 57–78, 01 2000.

[12] H. Peng, W. Peng, D. Zhao, and W. Wang, "Impact of the heterogeneity of adoption thresholds on behavior spreading in complex networks," *Applied Mathematics and Computation*, vol. 386, p. 125504, 2020.

[13] K. Paarporn, B. Canty, P. N. Brown, M. Alizadeh, and J. R. Marden, "The impact of complex and informed adversarial behavior in graphical coordination games," *IEEE Transactions on Control of Network Systems*, vol. 8, no. 1, pp. 200–211, 2021.

[14] B. Canty, P. N. Brown, M. Alizadeh, and J. R. Marden, "The impact of informed adversarial behavior in graphical coordination games," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 1923–1928.

[15] P. N. Brown, H. P. Borowski, and J. R. Marden, "Security against impersonation attacks in distributed systems," *IEEE Transactions on Control of Network Systems*, vol. 6, no. 1, pp. 440–450, 2018.

[16] H. H. Nax and J. Newton, "Deep and shallow thinking in the long run," *Available at SSRN*, 2020.

[17] M. Peski, "Generalized risk-dominance and asymmetric dynamics," *Journal of Economic Theory*, vol. 145, no. 1, pp. 216–248, 2010.

[18] G. Ellison, "Basins of attraction, long-run stochastic stability, and the speed of step-by-step evolution," *The Review of Economic Studies*, vol. 67, no. 1, pp. 17–45, 2000.

[19] C. Alós-Ferrer and N. Netzer, "The logit-response dynamics," *Games and Economic Behavior*, vol. 68, no. 2, pp. 413–427, 2010.

[20] A. S. Akber, H. Jaleel, W. Abbas, and J. S. Shamma, "Robustness of learning in games with heterogeneous players," *to appear in IEEE Transaction on Automatic Control. Preprint @ https://sites.google.com/view/imass-lums/publications*, 2022.