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Set-Invariance for Discrete-Time Linear Systems with Time-Varying Delay: a Polyhedral Approach

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Abstract— **In this paper, we address the positive invariance property in linear discrete-time systems in the presence of time-varying delays in the states. We build the study on a recently proposed approach based on an appropriate model transformation which allows the derivation of delay-dependent invariance conditions. For polyhedral sets, we establish necessary and sufficient invariance conditions with respect to the transformed model and prove that such conditions imply the confinement of the state trajectories of the original system in the set for arbitrary realizations of the varying delay, as long as the initial states belong to a set which is positively invariant with respect to an augmented switching system without delay, and can be computed in a finite number of steps known in advance.**

I. Introduction

Positive invariance of sets is a concept that has been intensively applied for the analysis and control of dynamical systems subject to state and input constraints. Even though the literature on this topic is mature nowadays for linear dynamic systems [1], some problems remain open when time-delays in the system's states and/or inputs appear [2] in both continuous- and discrete-time settings. In continuous-time, it is well-known that the delays describe time-heterogeneity and represent one of the simplest way to model propagation and transport phenomena or the behavior of population dynamics, and it is commonly accepted that their presence may result in poor performance and/or instabilities in the corresponding systems (see, for instance, [3], [4], [5], and the references therein).

The classical and most used definition of positive invariance (also called $\mathcal{D}$-invariance) for linear discrete-time delay systems leads to *delay-independent* conditions, i.e. that do not depend on the size of the delay [6], [7], [8]. Knowing that the size of the delay may affect the stability of the time-delay system, the conditions for the existence of $\mathcal{D}$-invariant sets are very restrictive.

A different perspective has been proposed in [9], where a model transformation allowed for a delay-dependent analysis of linear delay systems with a single delayed state and fixed value of the delay. This idea originates in some earlier works in the analysis of continuous-time linear delay systems [10] (see also [11]). A parameterized transformation of the original dynamical systems exploits its structure and allows a better decoupling between the so-called "delay-independent" and "delay-dependent" modes. Algebraic conditions for $\mathcal{D}$-invariance of polyhedral sets with respect to (w.r.t.) this transformed model have been derived, which depend on the size of the delay. A Linear Programming (LP) problem has been formulated to check if a given polyhedron is $\mathcal{D}$-invariant and to establish a range of values of the delay that preserve invariance. Furthermore, it has been established that invariance w.r.t. the transformed model implies confinement of the state trajectory in the set for the original model provided that the initial conditions of the system belong to an admissible set, which has also been characterized. These results have been extended to systems with multiple delays in the state in [12].

In the present work, we characterize positive invariance of polyhedral sets for linear discrete-time systems with time-varying delay in the states, subject to standard assumptions as, for instance, boundedness of the admissible delays. To the best of the authors' knowledge, this problem has not been object of study in the literature and this characterization is, in itself, the main contribution of this paper. Approaches based on Linear Matrix Inequalities (LMIs) exist, which are able to compute ellipsoidal invariant sets associated to quadratic Lyapunov functions. In [13], for instance, an LMI approach was proposed for robust stabilization of state-delayed discrete-time systems with bounded delay variation and actuators saturation.

First, we derive conditions for $\mathcal{D}$-invariance of polyhedral sets w.r.t. a transformed model with time-varying delay. Then, we show that the trajectories of the original model are confined in the invariant set of the transformed model if the initial conditions belong to an admissible set. Furthermore, we prove that this admissible set is a maximal positive invariant set of an augmented linear switched system without delays, and finitely determined in a number of steps equal to the size of the maximal delay. To illustrate the approach, we present two numerical examples (first- and second-order systems) and close the exposition by drawing a few concluding remarks.

Notations: Throughout the paper, the notations are standard or explained the first time they are introduced. In particular, $\mathbb{Z}_{[a,b]}$, with $a, b \in \mathbb{Z}$ stands for the set of integers i such that $a \leq i \leq b$. $\mathbb{Z}_+$ represents the set of nonnegative integers.

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II. PRELIMINARIES

In the sequel, we introduce a few notions and results adapted from [1] to cope with our framework.

Definition 1: (Convex Polyedral Set) Any closed and convex polyhedral set $\Omega \subseteq \mathbb{R}^n$ *can be characterized by a shaping matrix* $F \in \mathbb{R}^{f\times n}$ *and a vector* $w \in \mathbb{R}^f$*, i.e.,*

$$\Omega = \{x \in \mathbb{R}^n : Fx \leq w\}. \tag{1}$$

The class of polyhedral sets including the origin will be of interest, and can be characterized by $w > 0$ in (1).

Consider now a linear discrete-time system given by:

$$x(k+1) = \mathbb{A}x(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, and a compact set Ω containing the origin.

Definition 2: Given a scalar $0 \leq \lambda \leq 1$, Ω is called *positively invariant* with respect to (2) if for any initial condition $x(0) \in \Omega$, it follows that $x(k) \in \lambda\Omega, \forall k \in \mathbb{Z}_+$.

If $0 \leq \lambda < 1$, Ω is additionally said to be λ*-contractive*.

To simplify the exposition, from now on we will not explicitly represent λ-contractivity (which amounts to consider $\lambda = 1$), but all the results remain valid with minor changes if we assume a given contraction factor $\lambda < 1$.

The *i-step pre-image* of a set Ω is defined as $\mathcal{P}^i(\Omega) = \{x \ : \ \mathbb{A}^i x \in \Omega\}$. $\mathcal{P}^i(\Omega)$ is the set of states $x(k)$ such that $x(k+i) \in \Omega$. By convention, $\mathcal{P}^0(\Omega) = \Omega$. Ω is positively invariant if, and only if $\Omega \subset \mathcal{P}^1(\Omega)$.

If Ω is closed, contains the origin but is not positively invariant with respect to a stable dynamics (2), there exists a *maximal positively invariant set* contained in Ω [14].

The set $\mathcal{P}_N(\Omega)$ of states $x(k)$ such that $x(k+i) \in \Omega$ $\forall i \in \mathbb{Z}_{[0,N]}$ is defined as the intersection:

$$\mathcal{P}_N(\Omega) = \bigcap_{i=0}^{N} \mathcal{P}^i(\Omega). \tag{3}$$

Let Ω be the convex polyhedral set containing the origin in (1). Then, $\mathcal{P}_N(\Omega)$ is given by:

$$\mathcal{P}_N(\Omega) \ = \{x \ : \ F\mathbb{A}^i x \leq w, \text{ for } i \in \mathbb{Z}_{[0,N]}\}. \tag{4}$$

$\mathcal{P}_\infty(\Omega)$ (called the *maximal output admissible set* in [15]) is the maximal positively invariant set contained in Ω and is given by: $\mathcal{P}_\infty(\Omega) \ = \{x \ : \ F\mathbb{A}^i x \leq w, \text{ for } i \in \mathbb{Z}_+\}$.

III. POSITIVE INVARIANCE IN LINEAR SYSTEMS WITH TIME-VARYING DELAY

Consider the discrete-time system with time-varying delay

$$x(k+1) = Ax(k) + A_d x(k-d(k)), \tag{5}$$

with $x(k) \in \mathbb{R}^n$, A and A_d are real constant $\mathbb{R}^{n\times n}$ matrices, and the time-varying delay may take any value in a given bounded interval: $d(k) \in \mathbb{Z}_{[1,d_m]}$. The initial conditions of (5) are given by $x(i), i \in \mathbb{Z}_{[-d_m,0]}$.

We extend the set invariance definition in [16] (see also [6], [7]) to cope with time-varying delay as follows:

Definition 3: A set $\Omega \subset \mathbb{R}^n$ containing the origin is called *positively* $\mathcal{D}$*-invariant* with respect to the time-delay system (5) if for any initial conditions $x(i) \in \Omega$, $i \in \mathbb{Z}_{[-d_m,0]}$, it follows that $x(k) \in \Omega, \forall d(k) \in \mathbb{Z}_{[1,d_m]}, \forall k \in \mathbb{Z}_+$.

Consider now the convex polyhedral set Ω described in (1). Necessary and sufficient conditions for $\mathcal{D}$-invariance have been established as follows [6]:

Theorem 1: The polyhedral set Ω is positively $\mathcal{D}$-invariant w.r.t. (5) if, and only if there exist nonnegative matrices $H_0, L_0 \in \mathbb{R}^{f\times f}$ such that: $\begin{cases} H_0F = FA, \quad L_0F = FA_d \\ (H_0 + L_0)w \leq w. \end{cases}$

These conditions are delay-independent. As a consequence, they are also valid for the time-varying delay case [2]. Often these conditions are very hard to meet, because most commonly the size of the delay affects stability and, consequently, positive invariance. Then, $\mathcal{D}$-invariant polyhedral sets may not exist even if (5) is stable. An alternative for avoiding this conservatism is to work with an equivalent undelayed augmented model. Consider the augmented state vector:

$$X(k) = \begin{bmatrix} x^T(k) & x^T(k-1) & \cdots & x^T(k-d_m) \end{bmatrix}^T \in \mathbb{R}^N, \tag{6}$$

with $N = n(d_m + 1)$. The dynamics of the time-varying delay system (5) can be represented equivalently by the augmented delay-free switched system given by [17]:

$$X(k+1) = \mathcal{A}(d(k))X(k), \tag{7}$$

$$\mathcal{A}(d(k)) = \\ = \left[\begin{array}{cccc|c} A & \Gamma_1(d(k)) & \ldots & \Gamma_{d_m-1}(d(k)) & \Gamma_{d_m}(d(k)) \\ \hline \mathbf{0}_{n\times n} & & \mathbf{I}_{nd_m} & & \mathbf{0}_{nd_m\times n} \end{array}\right], \tag{8}$$

$$\Gamma_i(d(k)) = \begin{cases} A_d, & \text{if } i = d(k), \\ \mathbf{0}_{n\times n}, & \text{otherwise,} \end{cases} \quad \forall i \in \mathbb{Z}_{[1,d_m]}.$$

The block A_d changes its position according to the delay $d(k)$. The switched system (7) has d_m modes $\mathcal{A}(i)$, one mode for each possible $d(k) \in \mathbb{Z}_{[1,d_m]}$: $\mathcal{A}(i) = \mathcal{A}(d(k))$ for $d(k) = i$.

The model (7) is equivalent to the original model (5) in the sense that an initial condition $x(i), i \in \mathbb{Z}_{[-d_m,0]}$ results in the same trajectory $x(k), k \geq 0$. Hence, the methods for analysis and construction of invariant sets for undelayed switched systems can be used. In particular, positively invariant sets can be constructed if the origin is stable for this augmented model.

Let us now define the following polyhedron in the augmented state space,

$$\Omega_a = \{X \in \mathbb{R}^N : F_aX \leq w_a\}, F_a \in \mathbb{R}^{f_a\times N}, w_a \in \mathbb{R}^{f_a}. \tag{9}$$

The *one-step pre-image* of Ω_a w.r.t. (7) is now given by: $\mathcal{P}(\Omega_a) = \{X \ : \ \mathcal{A}(i)X \in \Omega_a, \forall i \in \mathbb{Z}_{[1,d_m]}\}$.

This definition takes into account the fact that any mode i of the switched system (7) can be active. Note that we use here for the pre-image operator a similar notation to the one we used for the linear time-invariant dynamics (2). The system it refers to will become clear from the context.

The i-step pre-image of Ω_a is obtained by applying $\mathcal{P}(.)$ recursively: $\mathcal{P}^i(\Omega_a) = \{X(k) \ : \ X(k+i) \in \Omega_a\}$.

The reader will notice that $X(k+i)$ is obtained by left-multiplying $X(k)$ by the product of i matrices $\mathcal{A}(l)$, with l assuming all possible values in $\mathbb{Z}_{[1,d_m]}$. For instance, if Ω_a is the polyhedral set defined in (9), then

$$\mathcal{P}^2(\Omega_a) = \{X(k) : F_a\mathcal{A}(j)\mathcal{A}(l)x(k) \leq w_a,\ \forall j,l \in \mathbb{Z}_{[1,d_m]}\}.$$

The set $\mathcal{P}_N(\Omega_a) = \{X(k) \ : \ X(k+i) \in \Omega_a,\ \forall i \in \mathbb{Z}_{[0,N]}\}$ is given by the intersection:

$$\mathcal{P}_N(\Omega_a) = \bigcap_{i=0}^{N} \mathcal{P}^i(\Omega_a). \tag{10}$$

In the polyhedral case, $\mathcal{P}_N(\Omega_a)$ is a polyhedral set given by a set of inequalities similar to those in (4), but defined in the augmented state space and with $\mathbb{A}^i$ replaced by the product of i matrices $\mathcal{A}(l)$, with l assuming all possible values in $\mathbb{Z}_{[1,d_m]}$. Theoretically, the maximal positively invariant set contained in Ω_a is the set $\mathcal{P}_\infty(\Omega_a)$, but it can be obtained from (10) in a finite number of steps. In general, the number of steps is not known in advance though, and can be very large.

Note that positive invariance w.r.t. the switched model (7) must satisfy the same conditions as positive invariance w.r.t. a linear system subject to polytopic uncertainties having the matrices $\mathcal{A}(i)$ as the vertices of the set defining the uncertainties [18].

By writing the delayed system as a switched undelayed one in an augmented state space, we can use known methods to compute polyhedral invariant sets. This way we detour from the conservatism of $\mathcal{D}$-invariance conditions w.r.t. the delayed model, being able to compute invariant sets if the system is stable. On the other hand, depending on the size of the maximal delay, we can end up with a high-dimensional augmented model, for which the computation of invariant polyhedral sets may become intractable.

In what follows we develop a solution to the problem addressed in this text: characterizing invariance of polyhedral sets defined in the original $\mathbb{R}^n$ space w.r.t. the system with time-varying delay (5), but without the conservatism of $\mathcal{D}$-invariance.

IV. Set Invariance in a Transformed Model

In [9], [12], invariance of polyhedral sets has been studied for systems with fixed time-delay. As a novelty, we extend here the use of a transformation in model (5) similar to that proposed in [9], [12] as a tool to obtain delay-dependent invariance conditions for the time-varying delay case. By introducing an auxiliary variable $K \in \mathbb{R}^{n\times n}$, the time-delay model above can be rewritten as:

$$\begin{aligned} x(k+1) = (A+K)x(k) + (A_d - K)x(k-d(k)) \\ -K(x(k) - x(k-d(k))). \end{aligned}$$

By writing the difference in the last term as:

$$x(k) - x(k-d(k)) = \sum_{i=-d(k)}^{-1} (x(k+i+1) - x(k+i)),$$

and replacing $x(k+i+1)$ by the expression defining the model (5), after standard calculations we arrive to the following transformed model with time-varying delay:

$$\begin{aligned} x(k+1) = (A+K)x(k) + (A_d - K)x(k-d(k)) \\ -\sum_{i=-d(k)}^{-1} K(A-I)x(k+i) - \sum_{i=-2d(k)}^{-d(k)-1} KA_dx(k+i)) \end{aligned} \tag{11}$$

with the initial conditions $x(i),\ i \in \mathbb{Z}_{[-2d_m,0]}$.

Based on the construction principles of the transformed model, one can notice that the equivalence of the trajectories of the original model (5) with those of the transformed model holds only if, in this last model, $x(k+i+1) = Ax(k+i) + A_dx(k+i-d(k))$ for $i \in \mathbb{Z}_{[-d(k),-1]}$. This equivalence is formally established as follows:

Theorem 2: Consider systems (5) and (11) and let their state trajectories for $k \in \mathbb{Z}_+$ be denoted, respectively, by $x(k)$ with initial conditions $x(i),\ i \in \mathbb{Z}_{[-d_m,0]}$, and by $x_t(k)$ with initial conditions $x_t(i),\ i \in \mathbb{Z}_{[-d_m,d_m]}$. If the realization of the delay $d(k) \in \mathbb{Z}_{[1,d_m]}$ for $k \in \mathbb{Z}_+$ is the same for both systems, and the initial conditions of (11) are given by:

$$\begin{aligned} x_t(i) &= x(i),\ \text{for } i \in \mathbb{Z}_{[-d_m,0]}, \\ x_t(i+1) &= Ax(i) + A_dx(i-d(i)),\ \text{for } i \in \mathbb{Z}_{[0,d_m-1]}, \end{aligned} \tag{12}$$

then $x_t(k) = x(k)\ \forall k \geq 0$.

Proof: The state trajectories of (5) and (11) coincide in the interval $\mathbb{Z}_{[-d_m,d_m]}$ as follows:

- For $k \in \mathbb{Z}_{[-d_m,0]}$, $x_t(k)$ coincide with the initial conditions of (5).
- For $k \in \mathbb{Z}_{[1,d_m]}$, $x_t(k)$ is given by the dynamics of (5).

For $k = d_m + 1$, the state x_t is given by (11):

$$x_t(d_m+1) = (A+K)x_t(d_m) + (A_d-K)x_t(d_m-d(d_m)) - \sum_{i=-d(d_m)}^{-1} K(Ax_t(d_m+i)+A_dx_t(d_m+i-d(d_m)-x_t(d_m+i)).$$

From (12), $Ax_t(d_m+i)+A_dx_t(d_m+i-d(d_m)) = x_t(d_m+i+1)$ for $i \in \mathbb{Z}_{[-d_m,-1]}$. Hence:

$$\begin{aligned} x_t(d_m+1) &= (A+K)x_t(d_m) \\ &\quad +(A_d-K)x_t(d_m-d(d_m)) \\ &\quad -K\textstyle\sum_{i=-d(d_m)}^{-1}(x_t(d_m+i+1) - x_t(d_m+i)) \\ &= (A+K)x_t(d_m) + (A_d-K)x_t(d_m-d(d_m)) \\ &\quad -K(x_t(d_m) - x_t(d_m-d(d_m)))) \\ &= Ax_t(d_m) + A_dx_t(d_m-d(d_m)) \\ &= Ax(d_m) + A_dx(d_m-d(d_m)) = x(d_m+1). \end{aligned}$$

The same development can be made for $k = d_m + 2$ and, naturally by induction, for $k = d_m + i$, $\forall i \geq 2$, proving that the state trajectories $x(k)$ and $x_t(k)$ coincide for $k \geq 0$. $\square$

The main consequence of this result is that any trajectory of the original model (5) is an admissible trajectory of the transformed model (11). This will allow us to derive invariance results over (5) from $\mathcal{D}$-invariance of polyhedral sets in the transformed model (11).

The $\mathcal{D}$-invariance definition and conditions can be applied to the transformed system (11), but with the initial conditions defined by $x(i)$, with $i \in \mathbb{Z}_{[-2d_m,0]}$. Necessary and sufficient conditions for positive $\mathcal{D}$-invariance of the polyhedron Ω w.r.t. the transformed system (11) have been established in [9] for the fixed delay case. Here, we extend these results to system (11) with time-varying delay.

Theorem 3: Ω is positively $\mathcal{D}$-invariant w.r.t. system (11) if, and only if, there exist nonnegative matrices $H, L, M, N \in \mathbb{R}^{f\times f}$ such that:

$$HF = F(A+K), \quad LF = F(A_d - K) \tag{13}$$

$$MF = -FK(A-I), \quad NF = -FKA_d \tag{14}$$

$$(H + L + d_m(M+N))w \leq w. \tag{15}$$

Proof: A possible realization of $d(k)$ is $d(k) = d_m \ \forall k \in \mathbb{Z}_+$, leading to the fixed delay case, for which the necessity of the conditions above has been proven in [9].

For sufficiency, assume that

$$Fx(k+i) \leq w, \ i \in \mathbb{Z}_{[-2d_m,0]}. \tag{16}$$

Then, from (11) we have that:

$$Fx(k+1) = F((A+K)x(k) + (A_d - K)x(k-d(k)) - \sum_{i=-d(k)}^{-1} K(A-I)x(k+i) - \sum_{i=-2d(k)}^{-d(k)-1} KA_d x(k+i))).$$

And from (13)-(16):

$$\begin{aligned} Fx(k+1) &= HFx(k) + LFx(k-d(k)) \\ &+ \sum_{i=-d(k)}^{-1} MFx(k+i) + \sum_{i=-2d(k)}^{-d(k)-1} NFx(k+i)) \\ &\leq Hw + Lw + d(k)Mw + d(k)Nw \\ &\leq Hw + Lw + d_m Mw + d_m Nw \leq w. \end{aligned}$$

We have proved that $x(k+1) \in \Omega$ if $x(k+i) \in \Omega$, $i \in \mathbb{Z}_{[-2d_m,0]}$ for arbitrary $k \in \mathbb{Z}_+$. Since, from Definition 3, this hypothesis is true for $k=0$, we conclude, by induction, that $x(k) \in \Omega \ \forall k \in \mathbb{Z}_+$. $\square$

Simplified expressions for $\mathcal{D}$-invariance of polyhedral sets which are symmetrical w.r.t. the origin can be derived following arguments similar to those in [9].

Given the polyhedron Ω (matrices F, w) and the maximal delay d_m, conditions (13)-(15) are linear on the matrix variables H, L, M, N and K. Then, positive $\mathcal{D}$-invariance of Ω can be checked by solving the following linear programming (LP) problem, where the contraction factor λ is optimized:

$$\begin{aligned} &\min_{\lambda, K, H, L, M, N} \lambda \\ &\text{s.t.: } \begin{cases} \text{(13)-(14)}, \quad H, L, M, N \geq 0 \\ (H + L + d_m(M+N))w - \lambda w \leq 0 \end{cases} \end{aligned} \tag{17}$$

If the optimal solution λ^* is such that $\lambda^* \leq 1$, then Ω is positively $\mathcal{D}$-invariant for the given value of d_m. These conditions can also be used to find a maximal value of d_m that preserves invariance.

V. Set Invariance in the Original Model

The following result relates $\mathcal{D}$-invariance of Ω w.r.t. the transformed system with the confinement of state trajectories of the original model (5) in Ω.

Theorem 4: If the set Ω is positively $\mathcal{D}$-invariant w.r.t. (11), then, the state trajectory of (5) is such that $x(k) \in \Omega$ $\forall k \geq 0$, provided that:

$$\begin{cases} x(i) \in \Omega, \ \forall i \in \mathbb{Z}_{[-d_m,0]}, \\ x(i) \in \Omega, \ \forall i \in \mathbb{Z}_{[1,d_m]}, \forall x(i) \text{ given by} \\ x(i+1) = Ax(i) + A_d x(i - d(i)), \ i \in \mathbb{Z}_{[0,d_m-1]}, \\ \forall \text{ possible realizations of } d(i) \in \mathbb{Z}_{[1,d_m]}, i \in \mathbb{Z}_{[0,d_m-1]}. \end{cases} \tag{18}$$

Proof: Consider systems (5) and (11) and let their state trajectories be denoted, respectively, by $x(k)$, $k \in \mathbb{Z}_+$ with initial conditions $x(i)$, $i \in \mathbb{Z}_{[-d_m,0]}$, and by $x_t(k)$, $k \in \mathbb{Z}_+$ with initial conditions $x_t(i)$, $i \in \mathbb{Z}_{[-d_m,d_m]}$. Since Ω is $\mathcal{D}$-invariant w.r.t. (11), if $x_t(i) \in \Omega$, $\forall i \in \mathbb{Z}_{[-d_m,d_m]}$, then, $x_t(k) \in \Omega \ \forall k \geq 0$. From Theorem 2, under the conditions (18), for any possible realization of $d(i), i \in \mathbb{Z}_{[0,d_m-1]}$ there exists a trajectory of (11) such that $x(k) = x_t(k)$, $\forall k \geq 0$. Due to the $\mathcal{D}$-invariance of Ω, we have that $x(k) = x_t(k) \in \Omega$, $\forall k \geq 0$. $\square$

A given polyhedral set Ω that is not $\mathcal{D}$-invariant w.r.t. the original model (5) can be so w.r.t. the transformed model (11). In this case, Theorem 4 states that the confinement of $x(k)$ in Ω w.r.t. the original model is achieved if the initial conditions result in a state trajectory belonging to Ω in the interval $\mathbb{Z}_{[1,d_m]}$, for all possible realizations of $d(k)$. This condition induces the following definition of admissible initial states w.r.t. state confinement in the set Ω.

Definition 4: Consider the system (5) and a polyhedral set Ω which is positively $\mathcal{D}$-invariant w.r.t. the transformed system (11). The *set of admissible initial states* of Ω w.r.t. (5) is defined as follows:
$\mathcal{I}(\Omega) = \{x(i) \in \Omega, \ i \in \mathbb{Z}_{[-d_m,0]} \text{ s.t. } x(i) \in \Omega, \ i \in \mathbb{Z}_{[1,d_m]}\}$.

$\mathcal{I}(\Omega)$ is the set of initial states $\big(x(i)$ for $i \in \mathbb{Z}_{[-d_m,0]}\big)$ which belong to Ω and make the next states $x(i)$ for $i \in \mathbb{Z}_{[1,d_m]}$ also belong to Ω, for all possible realization of $d(i)$.

For a polyhedral set $\Omega = \{x : \ Fx \leq w\}$, $\mathcal{I}(\Omega)$ represents a polyhedral set in the extended state space of (7). Indeed, given that the trajectories $x(k)$, $k \in \mathbb{Z}_+$ generated by (5) and (7) are identical, we can characterize $\mathcal{I}(\Omega)$ using the augmented model. To this end, let us define the cartesian product of Ω over the augmented state space as:

$$\Omega^{d_m} = \Omega \times \Omega \times \ldots \times \Omega \in \mathbb{R}^{(d_m+1)n}.$$

Notice that $X(0) \in \Omega^{d_m}$ is equivalent to the constraints $Fx(i) \leq w$, $i \in \mathbb{Z}_{[-d_m,0]}$ in (18). Furthermore, the states $x(i)$, $i \in \mathbb{Z}_{[1,d_m]}$ in (18) are the n first elements of $X(i+1) = \mathcal{A}(d(i))X(i)$, $i \in \mathbb{Z}_{[0,d_m-1]}$ (7). Hence, the constraints $x(i) \in \Omega \ \forall i \in \mathbb{Z}_{[1,d_m]}$, for all possible realizations of $d(i)$, $i \in \mathbb{Z}_{[0,d_m-1]}$ correspond to $X(i) \in \Omega^{d_m} \ \forall i \in \mathbb{Z}_{[1,d_m]}$. Since $X(0) \in \Omega^{d_m}$, we have deduced that the admissible initial states are characterized by $X(i) \in \mathcal{P}_{d_m}(\Omega^{d_m})$, where $\mathcal{P}_{d_m}(\Omega^{d_m})$ is a polyhedral set defined as in (10). The set of admissible initial conditions $\mathcal{I}(\Omega)$ is then given by $\mathcal{P}_{d_m}(\Omega^{d_m})$, which is the set of augmented states $X(k)$ such that $X(k+i) \in \Omega$, $\forall i \in \mathbb{Z}_{[0,d_m]}$. Now, we will prove that this set is positively invariant w.r.t. the augmented switched system (7).

Corollary 1: If $\Omega = \{x \ : \ Fx \leq w\}$ is $\mathcal{D}$-invariant w.r.t. (11), then $\mathcal{P}_{d_m}(\Omega^{d_m})$ is positively invariant w.r.t. (7).

Proof: Assume that $X(k) \in \mathcal{P}_{d_m}(\Omega^{d_m})$. Then, the constraints (18) are satisfied. The trajectories of $x(k)$ in (5) and (7) coincide, because both models are equivalent. From Theorem 4, if Ω is $\mathcal{D}$-invariant w.r.t. (11), then, for the original model (5), $x(k+d_m+1) \in \Omega$ if the initial conditions of (5) at time k belong to $\mathcal{I}(\Omega)$, i.e., if $X(k) \in \mathcal{P}_{d_m}(\Omega^{d_m})$. Then, we have that $x(k+i) \in \Omega$ for $i \in \mathbb{Z}_{[-d_m+1,1]}$ are such that $x(k+i) \in \Omega$ for $i \in \mathbb{Z}_{[2,d_m+1]}$. This proves that $X(k+1) \in \mathcal{I}(\Omega) = \mathcal{P}_{d_m}(\Omega^{d_m})$ and, hence, the positive invariance of the set of admissible initial conditions $\mathcal{P}_{d_m}(\Omega^{d_m})$. □

The next result shows that, in addition, $\mathcal{P}_{d_m}(\Omega^{d_m})$ is the maximal positively invariant set contained in Ω^{d_m}

Corollary 2: If $\Omega = \{x \ : \ Fx \leq w\}$ is positively $\mathcal{D}$-invariant w.r.t. (11), then $\mathcal{P}_{d_m}(\Omega^{d_m})$ is the maximal positively invariant set contained in Ω^{d_m}.

Proof: From the definition of $\mathcal{P}_{d_m}(\Omega^{d_m})$ we have that $\mathcal{P}_\infty(\Omega^{d_m}) \subseteq \mathcal{P}_{d_m}(\Omega^{d_m})$. Since $\mathcal{P}_{d_m}(\Omega^{d_m})$ is positively invariant and $\mathcal{P}_\infty(\Omega^{d_m})$ is the maximal invariant set, then, it is clear that $\mathcal{P}_\infty(\Omega^{d_m}) = \mathcal{P}_{d_m}(\Omega^{d_m})$. □

We can summarize the preceding results as follows. $\mathcal{D}$-invariance of Ω w.r.t. the transformed model implies the confinement in Ω of the state trajectories of the original time-varying delay system, provided that the initial conditions belong to a set, that turns out to be positively invariant w.r.t. an equivalent switching augmented model. Furthermore, this set is the maximal positively invariant set contained in the extension of Ω to the augmented model. Differently from the general case, the number of steps to compute this maximal set from (10) is known in advance and equal to d_m.

The reader will notice that the transformed model (11) is a tool that enables a delay-dependent analysis of invariance w.r.t. the original model (5), whereas the augmented model (7) is a tool to characterize the set of admissible initial states that ensures state confinement in the invariant set.

From the computational point of view, the previous construction starts with a $\mathcal{D}$-invariant set defined over the n-dimensional state space of the transformed model (11), less complex than directly working with an augmented model. The set $\mathcal{P}_{d_m}(\Omega^{d_m})$ is constructed over the augmented state space, but with only d_m steps of the general algorithm (10). Moreover, due to the particular structure of matrices $\mathcal{A}(i)$ (8), many inequalities from a given step are repeated in the next one, and can be easily removed. Also, numerical methods that progressively eliminate redundant inequalities along the recursion, as in [19], can be used to alleviate the computational burden.

We close this section by pointing out that these results developed for the varying delay case can be easily particularized to the fixed delay case, treated in [9], [12], where the augmented model is a non-switching linear time-invariant model. The characterization of the set of admissible initial conditions as a maximal positively invariant set is a novelty proposed in the present paper and is beneficial for the computations in the constant delay case.

VI. Illustrative Examples

A. Example: first-order model

Consider the following system, analysed in [9] in the fixed delay case: $x(k+1) = 0.8x(k) - 0.4x(k-d)$, for which no $\mathcal{D}$-invariant polyhedron containing the origin in its interior exists. For $d > 4$ this system is unstable, and this is one of the reasons why delay-independent invariance cannot be achieved.

The polyhedral set Ω is given by $\Omega = \{x \ : \ |x| \leq 1\}$. The solution of the LP problem (17) (adapted to the symmetrical case), after a trial-and-error adjustment of the value of d_m, gives $K^* = -0.4, \quad d_m = 2$, implying that Ω is $\mathcal{D}$-invariant w.r.t. the time-varying transformed model (11) with $d(k) \in \{1, 2\}$. Hence, the state trajectories of the original varying-delay model (5) will remain in Ω if its initial conditions belong to the set of admissible initial conditions $\mathcal{P}_2(\Omega^2)$.

The set $\mathcal{P}_2(\Omega^2)$ has been computed as described in section V. In Figure 1, a trajectory is depicted starting from the initial sequence $\{-0.375, -1, 0.75\}$ of $x(k)$. This sequence belongs to the intersection of the sets of admissible initial states associated to each mode composing the switching system (7) (which amounts to consider fixed delays $d = 1$ and $d = 2$). However, the initial sequence does not belong to $\mathcal{P}_2(\Omega^2)$. The three first values of the realization of $d(k) \in \mathbb{Z}_{[1,2]}$ are $\{2, 1, 2\}$, and the remaining values are random. With this realization $x(k)$ leaves Ω, as depicted in Figure 1. The polyhedron $\mathcal{P}_2(\Omega^2) \in \mathbb{R}^3$ is defined by 12 inequalities.

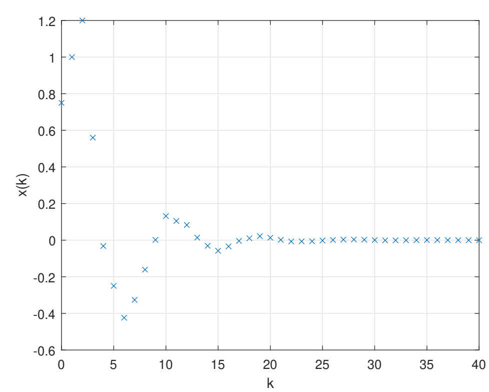

Fig. 1. Example 1: a state trajectory starting from initial conditions not belonging to $\mathcal{P}_\infty$.

B. Example: second-order model

Consider the system (5) with matrices A and A_d borrowed from [17]: $A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}$.

We obtained a symmetrical polyhedral set $\{x \ : \ |\bar{F}x| \leq \bar{w}\}$ with $\bar{F} = \begin{bmatrix} 0 & 1 \\ 2.2443 & 0.1189 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is $\mathcal{D}$-invariant w.r.t. the transformed model (11) with $d_m = 5$, using the invariance conditions (13)-(15) as constraints of a bilinear programming problem, as proposed in [20].

The solution of the LP problem (17) (adapted to the symmetrical case) leads to $K^* = \begin{bmatrix} -0.0948 & -0.0023 \\ 0 & -0.0947 \end{bmatrix}$.

The computation of the set of admissible initial states $\mathcal{P}_5(\Omega^5)$ resulted in a polyhedron defined over $\mathbb{R}^{12}$ by 86 inequalities. We simulated the state trajectories of the system with initial conditions given by 2 different vertices of $\mathcal{P}_5(\Omega^5)$. For each of them, we simulated the trajectories for 20 randomly generated realizations of the varying delay $d(k) \in \mathbb{Z}_{[1,5]}$. These trajectories are depicted in Figure 2. It can be seen that, as expected, none of them left the invariant polyhedron, even though some of them reached its boundary.

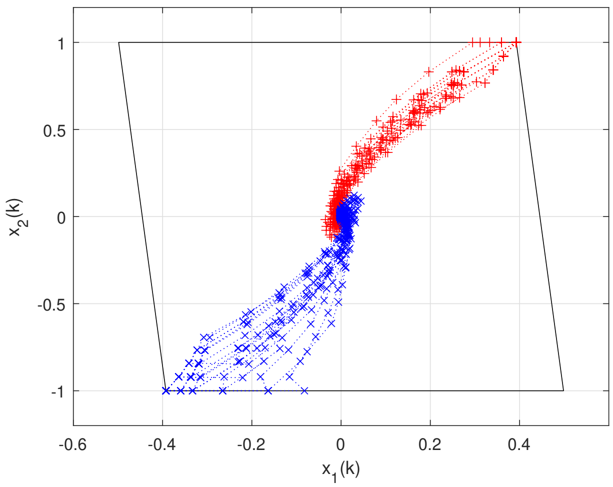

Fig. 2. Example 2: state trajectories starting from 2 different initial conditions, resulting from 20 different realizations of the varying delay.

VII. Conclusions

In this paper we analysed set invariance of polyhedral sets for linear discrete-time systems with time-varying delay on the states. We derived conditions that allow to check if invariance can be achieved for a given maximum size of the varying delay, and this is an advantage when compared to approaches that use classical delay-independent conditions. To the best of our knowledge, this is one of the first works that treat polyhedral invariance for varying-delay systems under this perspective. The set invariance conditions were established w.r.t. the original n-dimensional state space, and that is an advantage over approaches that use an augmented state space. We showed that confinement of the state in the considered set is achieved if the initial conditions belong to an admissible set, which is a maximal positively invariant set of a switched model and can be computed exactly in a number of steps equal to the size of the maximal delay. This result paves the way to the extension of this study to systems for which this maximal positively invariant set exists and can be computed, for instance, systems with polytopic uncertainties and bounded additive disturbance. The present paper, through the novel connections made between invariant sets for time-delay systems and switched systems, offers new perspectives on a relationship that raised attention in other recent studies [21]. Another possible extension regards the design of controllers for systems with delayed inputs and subject to state and control constraints, one of the major applications of invariant sets. The case of varying delay with bounded variation is also of interest.

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