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A Coupled Optimization Framework for Correlated Equilibria in Normal-Form Games

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***Abstract*— In competitive multi-player interactions, simultaneous optimality is a key requirement for establishing strategic equilibria. This property is explicit when the game-theoretic equilibrium is the simultaneously optimal solution of coupled optimization problems. However, no such optimization problems exist for the correlated equilibrium, a strategic equilibrium where the players can correlate their actions. We address the lack of a coupled optimization framework for the correlated equilibrium by introducing an *unnormalized game*—an extension of normal-form games in which the player strategies are lifted to unnormalized measures over the joint actions. We show that the set of fully mixed generalized Nash equilibria of this unnormalized game is a subset of the correlated equilibria of the normal-form game. Furthermore, we introduce an entropy regularization to the unnormalized game and prove that the entropy-regularized generalized Nash equilibrium is a sub-optimal correlated equilibrium where the degree of sub-optimality depends on the magnitude of regularization. We derive a closed form solution for an entropy-regularized generalized Nash equilibrium and verify via simulation its computational complexity.**

I. Introduction

As autonomous and artificial intelligence-assisted technology become ubiquitous in our daily lives, game theory emerges as an important tool for modeling and analyzing the interactions between autonomous agents. Within a game, player interactions are at a competitive equilibrium when their strategies are *simultaneously optimal*: no player can achieve a better objective by unilaterally deviating from its current strategy. For the Nash equilibrium and the Stackelberg equilibrium, simultaneous optimality is an explicit property: these equilibria solve optimization problems with coupled objectives and constraints. The connection to optimization has directly led to the application of gradient-based algorithms to computing game-theoretic equilibria in autonomy and artificial intelligence [1]–[3].

The correlated equilibrium is an extension of the Nash equilibrium to the joint action space. By utilizing a *correlation device* to enable inter-player coordination, a correlated equilibrium achieves better social welfare than the Nash equilibrium without compromising on simultaneous optimality [4]. Furthermore, since correlated equilibria form a connected polytope [5], fairness and other system-level metrics can be optimized smoothly over these equilibrium strategies. Scenarios that allow correlation device-enabled coordination include urban mobility [6], [7], robotics [8], and power markets [9].

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Despite its advantages, the correlated equilibrium's computation complexity grows exponentially in the number of players and actions, and the lack of a coupled optimization framework describing has made it difficult to apply gradient-based algorithms to its computations. Presently, we pose and answer the following question: *can we construct a coupled optimization problem whose optimal solution is the correlated equilibrium of a normal-form game?*

Contributions. Our contribution is two-fold: 1) we introduce unnormalized games: an extension of normal-form games in which the player strategies are unnormalized measures, and prove that a strictly positive generalized Nash equilibrium of the unnormalized game is a correlated equilibrium of the normal-form game, 2) we formulate an entropy-regularized unnormalized game, prove that its generalized Nash equilibria are sub-optimal correlated equilibria of the normal-form game and find a closed-form expression of a generalized Nash equilibrium. We also evaluate the computation complexity of the generalized Nash equilibrium and its approximation to correlated equilibrium via simulations.

Relevant research. First introduced in [4], the correlated equilibrium exists in both finite and infinite games, including games that possess no Nash equilibria [10]. A correlated equilibrium definition requires both a correlation device and the resulting probability distribution over the joint action space [11]. The correlated equilibrium have been defined and formulated differently depending on the game formulation [2], [12]. Its extensions under further game structure include constrained correlated equilibrium [13], quantal correlated equilibrium [14], extensive-form correlated equilibrium [15], and coarse correlated equilibrium [16]. Learning dynamics that converge to the correlated equilibrium include uncoupled no-regret learning dynamics [17] and evolution dynamics [18]. However, they do not provide closed-form solutions of the correlated equilibrium.

II. Equilibria concepts in normal form games

We consider a normal-form game with N players. Let $[A_i](A_i \in \mathbb{N})$ denote the set of actions available to player i, and let $[A] = [A_1] \times \ldots \times [A_N](A = \prod_{i\in[N]} A_i)$ denote the set of all joint actions available in the game. We denote player i's action as $a_i \in [A_i]$, the action taken by player i's opponents as a_{-i}, and every player's joint action as $a := (a_1, \ldots, a_N) \in [A]$. Under a joint action $a \in [A]$, player i incurs a cost $\ell_i(a)$, where $\ell_i : [A] \mapsto \mathbb{R}$ for all $i \in [N]$.

We denote the A_i−dimensional probability simplex over $[A_i]$ as Δ_i, the joint probability simplex as $\Delta = \Delta_1 \times \ldots \times \Delta_N$, and the A-dimensional probability simplex over $[A]$ as Δ_A. Player i's **strategy** $x_i \in \Delta_i$ is a probability distribution over the action set $[A_i]$. Under the strategy x_i, player i selects an action a_i with the probability $x_i(a_i)$ for all $a_i \in [A_i]$. The **joint strategy** $x := (x_1, \ldots, x_N) \in \prod_{i\in[N]} \Delta_i$ is the collection of all of the players' strategies. Let the opponent strategy, action space, and strategy space be given by $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, $[A_{-i}] = \prod_{j\neq i}[A_j]$, $\Delta_{-i} = \prod_{j\neq i} \Delta_j$, respectively. Under the joint strategy x, the expected cost for player i is given by

$$\mathbb{E}_{a\sim x}[\ell_i(a)] = \sum_{a_i\in[A_i]} x_i(a_i) \sum_{a_{-i}\in[A_{-i}]} x_{-i}(a_{-i})\ell_i(a_i, a_{-i}). \tag{1}$$

We use $\hat{\ell}_i : [A_i] \times \Delta_{-i} \mapsto \mathbb{R}$ to denote player i's expected cost for playing action a_i conditioned on the other players playing the strategy x_{-i}:

$$\hat{\ell}_i(a_i; x_{-i}) = \mathbb{E}[\ell_i(a_i, a_{-i}) \; |a_j \sim x_j, \; \forall j \neq i], \forall i \in [N]. \tag{2}$$

Using the notation $\hat{\ell}_i$, player i's expected cost (1) under strategy x_i is given by $\mathbb{E}_{a\sim x}[\ell_i(a)] = \sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i})$ when the other players choose strategies x_{-i}.

Each player minimizes its expected cost $\mathbb{E}_{a\sim x}[\ell_i(a)]$ through unilateral changes in its own strategy $x_i \in \Delta_i$. At the joint strategy $x = (x_1, \ldots, x_N)$ and for each $i \in [N]$, if x_i minimizes $\sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i})$ simultaneously, x is a Nash equilibrium.

Definition 1 (**Nash equilibrium**)**.** *The joint strategy $x^\star = (x_1^\star, \ldots, x_N^\star) \in \Delta$ is a Nash equilibrium if for each $i \in [N]$,*

$$\sum_{a_i\in[A_i]} x_i^\star(a_i)\hat{\ell}_i(a_i; x_{-i}^\star) \leq \sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i}^\star), \; \forall \; x_i \in \Delta_i. \tag{3}$$

The set of Nash equilibria is equivalent to the set of KKT points of the following coupled linear program for each $i \in [N]$. In general, this set is disconnected [5].

$$\begin{aligned} \min_{x_i\in\Delta_i} & \sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i}), \\ \text{s.t.} & \sum_{a_i\in[A_i]} x_i(a_i) = 1, x_i(a_i) \geq 0, \; \forall a_i \in [A_i], \end{aligned} \tag{4}$$

Nash equilibrium extends the notion of single-player optimality to simultaneous optimality under unilateral deviations in the players' strategies [19]. A Nash equilibrium strategy $(x_1^\star, \ldots, x_N^\star)$ ensures that, against the other players' strategy $x_{-i}^\star$, $x_i^\star$ minimize player i's expected cost within player i's own strategy space Δ_i.

Independent decision-making induces inequity. A Nash equilibrium implicitly assumes that the players make decisions independently—i.e., x_i, x_j are independent probability distributions for all $i, j \in [N]$, $j \neq i$. While this assumption holds for game-theoretic models such as the Prisoner's Dilemma [20], it fails to take advantage of the additional coordination structure that exists in large-scale cyber-physical systems. Furthermore, independent decision-making often induces inequity among players.

Example 1 (Vehicle standoff)**.** *Consider a single-lane road with bi-directional traffic and an unexpected pothole on its right side. Vehicles can choose to veer left or right to pass each other. The two pure Nash equilibria are (left, right) and (right, left), but the traffic direction that chooses the pothole side will constantly be at a disadvantage. A mixed Nash equilibrium can ensure that both traffic directions are equally likely to encounter unexpected potholes, but it also means that with positive probability, both directions' vehicles will choose the same roadside and stall traffic.*

A more robust solution is coordinating both traffic directions to alternate between the Nash equilibria (left, right) and (right, left). This requires correlation between the vehicles.

Definition 2 (Correlated strategy)**.** *The A-dimensional probability distribution $y \in \Delta_A$ is a correlated strategy if $y(a) \geq 0$ denotes the probability of the joint action $a = (a_1, \ldots, a_N)$ occurring, for all $a \in [A]$ and $\sum_{a\in[A]} y(a) = 1$ [5], [21].*

To use correlated strategies, players must have the incentive and the means to coordinate. As Example 1 illustrates, one incentive is to improve fairness among players, and one possible coordination method is a traffic operator. In general, inter-player coordination is conducted via a *correlation device* [11]. Presently, we assume that there exists a correlation device that can realize all correlated strategies given by Definition 2.

Every joint strategy induces a correlated strategy, but not every correlated strategy can be reduced to a joint strategy. Furthermore, all correlated strategies induced by joint strategies are rank one in their tensor form.

Example 2 (Rank of correlated strategy tensors)**.** *Consider a two-player normal form game with finite action sets $[U]$ and $[V]$. We will cast the correlated strategy $y \in \Delta_{UV}$ to a matrix $Y \in \mathbb{R}^{U\times V}$. For a joint strategy $(x_U, x_V) \in \Delta_U \times \Delta_V$, the corresponding correlated strategy Y is given by $Y = x_U x_V^\top$. Thus, all the joint strategies $x = (x_U, x_V)$ produce rank one correlated strategies in its matrix form.*

On the other hand, let Y_0 be any feasible correlated strategy, then the complete set of correlated strategies is given by $Y_0 + \mathcal{N}$ where $\mathcal{N}$ is defined as

$$\mathcal{N} = \{Y \in \mathbb{R}^{U\times V} | \mathbb{1}^\top Y \mathbb{1} = 0, \; Y + Y_0 \geq 0\},$$

From the constraint $\mathbb{1}^\top Y \mathbb{1} = 0$, matrices in $\mathcal{N}$ have a maximum rank of $\min\{U, V\} - 1$.

Example 2's tensor formulation of correlated strategies extends to the N-player setting: every joint strategy $(x_1, \ldots, x_N)$, where $x_i \in \Delta_i$ for all $i \in [N]$, induces a correlated strategy $\hat{y}$ given by

$$\hat{y}(a_1, \ldots, a_N) = \textstyle\prod_{i\in[N]} x_i(a_i), \; \forall (a_1, \ldots, a_N) \in [A]. \tag{5}$$

If we cast $\hat{y} \in \Delta_A$ to an N-dimensional tensor $Y \in \mathbb{R}^{A_1\times\ldots\times A_N}$, we observe that $\hat{y}$ is again a rank one tensor.

Comparison of solution spaces Δ and Δ_A. The joint strategy's and correlated strategy's solution spaces differ significantly in size. A joint strategy is given by N independent probability distributions $x_i \in \Delta_i$, and its overall dimension is $\sum_i A_i$. On the other hand, a correlated strategy has dimension $A = \prod_{i\in[N]} A_i$. When the number of players or the number of player actions increases, the joint strategy space Δ's dimension scales linearly, while the correlated strategy space Δ_A's dimension scales *exponentially*.

Optimality in correlated strategy space Δ_A. A correlated strategy is optimal for player i if no action $\hat{a}_i \in [A_i]$ can replace $a_i \in [A_i]$ in all joint actions involving a_i, such that player i's expected cost ℓ_i improves. This notion of optimality is the same as the Nash equilibrium (3). However, unlike the Nash equilibrium, defining the optimality of an independent strategy is no longer sufficiently descriptive. We formally define correlated equilibrium as below.

Definition 3 (**Correlated equilibrium [4]**). *The correlated strategy $y \in \Delta_A$ (Definition 2) is a correlated equilibrium if for all $i \in [N]$ and $a_i, \hat{a}_i \in [A_i]$,*

$$\textstyle\sum_{a_{-i}\in[A_{-i}]} \Big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i})\Big) y(a_i, a_{-i}) \le 0. \quad (6)$$

Intuitively, condition (6) implies that player i cannot independently swap action a_i for $\hat{a}_i$ while the other players play a_{-i} and achieve a lower expected cost. On the set of correlated strategies induced by joint strategies, the correlated equilibrium condition (6) is equivalent to the Nash equilibrium condition (3).

Lemma 1. *Over the set of correlated strategies induced by joint strategies as in* (5)*, the correlated equilibrium condition* (6) *is equivalent to the Nash equilibrium condition* (3)*.*

See [22, App.A] for proof.

Correlated equilibrium polytope. In the original formulation of a correlated equilibrium [4], the set of correlated equilibria is shown to be equivalent to the following linear polytope on the joint action space.

$$\begin{aligned}\mathcal{P}_{CE} := \Big\{ & y \in \Delta \mid \mathbb{1}^\top y = 1, y \ge 0, \\ & \textstyle\sum_{a_{-i}\in[A_{-i}]} y(a_i, a_{-i}) \Big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i})\Big) \le 0, \\ & \forall a_i, \hat{a}_i \in [A_i],\ i \in [N] \Big\}. \end{aligned} \quad (7)$$

In [5], the authors showed that in addition to being a connected polytope, $\mathcal{P}_{CE}$'s boundary set $\partial\mathcal{P}_{CE}$ contains the correlated strategies induced by Nash equilibria. However, computing the $\mathcal{P}_{CE}$ suffers from the curse of dimensionality both due to the dimension of Δ being exponential in the N and the number of $\mathcal{P}_{CE}$'s constraints, $\sum_{i\in[N]} \binom{A_i}{2}$, being exponential in A_i.

III. Lifting correlated equilibrium

While a correlated equilibrium has the interpretation of being a 'simultaneously optimal' strategy in literature, this interpretation lacks an explicit optimization formulation like the one that exists for Nash equilibrium in the form of (4). In this section, we formulate a novel game in which the player strategy spaces are lifted from the probability measure spaces over $[A_i]$ to an unnormalized measure space over $[A] = \prod_{i\in[N]}[A_i]$. We show that a fully mixed generalized Nash equilibrium of the lifted game corresponds to a fully mixed correlated equilibrium of the normal form game.

We first relax probability measures to unnormalized measures with finite mass [23], [24]. A vector α is an unnormalized measure if

$$\alpha \in \mathbb{R}^A_+/\{0\},\ \alpha(a) \ge 0,\ \forall a \in [A]. \quad (8)$$

Given two unnormalized measures α_1, α_2 over $[A]$, we denote their **element-wise product** by $\alpha_1 \circ \alpha_2$, such that

$$(\alpha_1 \circ \alpha_2) \in \mathbb{R}^A_+, (\alpha_1 \circ \alpha_2)(a) = \alpha_1(a)\alpha_2(a),\ \forall a \in [A].$$

We consider the decomposition of a correlated strategy y (Definition 2) into N unnormalized measures.

Definition 4 (Normalized Decomposition)**.** *Given a correlated strategy $y \in \Delta_A$, $(\alpha_1, \ldots, \alpha_N)$ is a normalized decomposition of y and y is a product of $(\alpha_1, \ldots \alpha_N)$ if*

$$y = \alpha_1 \circ \ldots \circ \alpha_N,\ \alpha_i \in \mathbb{R}^A_+,\ \forall i \in [N]. \quad (9)$$

Definition 4 decomposes y element-wise rather than factoring it vector-wise into a joint strategy (5). Furthermore, the mapping $(\alpha_1, \ldots, \alpha_N) \mapsto y$ is surjective but not injective.

Lemma 2. *Every correlated strategy $y \in \Delta_A$ has an infinite number of decompositions in the form of* (9)*. Furthermore, if $(\alpha_1, \ldots, \alpha_N)$ satisfies*

$$\mathbb{1}^\top(\alpha_1 \circ \ldots \circ \alpha_N) = 1, \alpha_i \ge 0,\ \forall i \in [N], \quad (10)$$

then $y = \alpha_1 \circ \ldots \circ \alpha_N$ is a correlated strategy.

See [22, App.B] for proof.

Example 3 (Normalized decompositions)**.** *We can represent the unnormalized measures of a two player game where $A_1 = A_2 \in \mathbb{N}$ by $A_1 \times A_2$-dimensional matrices, $\alpha \in \mathbb{R}^{A_1\times A_2}$, such that any element-wise product $\alpha_i \circ \alpha_j$ is equivalent to the Hadarmard product between their matrix counterparts. The following are all valid normalized decompositions and their correlated strategy product.*

$$\begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix} \circ \begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \circ \begin{pmatrix}1&0&0\\0&1&2\\0&3&4\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&2\\0&3&0\end{pmatrix}.$$

A. Unnormalized game

We define an **unnormalized game** that extends the normal-form game as follows: instead of choosing probability distributions supported on each player's individual action space, each player i chooses an unnormalized measure α_i over the joint action space $[A]$ as defined in Definition 8, constrained by the condition fact that $y = \alpha_1 \circ \ldots \alpha_N$ is a correlated strategy. The player objectives remain the expected cost incurred by each player (2), which is a multi-linear function of the unnormalized measures through (9) and (2). Each player's optimization problem is given by

$$\begin{aligned} \min_{\alpha_i \in \mathbb{R}^A_+} & \textstyle\sum_{a\in[A]} \ell_i(a)\alpha_1(a)\dots\alpha_N(a), \\ \text{s.t. } & \textstyle\sum_{a\in[A]} (\alpha_1 \circ \dots \circ \alpha_N)(a) = 1. \end{aligned} \tag{11}$$

In the unnormalized game, each player's strategy α_i has the same dimension as the correlated strategy of the original finite game. Given the other players' strategies α_{-i}, player i uses their strategy α_i to optimize the expected cost $\sum_{a\in[A]} \ell_i(a)\alpha_1(a)\dots\alpha_N(a)$, constrained by a mass constraint: $\sum_{a\in[A]}(\alpha_1 \circ \dots \circ \alpha_N)(a) = 1$. The optimal solution here is a generalized Nash equilibrium, where, in addition to minimizing their expected cost, each player's strategy is feasible with respect to the other players' strategies.

Definition 5 (Generalized Nash equilibrium [25])**.** *A joint strategy $(\alpha_1^\star, \dots, \alpha_N^\star)$ is a generalized Nash equilibrium if for all $i \in [N]$, $\alpha_i^\star$ is optimal for* (11) *with respect to $\alpha_{-i}^\star$.*

A generalized Nash equilibrium extends the Nash equilibrium (3) to games where each player's strategy feasibility depends on the other players' strategies. In the unnormalized game, all players share the mass constraint given by (10). We further restrict our analysis to fully mixed measures.

Assumption 1 (Fully mixed measures)**.** *The unnormalized measure $\alpha \in \mathbb{R}^A_+$ satisfies $\alpha(a) > 0, \ \forall a \in [A]$.*

When a correlated strategy is fully mixed, all of its normalized decompositions $(\alpha_1, \dots, \alpha_N)$ satisfy Assumption 1. Games with certain player cost structures, such as zero-sum games and games with non-dominant strategies, tend to have fully mixed Nash and correlated equilibria [26], [27].

B. Generalized Nash equilibrium and correlated equilibrium

We show that y is a correlated equilibrium of the normal form game if its decomposition $(\alpha_1, \dots, \alpha_N)$ is a generalized Nash equilibrium of the unnormalized game.

Proposition 1. *If $(\alpha_1, \dots, \alpha_N)$ is a generalized Nash equilibrium of the unnormalized game* (11) *and α_i satisfies Assumption 1 for all $i \in [N]$, then the product $y = \alpha_1 \circ \dots \circ \alpha_N$ is a correlated equilibria of the normal form game* (6)*.*

Proof. We prove this proposition by showing that Assumption 1 and the coupled KKT conditions of (11) together imply the correlated equilibrium condition in (6). From [25, Thm.3.3], the coupled KKT conditions of (11) are necessary and sufficient for $(\alpha_1, \dots, \alpha_N)$ to be a generalized Nash equilibrium of the unnormalized game.

From the unnormalized game (11) for player i, we assign the Lagrange multipliers $\sigma_i \in \mathbb{R}$ for the constraint $\sum_{a\in[A]}(\alpha_1 \circ \dots \circ \alpha_N)(a) = 1$ and $\mu_i(a)$ for the constraints $\alpha_i(a) \geq 0$. The first-order gradient condition and the complementarity condition of the KKT are given by

$$\begin{aligned} 0 =& \ell_i(a)\alpha_{-i}(a) - \sigma_i\alpha_{-i}(a) - \mu_i(a) = 0, \\ \mu_i(a) =& \begin{cases} \geq 0 & \alpha_i(a) = 0 \\ = 0 & \alpha_i(a) > 0 \end{cases}, \forall (i,a) \in [N] \times [A]. \end{aligned} \tag{12}$$

When $\alpha_{-i}(a) > 0$, the KKT conditions above imply that

$$\ell_i(a) \begin{cases} = \sigma_i, & \text{if } \alpha_i(a) > 0 \\ \geq \sigma_i, & \alpha_i(a) = 0 \end{cases}, \ \forall (i,a) \in [N] \times [A]. \tag{13}$$

From Assumption 1, $\alpha_{-i}(a) > 0$ and $\alpha_i(a) > 0$ for all $a \in [A]$. Therefore, $\mu_i(a) = 0$, $\sigma_i = \ell_i(a)$ for all $a \in [A]$. In particular, $\ell_i(a_i, a_{-i}) = \ell_i(\hat{a}_i, a_{-i})$ for all $a_i, \hat{a}_i \in [A_i]$. The correlated equilibrium condition (6) $(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i}))y(a_i, a_{-i})$ will then evaluate strictly to 0 for all $\hat{a}_i \in [A_i]$ and $i \in [N]$. We conclude that $y = \alpha_1 \circ \dots \circ \alpha_N$ is a correlated equilibrium. □

Remark 1. *Proposition 1 suggests that a correlated equilibrium is fully mixed only if $\ell_i(a)$ evaluates to the same value for all $a \in [A]$. While this may seem restrictive, we use entropy regularizations in Section IV to produce games in which the regularized costs are all equal for each opponent action a_{-i}. We can show that the generalized Nash equilibrium under regularization will approximate the correlated equilibrium of the normal-form game even if no fully mixed correlated equilibrium exists.*

Proposition 1's implication does not hold in reverse: if y is a fully mixed correlated equilibrium, its normalized decomposition may not be a generalized Nash equilibrium of the unnormalized game.

Example 4 (Correlated equilibria not captured by gNE)**.** *Consider a 2×2 matrix game where player one chooses the row and player two chooses the column. The player costs are given by matrices A and B, respectively, defined as*

$$P_1 = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

We vectorize the joint action space as $[A] = \{a_1, a_2, a_3, a_4\}$, corresponding to the counterclockwise sequence of joint actions in matrix P_i starting from the top left. For $y \in \Delta_4$ to be a correlated equilibrium as defined in (6)*, it must satisfy*

$$\begin{aligned} 3y(a_1) + 3y(a_4) - 2y(a_2) - 4y(a_3) &\leq 0, \\ 1y(a_1) + 1y(a_2) - 0y(a_3) - 2y(a_4) &\leq 0. \end{aligned} \tag{14}$$

We can verify that $y_{CE} = [\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}]$ satisfies (14)*. If player two uses strategy $\alpha_2(a_j) = \frac{1}{2}$ for all $j \in [4]$, player one's unnormalized game is given by*

$$\begin{aligned} &\min_{\alpha_1} \frac{1}{2}\big(3\alpha_1(a_1) + 2\alpha_1(a_2) + 4\alpha_1(a_3) + 3\alpha_1(a_4)\big) \\ &\text{s.t. } \textstyle\sum_j \alpha_1(a_j) = 2, \alpha_1(a_i) \geq 0, \ \forall i \in [4]. \end{aligned} \tag{15}$$

We verify that $\alpha_1 = [1/2 \quad 1/2 \quad 1/2 \quad 1/2]$ satisfies $\alpha_1 \circ \alpha_2 = y_{CE}$ but it does not minimize (15)*. Specifically, $\hat{\alpha}_1 = [0 \quad 1 \quad 1/2 \quad 1/2]$ can achieve a lower objective than α_1 against $\alpha_2 = [1/2 \quad 1/2 \quad 1/2 \quad 1/2]$.*

From Proposition 1, we conclude that if the generalized Nash equilibrium of (11) is strictly positive, then their product y is a fully mixed correlated equilibrium of the original normal-form game. A natural follow-up question is, when do strictly positive correlated strategies exist? We explore this in the following section using entropy regularizations.

IV. Entropy-Regularized Correlated Equilibria

We demonstrate how entropy regularization can be applied to the unnormalized game to the find ϵ-correlated equilibria of the original normal-form game. Consider the entropy-regularized counterpart of the unnormalized game (11), where each player solves the optimization problem given by

$$\begin{aligned} \min_{\alpha_i \in \mathbb{R}_+^A} & \sum_{a\in[A]} \Big(\ell_i(a) + \tfrac{1}{\lambda_i}\Big(\log\big(\alpha_i(a)\big) - 1\Big)\Big)\alpha_i(a)\alpha_{-i}(a), \\ \text{s.t. } & \sum_{a\in[A]}(\alpha_1 \circ \ldots \circ \alpha_N)(a) = 1. \end{aligned} \tag{16}$$

Here, $\lambda_i \geq 0$, $\lambda_i \in \mathbb{R}$ denotes the magnitude of the entropy regularization. The total entropy of the correlated strategy $y = \alpha_1 \circ \ldots \circ \alpha_N$ is given by $\sum_{a\in[A]} y(a)\log\big(y(a)\big)$, such that $\sum_{a\in[A]} y(a)\log(\alpha_i)$ is equivalent to player i's contribution to the total entropy. For two unnormalized measures that achieve equal costs $\sum_{a\in[A]} \ell_i(a)\alpha_i(a)\alpha_{-i}(a)$, (16) favors the measure with the greater entropy and thus achieving a lower cost as defined by (11).

Remark 2. *In game models of multi-robot systems [6], [8], the model parameters are often obtained from noisy and imperfect data. Given imperfect models, policies with higher entropy can protect players against any individual modeling flaw. Entropy regularization is often used to improve the policy robustness to modeling inaccuracies [28], [29]. In games, a entropy-regularized Nash equilibrium is a* logit quantal response equilibrium *[30] and is more robust than the Nash equilibrium in systems involving human players [31].*

We can show that the entropy-regularized unnormalized distribution game has the following closed-form solution.

Proposition 2. *In the entropy-regularized unnormalized game where each player solves* (16)*, there exists a generalized Nash equilibrium* $(\alpha_1, \ldots, \alpha_N)$*, where* α_i *is given by*

$$\alpha_i(a) = \frac{\exp\big(-\lambda_i \ell_i(a)\big)}{\Big(\sum_a \exp\big(-\sum_j \lambda_j \ell_j(a)\big)\Big)^{\frac{\lambda_i}{\sum_j \lambda_j}}}, \ \forall i, a \in [N]\times[A]. \tag{17}$$

See [22, App.C] for proof. The correlated equilibrium corresponding to (17) is

$$y(a) = \frac{\exp\big(-\sum_j \lambda_j \ell_j(a)\big)}{\sum_a \exp\big(-\sum_j \lambda_j \ell_j(a)\big)}, \ \forall a \in [A]. \tag{18}$$

The resulting correlated equilibrium is a softmax function over the regularized and weighted sum of individual player costs, where the level of entropy introduced is controlled by λ_i: as $\lambda_i \to 0$ is, the resulting correlated equilibrium converges to the completed mixed correlated strategy. We note that while Proposition 2 provides a solution for the entropy-regularized unnormalized game (16), other generalized Nash equilibria exist. In particular, each strictly positive generalized Nash equilibrium is an ϵ-correlated equilibrium of the original normal form game (11).

Corollary 1 (ϵ-correlated equilibrium)**.** *If the entropy-regularized generalized Nash equilibrium* $(\alpha_1, \ldots, \alpha_N)$ *satisfies Assumption 1 for each* $i \in [N]$*, their product* y (18) *is an* ϵ*-correlated equilibria—i.e., for all* $i \in [N]$,

$$\sum_{a_{-i}} y(a_i, a_{-i})\big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i})\big) \leq \frac{\epsilon_i}{\lambda_i}, \ \forall a_i, \hat{a}_i \in [A_i], \tag{19}$$

where $\epsilon_i = \max_{a,\hat{a}\in[A]} \log\Big(\alpha_i(a)/\alpha_i(\hat{a})\Big)$ *and* $\epsilon = \max_i \epsilon_i$.

See [22, App.D] for proof.

V. Computing ϵ-Correlated Equilibrium

We apply the results of Section IV to compute the generalized Nash equilibrium of the unnormalized game (11) and evaluate its feasibility as an ϵ-correlated equilibrium of the original normal-form game.

In Figure 1, we simulate normal-form games (4) with $N = \{2, 3\}$ players and individual action spaces of size $A = \{2, 5, 10\}$ and evaluate the accuracy of the entropy-regularized generalized Nash equilibrium (18) over $K = 1000$ randomly generated costs. We plot the empirical violation with a 5% standard deviation range of the correlated equilibrium condition (6) under ϵ (empirical) and the theoretical bound $\max_i \epsilon_i/\lambda_i$ (19) under ϵ (bound) for the regularization values $\lambda = \{0.1, 10, 30, 100, 1000, 1e4\}$. We assume that all players use the same entropy regularization, $\lambda_i = \lambda$, $\forall i \in [N]$. Shown in Figure 1, the results demonstrate statistical decrease in correlated equilibrium violations as the entropy regularization increases.

For each game, we compute its entropy-regularized generalized Nash equilibrium $y^\star$ via (18) and evaluate $y^\star$'s empirical sub-optimality $\epsilon_{ce} = \epsilon$ (empirical) as

$$\max_{\substack{i\in[N] \\ a_i, a_i' \in [A_i]}} \ \sum_{a_{-i}\in[A_{-i}]} y^\star(a_i, a_{-i})\big(\ell_i(a_i, a_{-i}) - \ell_i(a_i', a_{-i})\big). \tag{20}$$

We note that ϵ_{ce} is equivalent to the distance between y_{ce} and the correlated polytope in ∞ vector norm.

Finally, we note that a key challenge in applying correlated equilibrium for autonomous interactions is its poor scalability in the number of agents and actions. To this end, (18) provides an approximation that significantly reduces the computation complexity. We evaluate the computation complexity of (18) in Figure 2, and observe that while its computation time scales poorly in the number of players, it is significantly lower than the computation time of solving for a correlated equilibrium polytope via linear programming: CVXPY computes a correlated equilibrium for a $N = 3$ player game with action space $A_i = 3$ in ~ 1.87 seconds, whereas approximating it via (18) only takes $4e-3$ seconds.

VI. Conclusion

We introduce an extension of normal-form games and show that for fully mixed unnormalized measures, the set of generalized Nash equilibria of the unnormalized measure game produces fully mixed correlated equilibria in the original normal-form game. Additionally, we use entropy regularization to compute correlated strategies that are within ϵ

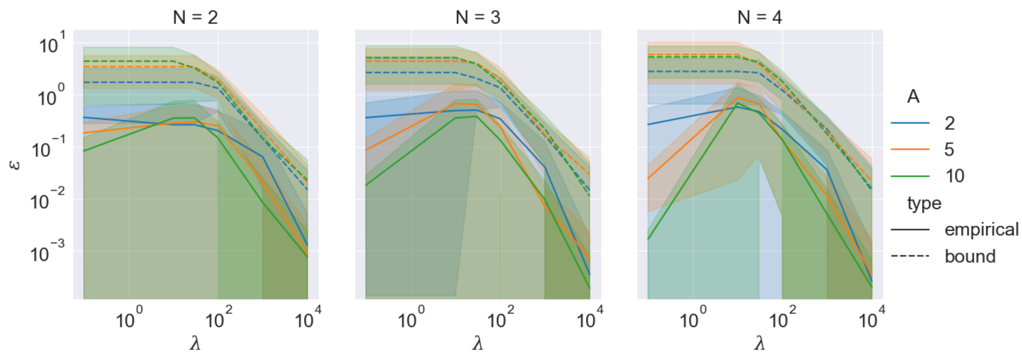

Fig. 1: Empirical vs theoretical sub-optimality of the entropy-regularized generalized Nash equilibrium (18) as a correlated equilibrium.

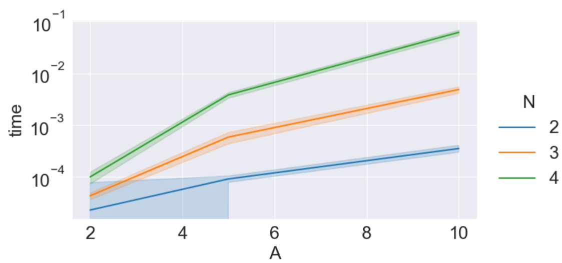

Fig. 2: Computation time (seconds) of the ϵ-correlated equilibrium for different numbers of players and actions.

distance of the correlated equilibrium polytope. Future work involves exploring the set of generalize Nash equilibrium and optimizing fairness metrics over it.

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