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PWA control functions: from the projection of mpQP solution and back to the convexification by lifting

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Abstract— **This paper focuses on the geometric properties of the Piece-Wise Affine (PWA) feedback function as they appear from the optimal solution of the multi-parameter quadratic programming (mpQP) problem. Such optimization problems are popular formulations, for example, in the design of model-based predictive controllers (MPC) for discrete linear systems subject to input and state constraints. The paper considers such a PWA function as input data and provides a method for reconstructing a feasible convex set and a PWA curve within it, which retrieves the identical structure of the solution in the original parametric feasible set. The proposed method involves establishing and decomposing the topology structure of the polyhedral critical regions, which form the domain of the PWA function by means of a graph of interconnections. The regions are split into the boundary and interior collections using convex-concave lifting. The explicit solution is then merged based on the convex-concave liftings to reconstruct the feasible domain and the PWA curves.**

I. INTRODUCTION

Model predictive control (MPC) has captured significant attention in the academic community due to its receding horizon optimal control formulation [1]. MPC uses a dynamic model of the process to predict the future evolution over a finite-time horizon and selects the optimal control actions with respect to a specified performance index. Furthermore, MPC can handle process constraints and multi-variable interactions in a unified formulation. Various researchers have recently studied MPC and their optimization-based control alternatives from different angles [2]. The present paper focuses on the geometric characteristics of MPC and aims to propose inverse optimal solutions for the predictive controllers [3], [4].

Explicit MPC simplifies online computation by replacing real-time quadratic programming with a PWA function [5]. Researchers have focused on enhancing Explicit MPC and the PWA function acquisition process. In [6], the authors reduce storage and evaluation time by assuming the initial state is contained in a given set, omitting irrelevant predictive trajectory regions. In [7], by enumerating the possible optimal active sets, the authors showed that Explicit MPC solutions admit a closed-form solution which does not require the storage of critical regions. In [8], the constrained problem utilizes geometric principles to eliminate redundancy within the set of constraints for the associated QP problem. A survey on the recent developments of Explicit MPC is conducted in [9]. A book about the characters of constrained control problems was summarised in [10].

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Once the advantages of implicit and explicit MPC have been characterized, a series of studies [11] proved that explicit PWA formulations could be used to obtain inverse optimal QP formulations with lower computation footprint. Remarkably, inverse-optimal formulations have been constructed with only one supplementary dimension of the vector of arguments [3], [4], thus offering a compact QP for the PWA controller originated by the MPC formulation.

The inverse optimal solutions for PWA feedback functions obtained in the literature exploit constraint activation at the optimum for the entire range of the parameters. However, the original MPC formulation has a solution that is unconstrained around the equilibrium whenever this is located in the interior of the (state-input) feasible domain. The construction reported in [4] has succeeded in one-dimensional systems but needs more validation in higher dimensions. In this context, and in contrast to prior research results, such as in [3] and [12], this paper aims to devise a new PWA function and a corresponding higher-dimensional polytope to uphold the geometric structure of the original control action, including the feasible domain and the constrained-unconstrained characteristics. Specifically, the PWA control is partitioned into two components: a linear function corresponding to the unconstrained critical region in the parameters' space and a constrained solution that is essentially a projection onto the boundary of the feasible set.

- A higher-dimensional polytope and a suitable PWA function are defined starting from an original continuous PWA control and its corresponding polyhedral critical region. Notably, the newly created polytope and the extended PWA function exhibit the same projection as the original PWA function.
- Drawing inspiration from the concept of convex lifting [13], we introduce a novel approach called convex-concave lifting to facilitate the construction of the higher-dimensional polytope.
- We examine the connection between the constructed polytope and the original mpQP problem and point out that the corresponding mpQP problem has a lower complexity.

Notation: In this paper, $\mathbb{R}^n$ and $\mathbb{N}$ denote the set of real numbers in n-dimensional space, the set of nonnegative integers, and $\mathcal{I}_N = [1, N] \cap \mathbb{N}$. The symbols $\mathbf{0}_{m\times n}$ and I_m represent a matrix of size $m \times n$ with all elements equal to zero and an m-dimensional identity matrix, respectively. If

$\mathcal{P} \subset \mathbb{R}^n$ is a polyhedral set, then $int(\mathcal{P})$ denotes the set of interior points of $\mathcal{P}$ and $bd(\mathcal{P})$ denotes the set of boundary points of $\mathcal{P}$, $\text{Proj}_{\mathbb{R}^m} \mathcal{P}$ represents the orthogonal projection of $\mathcal{P}$ onto the subspace $\mathbb{R}^m$. The convex hull of a set $\{*\}$ is denoted by conv$\{*\}$. $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$.

II. BACKGROUND AND PROBLEM FORMULATION

A. From MPC to PWA control

Consider a discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

where the states $x_k \in \mathbb{R}^n$, and the inputs $u_k \in \mathbb{R}^m$ at time k are bounded by polyhedral sets:

$$\mathcal{X} = \{x \in \mathbb{R}^n | H_x x \leq b_x\}, \mathcal{U} = \{u \in \mathbb{R}^m | H_u u \leq b_u\},$$

where b_x, b_u, H_x, and H_u are known constant matrices, such that $0 \in int(\mathcal{X})$ and $0 \in int(\mathcal{U})$.

The model-based predictive controller with a N-steps receding horizon can be obtained by the iterative construction of the control sequence $\kappa_u^*(x) = [u_0^T, \cdots, u_{N-1}^T]^T$ which solves the finite-time optimal control problem for the given initial state $x_0 = x$:

$$\kappa_u^*(x) = \arg\min_{\kappa_u} \sum_{k=0}^{N-1} u_k^T R u_k + \sum_{k=1}^{N} x_k^T Q x_k \tag{2a}$$

$$\text{s. t. } x_{k+1} = Ax_k + Bu_k, \tag{2b}$$

$$u_k \in \mathcal{U}, x_k \in \mathcal{X}, x_N \in \Omega \tag{2c}$$

with $R \succ 0, Q \succeq 0$ and $\Omega \subseteq \mathbb{R}^n$ a control invariant set with $0 \in int(\Omega)$. Without loss of generality, the problems to be solved at each sampling instant can be written:

$$\kappa_u^*(x) = \arg \min_{\kappa_u} \quad \kappa_u^T H \kappa_u + x^T F \kappa_u, \tag{3a}$$

$$\text{s.t.} \quad [x^T, \kappa_u^T] \in \mathcal{P}, \tag{3b}$$

where $x \in \underline{\mathcal{X}} = \text{Proj}_{\mathbb{R}^n} \mathcal{P} \subseteq \mathcal{X}$ is the current state playing the role of a parameter, $\kappa_u(x) \in \mathbb{R}^{mN}$ is the optimization vector and $H \succ 0$. The set $\mathcal{P}$ is the parameterized feasible domain with the formulation

$$\mathcal{P} = \{[x^T, \kappa_u^T]^T | G\kappa_u \leq W + Ex, x \in \underline{\mathcal{X}}\}$$

where G, W, E are constructed from the mpQP problem (2). From the above notations it follows $\forall x \in \underline{\mathcal{X}}, \mathcal{P} \neq \emptyset$.

The solution to problem (3) takes the form of a piecewise affine (PWA) function of the system state x. In order to deal with those properties, the notion of polyhedral partition needs to be introduced.

Definition 1. A collection of polyhedral sets $\{\mathcal{X}_1, \ldots, \mathcal{X}_N\}$ is called a polyhedral partition of $\underline{\mathcal{X}}$ if $\forall i, j \in \mathcal{I}_N, i \neq j$, $\underline{\mathcal{X}} = \cup_{i=1}^N \mathcal{X}_i$ and $int(\mathcal{X}_i) \cap int(\mathcal{X}_j) = \emptyset$.

Such a polyhedral partition is next denoted as $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

Building upon the results presented in [5], the solution to the mpQP problem (2) can be expressed as a continuous PWA function of x, denoted as $\kappa_u^*(x)$.

$$\kappa_u^*(x) = f_i(x), x \in \mathcal{X}_i,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^{mN}$ is piecewise affine and $\mathcal{X}_i$ is a polyhedron of polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

B. Properties of the MPC controller

Similar to the literature on the geometric structure of MPC, the relationship between the unconstrained optimum and the feasible domain is privileged in the present work.

Proposition 1. If $0 \in int(\mathcal{X})$, $0 \in int(\mathcal{U})$ and $0 \in int(\Omega)$ in (2) then with respect to the solution of (3):

$$\mathcal{X}_1 = \{x \in \underline{\mathcal{X}} | \kappa_u^*(x) = -H^{-1}Fx\}$$

is a polyhedron set with $int(\mathcal{X}_1) \neq \emptyset$, and additionally,

$$\forall x \in int(\mathcal{X}_1), [x^T, \kappa_u^*(x)^T] \in int(\mathcal{P}).$$

Proof. See Proposition 1 of [4]. □

The set $\mathcal{X}_1$ is denoted as the unconstrained critical region of the Explicit MPC. In the geometry of the PWA control function, the importance of this set is underlined by the following result.

Proposition 2. If $0 \in int(\mathcal{X})$, $0 \in int(\mathcal{U})$ and $0 \in int(\Omega)$ in (2), then for $x \in \mathcal{X} \setminus \mathcal{X}_1$, the optimal solution for the mpQP problem (3) satisfies $[x^T, \kappa_u^*(x)^T] \in bd(\mathcal{P})$.

Proof. See Proposition 1 of [4]. □

Corollary 1. If $0 \in int(\underline{\mathcal{X}})$ and $0 \in int(\mathcal{U})$ then

$$\forall x \in int(\mathcal{X}_1), \left[x^T, \kappa_u^*(x)^T\right]^T \in int(\mathcal{P}_N),$$

where

$$\mathcal{P}_N := \text{conv}\left\{[x^T, \kappa_u^*(x)^T]^T : x \in \underline{\mathcal{X}}\right\}. \quad \square$$

Two important features of an MPC controller emerge from the aforementioned properties.

- The optimal solution $\kappa_u^*(x)$ consists of two parts: the unconstrained optimum for $x \in \mathcal{X}_1$, and the boundary solution for $x \in \mathcal{X}_i$ where $i \in \mathcal{I}_N \setminus \{1\}$.
- Only the first component of the sequence $\kappa_u^*(x)$ is used by MPC control action $\kappa_{pwa}(x)$ and will be denoted as

$$\kappa_{pwa}(x) = \left[I_m, \mathbf{0}_{m \times (mN-m)}\right] \kappa_u^*(x),$$

  or explicitly as $\kappa_{pwa}(x) = F_i x + g_i, x \in \mathcal{X}_i$ with $F_i \in \mathbb{R}^{m \times n}$, $g_i \in \mathbb{R}^m$.

In the following sections, we combine these two properties to propose a problem of interest and its solution strategy.

C. Problems formulation

Starting from an mpQP problem (3) with a parameter vector $x \in \underline{\mathcal{X}} \subset \mathbb{R}^n$ and the parameterized feasible domain $\mathcal{P} \subset \mathbb{R}^{n+mN}$, the corresponding solution $\kappa_u^*(x)$ for the problem (3) and control action $\kappa_{pwa}(x) : \mathbb{R}^n \to \mathbb{R}^m$ for the system (1) can be obtained with standard methods. For the current developments, the control action $\kappa_{pwa}(x)$ will be considered to be available in an explicit form.

The goal is to construct a set $\mathcal{P}_z \subset \mathbb{R}^{n+m+n_z}$ starting from the continuous PWA control action $\kappa_{pwa}(x)$, such that

Prop 1: $\mathcal{P}_z$ is a polytope, and it satisfies:

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P}_z = \underline{\mathcal{X}}.$$

Prop 2: There exists a continuous PWA vector function $\kappa_z(x) : \mathbb{R}^n \to \mathbb{R}^{m+n_z}$ subject to:

$$\kappa_{pwa}(x) = \left[I_m, \mathbf{0}_{m\times n_z}\right] \kappa_z(x). \tag{4}$$

Prop 3: $\forall x \in int(\mathcal{X}_1), [x^T, \kappa_z^T(x)]^T \in int(\mathcal{P}_z)$.
Prop 4: $\forall x \in \mathcal{X} \setminus int(\mathcal{X}_1), [x^T, \kappa_z^T(x)]^T \in bd(\mathcal{P}_z)$.

By imposing these four conditions, implicitly, a reformulation of the MPC feasible domain and its optimal solution $(\mathcal{P}, \kappa_u^*)$ is achieved:

$$\mathcal{P} \subset \mathbb{R}^{n+mN} \longrightarrow \mathcal{P}_z \subset \mathbb{R}^{n+m+n_z},$$
$$\kappa_u^* : \mathbb{R}^n \to \mathbb{R}^{mN} \longrightarrow \kappa_z : \mathbb{R}^n \to \mathbb{R}^{m+n_z}.$$

If $m + n_z \ll mN$, the new pair $(\mathcal{P}_z, \kappa_z)$ preserves the topological structure of the pair $(\mathcal{P}, \kappa_u^*)$ while reducing its complexity. One can employ $(\mathcal{P}_z, \kappa_z)$ to formulate a new problem that is equivalent to (3) but has a lower complexity.

III. Preliminary Results

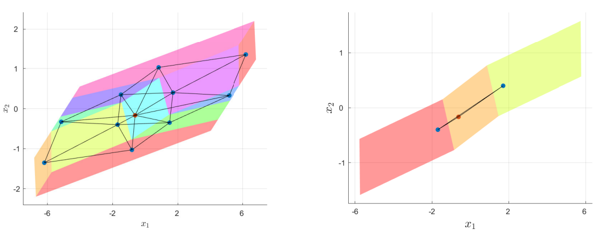
Two instrumental notions are introduced in this section.

A. Convex lifting

Definition 2. [13] Given a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ of a polyhedron $\underline{\mathcal{X}} \subset \mathbb{R}^n$, $l(x) : \underline{\mathcal{X}} \to \mathbb{R}$ is called a PWA convex lifting if the following conditions hold: 1) $l(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$; 2) $l(x)$ is continuous over $\underline{\mathcal{X}}$; 3) $l(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ with $i, j \in \mathcal{I}_N$ and $i \neq j$.

B. Interconnection graph corresponding to the topology of the PWA partition

The topological structure of a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ characterizes the interconnectivity and distribution of the constituent regions $\mathcal{X}_i$ according to the non-emptiness of their intersection. This is represented in terms of as a non-directional graph $(\mathscr{N}, \mathscr{E})_{\mathcal{I}_N}$, with $\mathscr{N}$ denoting the regions and $\mathscr{E}$ representing the links between these regions. If $\mathcal{X}_i \cap \mathcal{X}_j \neq \emptyset$, $\mathscr{E}(i,j) = \mathscr{E}(j,i) = 1$. Otherwise, $\mathscr{E}(i,j) = \mathscr{E}(j,i) = 0$. Fig. 1.a illustrates a topological structure graph for a 2-dimensional polyhedral partition.



(a) The whole structure (b) An interior subgraph

Fig. 1. A topological structure graph in the 2-dimensional partition

IV. A Basic Constructive Result

To address the problem presented in subsection II.C, the topological structure graph of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is classified and a novel convex-concave lifting technique to extend the variable to a higher-dimensional space for convexification purposes. The constructive process is completed with a third step where the PWA control law $\kappa_{pwa}(x)$ is combined with the lifting to establish a feasible set $\mathcal{P}_z$.

A. Decomposable interconnection graph within the partition

First, let us refer to the dimensional expansion lemma proposed in [3].

Lemma 1. [3] Let $\Gamma_s \subset \mathbb{R}^{d_s}$ be a full-dimensional polytope with the set of vertices $\mathcal{V}(\Gamma_s) = \{s^{(1)}, \cdots, s^{(q)}\}$. For any finite set of points $\{s^{(1)}, \cdots, s^{(q)}\} \subset \mathbb{R}^{d_t}$ defining a full-dimensional polytope, an extension of the family $\mathcal{V}(\Gamma_s)$ can be obtained in higher-dimensional space $\mathbb{R}^{d_s+d_t}$ for the concatenated vectors $[s^T, t^T]^T$ defining the set:

$$V_{s,t} := \left\{ \begin{bmatrix} s^{(1)} \\ t^{(1)} \end{bmatrix}, \cdots, \begin{bmatrix} s^{(q)} \\ t^{(q)} \end{bmatrix} \right\}.$$

The polytope $\Gamma_{s,t} = \text{conv}(V_{s,t})$ satisfies: $V_{s,t} = \mathcal{V}(\Gamma_{s,t})$.

Remark 1. For two sets Γ_s and $\Gamma_{s,t}$, the following holds:

$$\text{Proj}_{\mathbb{R}^{d_s}} \Gamma_{s,t} = \Gamma_s.$$

A set $\mathcal{P}_u$ is defined similarly as $\mathcal{P}_N$ in Corollary 1:

$$\mathcal{P}_u := \text{conv}\left\{[x^T, \kappa_{pwa}(x)^T]^T : x \in \underline{\mathcal{X}}\right\}. \tag{5}$$

Apparently, $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_u = \underline{\mathcal{X}}$. But $\mathcal{P}_u$ cannot satisfy the *Prop* 1–4 required to construct the feasible set $\mathcal{P}_z$. Consequently, an extra step is necessary to embed $\mathcal{P}_u$ in a higher-dimensional polytope.

Definition 3. Let $(\mathscr{N}, \mathscr{E})_{\mathcal{I}_N}$ be a graph associated with $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. A PWA function $f(x) : \bigcup_{i\in\mathcal{N}_{bd}} \mathcal{X}_i \to \mathbb{R}$ is called a boundary lifting function (BLF) if the following holds:

$$\mathcal{P}_f^i \subset bd(\mathcal{P}_f)$$

with $i \in \mathcal{N}_{bd} \subseteq \mathcal{I}_N$ and

$$\mathcal{P}_f^i := \text{conv}\left\{[x^T, f(x)]^T : x \in \mathcal{X}_i\right\},$$
$$\mathcal{P}_f := \text{conv}\left\{[v^T, f(v)]^T : v \in \bigcup_{i\in\mathcal{N}_{bd}} \mathcal{V}(\mathcal{X}_i)\right\}.$$

Remark 2. The existence of a BLF over $\bigcup_{i\in\mathcal{N}_{bd}} \mathcal{X}_i$ doesn not solve the stated problem but some connections between $\mathcal{P}_f$ and $\mathcal{P}_z$ can be estabished:

- $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_z = \underline{\mathcal{X}}$. However, the set $\mathcal{P}_f$ only guarantees that $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_f \supseteq \bigcup_{i\in\mathcal{N}_{bd}} \mathcal{X}_i$.
- The set $\bigcup_{i\in\mathcal{N}_{bd}} \mathcal{X}_i$ is not required to be both compact and connected. Therefore, the PWA function $f(x)$ may be discontinuous.

Subsequently, we aim to identify multiple appropriate BLFs to build a polytope $\mathcal{P}_z$ in a higher-dimensional space.

The domain of $f(x)$ allows us to partition the topology structure graph $(\mathscr{N},\mathscr{E})_{\mathcal{I}_N}$ into two subgraphs: the boundary subgraph $(\mathscr{N},\mathscr{E})_{\mathcal{N}_{bd}}$ and interior subgraph $(\mathscr{N},\mathscr{E})_{\mathcal{N}_{in}}$. Here, $\mathcal{N}_{in} := \mathcal{I}_N \setminus \mathcal{N}_{bd}$. Fig. 1.b illustrates an interior subgraph of $(\mathscr{N},\mathscr{E})_{\mathcal{I}_N}$, as presented in Fig. 1.a.

B. Convex-concave construction

In order to create the feasible set $\mathcal{P}_z$, it is necessary to devise a BLF that elevates the regions in the interior subgraph $(\mathscr{N},\mathscr{E})_{\mathcal{N}_{in}}$ to the boundary. A convex-concave lifting approach, derived from convex lifting, is suggested to facilitate this process.

Definition 4. Let $\{\mathcal{X}_i\}_{\mathcal{N}_{in}} \subseteq \{\mathcal{X}_i\}_{\mathcal{I}_N}$ be a collection of regions. A piecewise affine lifting function $g : \mathbb{R}^n \to \mathbb{R}$, defined as $g(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$ where $i \in \mathcal{N}_{in}$, is categorized as a convex-concave lifting if it satisfies the following conditions:

- The set $\mathcal{N}_{in}$ is partitioned into two groups:
  1) $\mathcal{N}_{in}^{vex}$, containing the convex items, and $\mathcal{N}_{in}^{cave}$, containing the concave items. These groups fulfil the conditions $\mathcal{N}_{in}^{vex} \cup \mathcal{N}_{in}^{cave} = \mathcal{N}_{in}$ and $\mathcal{N}_{in}^{vex} \cap \mathcal{N}_{in}^{cave} = \{1\}$.
- For all $i \in \mathcal{N}_{in}^{vex}$, the following conditions are satisfied:
  2) $g(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ and all $j \neq i, j \in \mathcal{N}_{in}^{vex}$;
  3) $g(x) < a_j^T x + b_j$, $\forall x \in \mathcal{X}_i$ and $\forall j \in \mathcal{N}_{in}^{cave} \setminus \{1\}$.
- For all $i \in \mathcal{N}_{in}^{cave}$, the following conditions are satisfied:
  4) $g(x) < a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ and all $j \neq i, j \in \mathcal{N}_{in}^{cave}$;
  5) $g(x) > a_j^T x + b_j$, $\forall x \in \mathcal{X}_i$ and $\forall j \in \mathcal{N}_{in}^{vex} \setminus \{1\}$.
- For all $i, j \in \mathcal{N}_{in}, i \neq j$, the following condition holds:
  6) $a_i^T x + b_i = a_j^T x + b_j, x \in \mathcal{X}_i \cap \mathcal{X}_j$.

Remark 3. A convex lifting is a convex-concave lifting with $\mathcal{N}_{in}^{cave} = \{1\}$.

The proposed convex-concave liftable conditions and their connection to a BLF for state space $\mathcal{X}$ will be discussed next.

Definition 5. The collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ is said to be valid for convex-concave liftability if

$$\bigcup_{i \in \mathcal{N}_{in}^{vex} \setminus \{1\}} \mathcal{X}_i \subset \mathcal{H}^+, \bigcup_{i \in \mathcal{N}_{in}^{cave} \setminus \{1\}} \mathcal{X}_i \subset \mathcal{H}^-,$$

where $\mathcal{H}^+$ and $\mathcal{H}^-$ are two halfspaces generated by a hyperplane $\mathcal{H} \subset \mathbb{R}^n$.

Remark 4. According to the above definition, a necessary condition for the application of the convex-concave lifting is the existence of a hyperplane to partition $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ into two subsets. Fig. 1.a and Fig. 1.b illustrate this approach. For example, in the graph $(\mathscr{N},\mathscr{E})_{\mathcal{I}_N}$, it may not be possible to divide the regions, except for the original region 1, into two groups using a hyperplane. However, a hyperplane can divide the interior subgraph $(\mathscr{N},\mathscr{E})_{\mathcal{N}_{in}}$ into two groups once the region 1 is omitted. Thus, we refer to the set of regions $(\mathscr{N},\mathscr{E})_{\mathcal{N}_{in}}$ generated from Fig. 1.b as valid for convex-concave liftability.

An algorithmic procedure for convex-concave lifting can be consulted in [4].

C. Lifting a Polyhedral Partition

Suppose the collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}} \subseteq \{\mathcal{X}_i\}_{\mathcal{I}_N}$ is valid for convex-concave liftability. Let $g(x) : \bigcup_{\forall i \in \mathcal{N}_{in}} \mathcal{X}_i \to \mathbb{R}$ be a PWA function defined as $g(x) = a_i^T x + b_i, x_i \in \mathcal{X}_i$, which is a convex-concave lifting of $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$.

Proposition 3. Let $\{\mathcal{X}_i\}_{\mathcal{N}_{in}} \subseteq \{\mathcal{X}_i\}_{\mathcal{I}_N}$ be a convex-concave liftable collection, where $\mathcal{N}_{in}^{vex}$ and $\mathcal{N}_{in}^{cave}$ are the convex and concave items of $\mathcal{N}_{in}$, respectively. The following equation holds: $\mathcal{P}_g \subset \mathbb{R}^{n+1} : \mathrm{Proj}_{\mathbb{R}^n} \mathcal{P}_g = \underline{\mathcal{X}}$ with

$$\begin{aligned}\mathcal{P}_g =& \{\left[x^T, z\right]^T \text{|s.t. } z \geq a_i^T x + b_i, z \leq a_j^T x + b_j, \\ & \forall x \in \underline{\mathcal{X}}, i \in \mathcal{N}_{in}^{vex} \setminus \{1\}, j \in \mathcal{N}_{in}^{cave} \setminus \{1\}\},\end{aligned}$$

if a_i, b_i are the solution to the optimization problem:

$$\min_{a_i, b_i} \quad \sum_{\forall i \in \mathcal{N}_{in}} (a_i^T a_i + b_i^T b_i) \tag{6a}$$

$$\text{s.t.} \quad \text{Conditions 2)-6) in Definition 4} \tag{6b}$$

$$a_i^T x + b_i \leq a_j^T x + b_j, \tag{6c}$$

$$\forall x \in \underline{\mathcal{X}}, i \in \mathcal{N}_{in}^{vex} \setminus \{1\}, j \in \mathcal{N}_{in}^{cave} \setminus \{1\}. \tag{6d}$$

Proof. In the expression of $\mathcal{P}_g$, the condition $x \in \underline{\mathcal{X}}$ must be satisfied, which implies that: $\mathrm{Proj}_{\mathbb{R}^n} \mathcal{P}_g \subseteq \underline{\mathcal{X}}$. Moreover, for any $x \in \underline{\mathcal{X}}$, there exists a real number $z_0 \in \mathbb{R}$ such that $[x^T, z_0]^T \in \mathcal{P}_g$. Therefore, we have $\underline{\mathcal{X}} \subseteq \mathrm{Proj}_{\mathbb{R}^n} \mathcal{P}_g$. As a result, the proposition is proven to be true. □

Remark 5. To simplify the formulation, $\mathcal{P}_g$ will be equivalently denoted as:

$$\mathcal{P}_g = \{[x^T, z]^T | H_{z,x} x + H_{z,z} z \leq b_z\}. \tag{7}$$

Besides, to construct the convex-concave lifting $g(x)$ based on the Definition 4, a set of new constraints (6c) and (6d) must be considered. The aforementioned constraints correspond to boundary conditions and can be simplified to a finite set of constraints as follows:

- For $i \in \mathcal{N}_{in}^{vex} \setminus \{1\}, \forall j \in \mathcal{N}_{in}^{cave} \setminus \{1\}$, and $\forall v \in \mathcal{V}(\underline{\mathcal{X}})$:

$$a_i^T v + b_i \geq a_j^T v + b_j + \epsilon.$$

D. Composition of PWA functions

This subsection presents a methodology for building a set $\mathcal{P}_z$ that leverages the decomposable interconnection graph approach detailed in Subsection A, in conjunction with convex-concave lifting techniques in Subsections B and C.

Given a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ with topology $(\mathscr{N},\mathscr{E})_{\mathcal{I}_N}$, we seek to construct a polytope $\mathcal{P}_z$ that encapsulates the continuous PWA function $\kappa_{pwa} : \bigcup_{i=1}^{N} \mathcal{X}_i \to \mathbb{R}^m$. To achieve this objective, we begin by constructing a polytope $\mathcal{P}_u \subset \mathbb{R}^{n+m}$ defined in (5). The polytope can be represented equivalently in the half-space representation as:

$$\mathcal{P}_u = \left\{[x^T, u^T]^T \in \mathbb{R}^{n+m} | H_{u,x} x + H_{u,u} u \leq b_u\right\}. \tag{8}$$

We begin by noting that $\operatorname{Proj}_{\mathbb{R}^n} \mathcal{P}_u = \underline{\mathcal{X}}$. Additionally, we partition the topology $(\mathscr{N}, \mathscr{E})_{\mathcal{I}_N}$ into two subgraphs based on the following boundary condition:

$$(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{bd}} : \forall i \in \mathcal{N}_{bd}, \mathcal{P}_u^i \subset bd(\mathcal{P}_u); \tag{9a}$$

$$(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{in}} : \forall i \in \mathcal{N}_{in}, \mathcal{P}_u^i \cap int(\mathcal{P}_u) \neq \emptyset. \tag{9b}$$

with $\mathcal{P}_u^i = \operatorname{conv}\left\{[v^T, \kappa_{pwa}^T(v)]^T : v \in \mathcal{V}(\mathcal{X}_i)\right\}$. Besides, sets $\mathcal{N}_{bd}$ and $\mathcal{N}_{in}$ satisfy:

$$\mathcal{N}_{bd} \cup \mathcal{N}_{in} = \mathcal{I}_N, \mathcal{N}_{bd} \cap \mathcal{N}_{in} = \emptyset, 1 \in \mathcal{N}_{in}.$$

Assumption 1. For the subgraph $(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{in}}$, the relevant collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ is valid for convex-concave liftability.

Based on Proposition 3 and Assumption 1, we can design a convex-concave lifting $g : \bigcup_{\forall i \in \mathcal{N}_{in}} \mathcal{X}_i \to \mathbb{R}$, such that

$$\operatorname{Proj}_{\mathbb{R}^n} \mathcal{P}_g = \underline{\mathcal{X}} \text{ and } \forall i \in \mathcal{N}_{in} \setminus \{1\}, \mathcal{P}_g^i \subset bd(\mathcal{P}_g)$$

with $\mathcal{P}_g$ defined in (7) and $\mathcal{P}_g^i$ denoted as

$$\mathcal{P}_g^i = \operatorname{conv}\left\{[x^T, g^T(x)]^T : x \in \mathcal{X}_i\right\}. \tag{10}$$

Using PWA function $\kappa_{pwa}(x)$ and a convex-concave lifting $g(x)$, the following proposition constructs set $\mathcal{P}_z$.

Proposition 4. Let us consider a continuous PWA function $\kappa_{pwa}(x)$ defined over a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, the associated $\mathcal{P}_u$ as in (8) and a specialized convex-concave lifting $g(x)$ over a collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$. The polytope $\mathcal{P}_z \subset \mathbb{R}^{n+m+1}$ satisfies *Prop* 1–4 in II-C and can be expressed as:

$$\begin{aligned}\mathcal{P}_z =& \{\left[x^T, u^T, z\right]^T \in \mathbb{R}^{n+m+1} | \text{ subject to} \\ & H_{u,x}x + H_{u,u}u \leq b_u, H_{z,x}x + H_{z,z}z \leq b_z\}.\end{aligned}$$

Proof. The inclusion of $x \in \underline{\mathcal{X}}$ in $\mathcal{P}_z$ implies that $\operatorname{Proj}_{\mathbb{R}^n} \mathcal{P}_z \subseteq \underline{\mathcal{X}}$. Additionally, for any $x \in \underline{\mathcal{X}}$, there exist two vectors $u_0 \in \mathbb{R}^m$ and $z_0 \in \mathbb{R}$ such that:

$$[x^T, u_0^T]^T \in \mathcal{P}_u, [x^T, z_0]^T \in \mathcal{P}_g$$

Therefore, we have $[x^T, u_0^T, z_0]^T \in \mathcal{P}_z \Rightarrow \underline{\mathcal{X}} \subseteq \operatorname{Proj}_{\mathbb{R}^n} \mathcal{P}_z$. *Prop* 1 imposed on $\mathcal{P}_z$ according to II-C is proven to hold.

For *Prop* 2 in II-C, we define a PWA function $\kappa_z : \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \to \mathbb{R}^{m+1}$ with $\kappa_z(x) = \left[\kappa_{pwa}(x)^T, z(x)^T\right]^T$ and

$$z(x) = \begin{cases} g(x), & x \in \mathcal{X}_i, i \in \mathcal{N}_{in}, \\ g_c(x), & x \in \mathcal{X}_i, i \in \mathcal{N}_{bd}, \end{cases}$$

where function $g_c(x) : \mathbb{R}^n \to \mathbb{R}$ only needs to ensure the continuity of $z(x)$ over the entire state space $\underline{\mathcal{X}}$. Apparently, $\kappa_{pwa}(x) = \left[I_m, \mathbf{0}_{m\times 1}\right] \kappa_z(x)$.

Prop 3–4 in II-C are proven by analyzing various situations based on the value of x.

If $x \in int(\mathcal{X}_1)$, since $1 \in \mathcal{N}_{in}$, we have $[x^T, \kappa_{pwa}^T(x)]^T \in int(\mathcal{P}_u)$. This implies the existence of $\epsilon_1 > 0$ such that $[x^T, \kappa_{pwa}^T(x)]^T + \epsilon_1 \mathbb{B}^{n+m} \subset \mathcal{P}_u$. Based on the design process of convex-concave lifting, if $x \in int(\mathcal{X}_1)$, then $[x^T, g(x)]^T \in int(\mathcal{P}_g)$ holds true. Similarly, there is $\epsilon_2 > 0$ such that $[x^T, g(x)]^T + \epsilon_2 \mathbb{B}^{n+1} \subset \mathcal{P}_g$. Thus, we can choose $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ so that $[x^T, \kappa_{pwa}^T(x), g(x)]^T + \epsilon_3 \mathbb{B}^{n+m+1} \subset \mathcal{P}_z$.

If $x \in \mathcal{X}_i$ with $i \in \mathcal{N}_{in} \setminus \{1\}$, due to $i \in \mathcal{N}_{in}$, $[x^T, \kappa_{pwa}^T(x)]^T \in int(\mathcal{P}_u)$. It means $\exists \epsilon_1 > 0, [x^T, \kappa_{pwa}^T(x)]^T + \epsilon_1 \mathbb{B}^{n+m} \subset \mathcal{P}_u$. According to the design process of convex-concave lifting, $[x^T, g(x)]^T \in bd(\mathcal{P}_g)$, which means $\nexists \epsilon_2 > 0, [x^T, g(x)]^T + \epsilon_2 \mathbb{B}^{n+1} \subset \mathcal{P}_g$. It means $\nexists \epsilon > 0, [x^T, \kappa_{pwa}^T(x), g(x)]^T + \epsilon \mathbb{B}^{n+m+1} \subset \mathcal{P}_z$. Thus, $[x^T, \kappa_{pwa}^T(x), g(x)]^T \in bd(\mathcal{P}_z)$.

If $x \in \mathcal{X}_i$ with $i \in \mathcal{N}_{bd}$, we can not find $\epsilon > 0, [x^T, \kappa_{pwa}^T(x)]^T + \epsilon \mathbb{B}^{n+m} \subset \mathcal{P}_u$, which also means $\nexists \epsilon > 0, [x^T, \kappa_{pwa}^T(x), g_c(x)]^T + \epsilon \mathbb{B}^{n+m+1} \subset \mathcal{P}_z$ and $[x^T, \kappa_{pwa}^T(x), g_c(x)]^T \in bd(\mathcal{P}_z)$. □

E. Iterative decomposition of the interconnection graph

If construction a convex-concave lifting holds for subgraph $(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{in}}$, i.e., the collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ is valid for convex-concave liftability, a convex-concave lifting procedure can be constructed for $\{\mathcal{X}_i\}_{\mathcal{N}_{in} \setminus \{1\}}$ to the boundary of the lifting set while leaving region 1 still as the unconstrained region.

If construction a convex-concave lifting doesn't hold, the basic approach involves dividing subgraph $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ into two new subgraphs, named$\{\mathcal{X}_i\}_{\mathcal{N}_{bd}^1}$ (i.e., liftable collection) and $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^1}$ (i.e., unliftable collection), and then repeating the iteration on $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c}$ until $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c} = \{\mathcal{X}_i\}_{\{1\}}$. Here, $c \in \mathbb{N}$, $\{\mathcal{X}_i\}_{\mathcal{N}_{bd}^c}$ and $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c}$ are divided from $\{\mathcal{X}_c\}_{\mathcal{N}_{in}^{c-1}}$, and specifically, $\mathcal{N}_{in}^0 := \mathcal{N}_{in}$. The Algorithm 1 summarizes the constructing of the set $\mathcal{P}_z$ up to the validation of the Assumption 1.

The main procedures involve finding a hyperplane $\mathcal{H}^c$ in the 6-th step and achieving the objective by combining $\kappa_{pwa}(x)$ with several convex-concave lifting functions $g^j(x)$, where $j \in \{1, \cdots, c\}$. While finding an optimal hyperplane is irrelevant to this study, it should be mentioned that the decomposition can also be achieved by excluding node 1 and decomposing the graph up to the partition in disjoint subgraphs. The following subsection presents a generic approach to deal with step 9.

F. General solution enabled by convex-concave liftings

For the j-th convex-concave lifting of the liftable collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^j}$ with $j \in \{1, \cdots, c\}$, we denote the lifting as $g^j(x)$ and the corresponding $\mathcal{P}_{g^j}$ similar to (7) and (10):

$$\begin{aligned} g^j :& \bigcup_{\forall i \in \mathcal{N}_{in}^j} \mathcal{X}_i \to \mathbb{R}, \\ \mathcal{P}_{g^j} =& \{[x^T, z_j]^T | H_{z_j,x}x + H_{z_j,z_j}z_j \leq b_{z_j}\}.\end{aligned}$$

Proposition 5. For a continuous PWA function $\kappa_{pwa}(x)$ defined over a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, we have constructed set $\mathcal{P}_u$ and a sequential convex-concave lifting $g^j(x)$ defined over a collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^j}$ with $j \in \{1, \cdots, c\}$. Polytope $\mathcal{P}_z \subset \mathbb{R}^{n+m+c}$ satisfies *Prop* 1–4 in II-C and can be expressed as:

$$\begin{aligned}\mathcal{P}_z =& \{\left[x^T, u^T, z^T\right]^T \in \mathbb{R}^{n+m+c} | \text{ subject to:} j \in \{1, \cdots, c\} \\ & H_{u,x}x + H_{u,u}u \leq b_u, H_{z_j,x}x + H_{z_j,z_j}z_j \leq b_{z_j}\}\end{aligned}$$

with z_j the j-th element of vector $z \in \mathbb{R}^c$.

Proof. The process is similar to that in Proposition 4. □

Algorithm 1 Construction a feasible set $\mathcal{P}_z$ for general case.

Input: A continuous PWA function $\kappa_{pwa} : \bigcup_{i=1}^{N} \mathcal{X}_i \rightarrow \mathbb{R}^m$ and its domain $\underline{\mathcal{X}} \subset \mathbb{R}^n$.

Output: A higher-dimensional polytope $\mathcal{P}_z \subset \mathbb{R}^{n+m+c}$ and a relevant continuous PWA function $\kappa_z(x) : \underline{\mathcal{X}} \rightarrow \mathbb{R}^{m+c}$ with $c \in \mathbb{N}$.

1: Initialize $c = 0$.

2: Establish the topology structure graph of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ denoted as $(\mathscr{N}, \mathscr{E})_{\mathcal{I}_N}$.

3: Construct the polytope $\mathcal{P}_u \subset \mathbb{R}^{n+m}$, denoted by (8), for the polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

4: The graph $(\mathscr{N}, \mathscr{E})_{\mathcal{I}_N}$ can be partitioned into two parts based on $\mathcal{P}_u$, resulting in $(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{bd}}$ and $(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{in}}$, as shown in (9).

5: **while** $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c}$ is not convex-concave liftable **do**

6: Split the graph $(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{in}^c \setminus \{1\}}$ into two subgraphs by finding a hyperplane $\mathcal{H}^c$:

$$(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{bd}^{c+1}} : \forall i \in \mathcal{N}_{bd}^{c+1}, \mathcal{H}^c \cap \mathcal{X}_i \neq \emptyset; \tag{11a}$$

$$(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{in}^{c+1}} : \forall i \in \mathcal{N}_{in}^{c+1}, \mathcal{H}^c \cap \mathcal{X}_i = \emptyset. \tag{11b}$$

7: Update parameters: $c = c + 1$.

8: Update the item set: $\mathcal{N}_{in}^c = \mathcal{N}_{in}^c \cup \{1\}$.

9: Construct a convex-concave lifting function $g^c(x)$ for subgraph $(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{bd}^c}$, ensuring the corresponding projection $\mathrm{Proj}_{\mathbb{R}^n} \mathcal{P}_{g^c} = \underline{\mathcal{X}}$ as defined in equation (7).

10: **end while**

11: Update parameters: $c = c + 1$.

12: Repeat Step 9 for subgraph $(\mathscr{N}, \mathscr{E})_{\mathcal{N}_{in}^c}$.

13: By combining the functions $\kappa_{pwa}(x)$ and $g^j(x)$ with $j \in \{1, \cdots, c\}$, we generate set $\mathcal{P}_z$ and define the appropriate function $\kappa_z : \underline{\mathcal{X}} \rightarrow \mathbb{R}^{m+c}$.

G. A link with low complexity mpQP alternatives to MPC

This subsection explains constructing an mpQP problem using $\mathcal{P}_z$ and the function $\kappa_z(x)$ proposed in the previous subsection. We also analyze the relationship between the new mpQP problem and the original problem in (3).

Consider an mpQP problem:

$$\kappa_z(x) = \arg \min_{u,z} f(x, u, z) \tag{12a}$$

$$\text{subject to:} [x^T, u^T, z^T]^T \in \mathcal{P}_z, \tag{12b}$$

where $\kappa_z(x)$ and $\mathcal{P}_z$ are provided by proposition 5.

Upon comparing problems (3) and (12), the follows hold:

- They share the same domain of parameter x:

$$\mathrm{Proj}_{\mathbb{R}^n} \mathcal{P} = \mathrm{Proj}_{\mathbb{R}^n} \mathcal{P}_z = \underline{\mathcal{X}}.$$

- The first m elements of the optimal solution coincide:

$$\kappa_{pwa}(x) = \left[I_m, \mathbf{0}_{m \times (m-1)N}\right] \kappa_u^*(x) = \left[I_m, \mathbf{0}_{m \times c}\right] \kappa_z(x).$$

- Their optimal solutions share the same geometric structures: the solution for the unconstrained region $\mathcal{X}_1$ remains within the feasible domain. In contrast, the solution for other regions lies on the boundary.
- The variables in problem (3) have a dimension of mN, while those in problem (12) have a dimension of $m + c$. Typically, we have $mN \gg m + c$, indicating that problem (12) is less complex and can be solved faster online than problem (3).

Remark 6. Designing a suitable cost function $f(x, u, z)$ to ensure the feasibility of problem (12) is a challenging task and will be the subject of further studies.

V. Conclusions

Starting from the optimal control law obtained by means of an mpQP problem, an analysis of the polyhedral partition's topology has been made to decompose the regions into the boundary (saturated one) and interior parts. Different strategies exploiting the convex-concave liftability of the interior regions help replace the interior regions on the boundary in a higher dimension. The polytope obtained by liftings is one of the most important steps for building inverse-optimal solutions, as it provides the feasible domain in terms of linear inequalities. Future work will build upon this method and detail the selection of the optimal index.

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