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Designing Cluster Consensus on Higher-order Interaction Networks

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***Abstract*—This paper examines the cluster consensus design problem on higher-order interaction networks. Specifically, the higher-order interaction mechanism is captured by matrix-weighted networks that allow the interdependency across the dimensions of the agents' states, and the matrix-valued weight matrices $A_{ij} \in \mathbb{R}^{d\times d}$ associated with specific edges are further assumed to share the same nullspace for design purposes. Under mild assumptions on network connectivity, we first examine the case that the nullspace of positive semi-definite edges is spanned by a nonzero vector $\boldsymbol{\xi} \in \mathbb{R}^d$ and show that the predictable cluster consensus can be achieved, which is eventually located in the $1-$dimensional linear space determined by $\operatorname{span}\{\boldsymbol{\xi}\}$ and the average of agents' initial states. Moreover, the transient state of agents in each cluster can also be explicitly characterized. Namely, the derivative of the average state of agents in each cluster is perpendicular to $\operatorname{span}\{\boldsymbol{\xi}\}$. To generalize the above results, we proceed to examine the case that the nullspace of positive semi-definite edges is spanned by more than one linearly independent $d-$dimensional vector, in which case, analogous results can be obtained, and the explicit geometric interpretation is also provided.**

I. INTRODUCTION

The fundamental principle of how collective behavior emerges from local interactions plays a vital role in both the analysis and design of multi-agent systems [1], [2], [3]. In literature, the scalar-weighted consensus-type protocol is a critical routine of distributed algorithms for control, optimization, and learning over networks [4], [5], [6], [7], [8]. However, the traditional scalar-weighted interaction protocols fail to characterize high-order interactions amongst agents, such as the interdependency across the dimensions of the agents' states, which is ubiquitously observed in both natural and artificial multi-agent systems [9], [10], [11]. For instance, in opinion dynamics analysis of social networks, the logic interdependency amongst a set of topics can be naturally captured by the higher-order interaction networks via employing a matrix-weighted inter-agent coupling mechanism [10].

In the linear case, matrix-weighted networks turn out to be a reasonable implementation for higher-order interaction networks [12], [13], [14], [15], [16], [17], [18], [19]. Recently, multi-agent systems on matrix-weighted networks have been examined [13], [14]. For instance, it has been pointed out that the network connectivity cannot guarantee the consensus in the matrix-weighted networks due to the existence of edges that are weighted by positive/negative semi-definite matrices, a notable difference from scalar-weighted networks [13], [16], [17]. Recently, a variety of works on matrix-weighted networks have emerged in the community, such as consensus on time-varying matrix-weighted networks [18], [19], controllability of matrix-weighted networks [20], matrix-weighted consensus subject to physical constraints [21], [22] and so forth.

Notably, the cluster consensus turns out to be a ubiquitous collective behavior for matrix-weighted networks, which can naturally capture the behavioral diversity of swarms such as schools of fish or flocks of birds [3]. Here, the algebraic structure of the nullspace of matrix-valued edge weights plays a central role [19], [23], [3]. In this paper, we examine the cluster consensus design problem on homogeneous higher-order interaction networks. By homogeneity, we mean that a specific subset of matrix-valued edge weights are homogeneous in the sense that they all share the same nullspace. The motivation for this assumption concerns the freedom of cluster consensus design.

A notable distinction of the matrix-weighted networks is that the nullspace of matrix-valued Laplacian (which determines the collective behaviors of matrix-weighted networks) is not only influenced by network topology but also by the nullspace of matrix-valued edge weights. Inspired by this fact, this paper proposes a novel interaction protocol design paradigm for cluster consensus by manipulating the nullspace corresponding to the matrix-valued weight associated with specific edges in the network. We first examine the case that the nullspace of positive semi-definite matrix-valued edges is spanned by a nonzero vector $\xi \in \mathbb{R}^d$ and show that the predictable cluster consensus can be achieved on a "line-shape" formation determined by $\operatorname{span}\{\boldsymbol{\xi}\}$ and the average of initial states of all agents. Moreover, the derivative of the average state of agents in each cluster is perpendicular to $\operatorname{span}\{\boldsymbol{\xi}\}$. Then, we proceed to examine the case that the nullspace of positive semi-definite matrix-valued edges is spanned by two linearly independent vectors, and analogous results are obtained and discussed.

The remainder of this paper is organized as follows. The preliminaries and interaction protocol are introduced in §2 and §3, respectively. We provide the main results on the

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collective behaviors of homogeneous higher-order interaction networks in terms of the dimension of nullspace of positive semi-definite matrix-valued weight in §4 and §5, respectively. The concluding remarks are finally given in §6.

II. Preliminaries

A. Notations

Let $\mathbb{R}$ and $\mathbb{N}$ be real and natural numbers, respectively. Denote $\underline{n} = \{1, 2, \ldots, n\}$ for an $n \in \mathbb{N}$. We use $M > 0$ (respectively, $M \geq 0$) to denote that a symmetric matrix M is positive definite (respectively, positive semi-definite). The nullspace of a matrix $M \in \mathbb{R}^{n\times n}$ is $\textbf{null}(M) = \{\boldsymbol{z} \in \mathbb{R}^n \mid M\boldsymbol{z} = 0\}$. For a vector $\boldsymbol{x} \in \mathbb{R}^d$, we use $[\boldsymbol{x}]_i$ to denote the component on i-th dimension in $\boldsymbol{x}$. Let $W^{\perp}$ denote the orthogonal complement of a subspace in vector space V, namely, $W^{\perp} = \{v \in V : \langle u, v\rangle = 0, \forall u \in W\}$ where $\langle \cdot, \cdot\rangle$ denotes the inner product.

B. Higher-order Interaction Networks

Consider a higher-order interaction network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ where the node set and the edge set of $\mathcal{G}$ are denoted by $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, respectively. The matrix-valued weight for each edge $(i, j) \in \mathcal{E}$ in $\mathcal{G}$ is a symmetric matrix $A_{ij} \in \mathbb{R}^{d\times d}$ such that $A_{ij} \geq 0$ or $A_{ij} > 0$ and $A_{ij} = 0$ otherwise. An edge $(i, j) \in \mathcal{E}$ is positive definite or positive semi-definite if the associated matrix-valued weight A_{ij} is positive definite or positive semi-definite. Thereby, the matrix-weighted adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{dn\times dn}$ is a block matrix such that the block located in the i-th row and the j-th column is A_{ij}. Note that a matrix-weighted network degrades into a scalar-weighted network if $A_{ij} = I_d$ for all $(i, j) \in \mathcal{E}$. We shall assume that $A_{ij} = A_{ji}$ for all $i \neq j \in \mathcal{V}$ and $A_{ii} = 0$ for all $i \in \mathcal{V}$. The neighbor set of an agent $i \in \mathcal{V}$ is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. Denote $C = \textbf{diag}\{C_1, C_1, \cdots, C_n\} \in \mathbb{R}^{dn}$ as the matrix-weighted degree matrix of a graph where $C_i = \sum_{j\in\mathcal{N}_i} A_{ij} \in \mathbb{R}^{d\times d}$. The matrix-weighted Laplacian matrix of a matrix-weighted network is defined as $L = C - A$. For any orientation of an edge $(i, j) \in \mathcal{E}$ in $\mathcal{G}$, the nodes j and i are referred to as head and tail, respectively. The edge set of $\mathcal{G}$ can also be referred to as $\mathcal{E} = \{e_1, e_2, \ldots, e_{|\mathcal{E}|}\}$ where e_i represents the i-th edge in $\mathcal{E}$ for an arbitrary order. The incidence matrix of $\mathcal{G}$ is denoted by $H = [h_{ij}] \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{V}|}$ such that $h_{ij} = 1$ if node j is the head of the i-th edge, $h_{ij} = -1$ if node j is the tail of the i-th edge, and $h_{ij} = 0$ otherwise. A positive path in a graph $\mathcal{G}$ is a path such that every edge in this path is a positive definite. A tree in a matrix-weighted network is a positive tree if every edge contained in this tree is a positive definite edge. A positive spanning tree of a matrix-weighted network $\mathcal{G}$ is a positive tree containing all nodes in $\mathcal{G}$. The induced subgraph $\mathcal{G}(\mathcal{S})$ of graph $\mathcal{G}$ is the graph whose node set is $\mathcal{S} \subset \mathcal{V}$ and whose edge set consists of all of the edges incident to nodes in $\mathcal{S}$.

III. Higher-order Interaction Protocol

Consider a higher-order interaction network consisting of $n \in \mathbb{N}$ agents, where the state of agent $i \in \underline{n}$ is denoted by $\boldsymbol{x}_i(t) = [x_{i1}, x_{i2}, \ldots, x_{id}]^T \in \mathbb{R}^d$ ($d \in \mathbb{N}$). The interaction protocol of agent $i \in \mathcal{V}$ admits

$$\dot{\boldsymbol{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} A_{ij}(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)), \tag{1}$$

and the overall dynamics of the network can be dictated by the *matrix-weighted Laplacian* as follows

$$\dot{\boldsymbol{x}}(t) = -L\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{x}(t) = [\boldsymbol{x}_1^T(t), \boldsymbol{x}_2^T(t), \ldots, \boldsymbol{x}_n^T(t)]^T \in \mathbb{R}^{dn}$. We shall also refer to (2) as the *matrix-weighted network* in the upcoming discussions. We note that the matrix-weighted Laplacian L can be viewed as a block matrix derived from matrix-valued edge weights A_{ij}. For matrix-weighted networks, the cluster consensus on agents' states can be ubiquitously achieved even if the underlying network is connected [13], [19], [4]. This is due to the nullspace expansion of matrix-weighed Laplacian L, as dictated by the following lemma.

Lemma 1. *[16], [17], [13] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a higher-order interaction network in the form of* (2). *Then the Laplacian matrix L of $\mathcal{G}$ is positive semi-definite and its nullspace is*

$$\textbf{null}(L) = \textbf{span}\{\mathcal{R}, \mathcal{H}\}$$

where

$$\mathcal{R} = \textbf{range}\{\mathbf{1} \otimes I_d\} \tag{3}$$

and

$$\begin{aligned}\mathcal{H} = \{&\boldsymbol{v} = [\boldsymbol{v}_1^T, \boldsymbol{v}_2^T, \cdots, \boldsymbol{v}_n^T]^T \in \mathbb{R}^{dn} \mid \\ &(\boldsymbol{v}_i - A_{ij}\boldsymbol{v}_j) \in \textbf{null}(A_{ij}), (i, j) \in \mathcal{E}\}.\end{aligned} \tag{4}$$

Definition 1 (Node partition)**.** A node partition of a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ consists of $s \in \mathbb{N}$ subsets of nodes $\mathcal{C}_l \subset \mathcal{V}$ such that $\mathcal{V} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_s$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_s = \emptyset$, where $l \in \underline{s}$.

Definition 2 (Cluster consensus)**.** The higher-order interaction network (2) admits a cluster consensus solution if there exists a partition $\{\mathcal{C}_l\}_{l=1}^s$ of node set $\mathcal{V}$ such that $\lim_{t\to\infty}\boldsymbol{x}_i(t) = \lim_{t\to\infty}\boldsymbol{x}_j(t)$, $\forall i, j \in \mathcal{C}_l$, for all $l \in \underline{s}$ and $s \in \mathbb{N}$.

We shall first make the following assumption in this paper.

Assumption 1. There exists a node partition $\{\mathcal{C}_l\}_{l=1}^s$ of $\mathcal{G}$ such that each induced subgraph $\mathcal{G}(\mathcal{C}_l)$ has a positive spanning tree for all $l \in \underline{s}$. Moreover, for any $(i, j) \in \mathcal{E}$ such that i and j are in different node partitions, we have $A_{ij} \geq 0$.

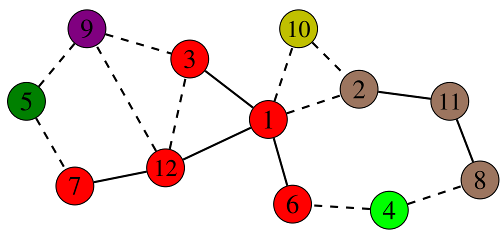


Fig. 1. An illustrative example for demonstrating Assumption 1. Solid lines highlight positive definite edges, while dashed lines represent positive semi-definite edges. Agents belonging to the same cluster are color-coded for clarity.

Example 1. We provide an example to illustrate Assumption 1. Consider a matrix-weighted network satisfying Assumption 1 (as shown in Figure 1). One can see that the associated node partition is

$$\begin{aligned}\{\mathcal{C}_l\}_{l=1}^s \quad = \quad & \{\{1,3,6,7,12\},\{2,8,11\},\\ & \{4\},\{5\},\{9\},\{10\}\}.\end{aligned}$$

Note that positive semi-definite edges are allowed within a node cluster, e.g., $A_{3,12} \geq 0$.

We introduce a preliminary lemma from [24] to show that the matrix-weighted network (2) admits a cluster consensus solution if Assumption 1 is satisfied.

Lemma 2. *[24] If there exists a positive tree $\mathcal{T}$ in a higher-order interaction network $\mathcal{G}$. Then, the higher-order interaction network* (2) *admits*

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t)$$

for all $i, j \in \mathcal{T}$.

According to Lamma 2, if there exists a positive tree within each cluster in a higher-order interaction network (Assumption 1), then the cluster consensus solution is feasible.

IV. Cluster Consensus on Higher-order Interaction Network: 1-dimensional Case

In this section, we first discuss the case where the dimension of the nullspace of semi-definite matrix-valued weight is one, which is dictated in the following assumption.

Assumption 2. There exists a nonzero vector $\boldsymbol{\xi} \in \mathbb{R}^d$ $(d \geq 2)$ such that $\mathbf{null}(A_{ij}) = \mathbf{span}\{\boldsymbol{\xi}\}$ for all $A_{ij} \geq 0$ where $(i,j) \in \mathcal{E}$.

We shall use $\boldsymbol{x}_{\mathcal{C}_l}(\infty)$ as the abbreviation for the consensus state of agents in cluster $\mathcal{C}_l$, namely, $\boldsymbol{x}_{\mathcal{C}_l}(\infty) = \lim_{t\to\infty}|\boldsymbol{x}_i(t)|$ for all $i \in \mathcal{C}_l$ where $l \in \underline{s}$. Moreover, let $\bar{\boldsymbol{x}}_{\mathcal{C}_l}(t) = \frac{1}{|\mathcal{C}_l|}\sum_{i\in\mathcal{C}_l}\boldsymbol{x}_i(t)$ and $\bar{\boldsymbol{x}}(t) = \dfrac{1}{n}\sum_{i\in\mathcal{V}}\boldsymbol{x}_i(t)$ denote the average state of agents in cluster $\mathcal{C}_l$ and all nodes at time $t \geq 0$, respectively, where $l \in \underline{s}$.

Theorem 1. *Let $\mathcal{G}$ be a higher-order interaction network satisfying Assumptions 1 and 2. Then the consensus state of agents in cluster $\mathcal{C}_l$ $(l \in \underline{s})$ satisfies*

$$\boldsymbol{x}_{\mathcal{C}_l}(\infty) - \bar{\boldsymbol{x}}(0) \in \mathbf{span}\{\boldsymbol{\xi}\}. \tag{5}$$

Moreover, the derivative of the average state of agents in cluster $\mathcal{C}_l$ $(l \in \underline{s})$ is perpendicular to $\mathbf{span}\{\boldsymbol{\xi}\}$, *namely,*

$$\dot{\bar{\boldsymbol{x}}}_{\mathcal{C}_l}(t) \in \mathbf{span}\{\boldsymbol{\xi}\}^{\perp}, \text{ for all } t \geq 0. \tag{6}$$

Remark 1. To better understand this theorem, we shall first discuss the system under $d = 2$. In this case,

$$\mathbf{dim}\{\mathbf{span}\{\boldsymbol{\xi}\}\} = \mathbf{dim}\left\{\mathbf{span}\{\boldsymbol{\xi}\}^{\perp}\right\} = 1,$$

which implies that $\boldsymbol{x}_{\mathcal{C}_l}(\infty) - \bar{\boldsymbol{x}}(0)$ is restricted to a line space spanned by $\boldsymbol{\xi}$. Similarly, the trajectory of $\bar{\boldsymbol{x}}_{\mathcal{C}_l}$ is restricted to move on the line that is orthogonal to $\boldsymbol{\xi}$. In fact, let $\boldsymbol{\xi}^* = [[\boldsymbol{\xi}]_2, -[\boldsymbol{\xi}]_1]^T$, the equation (5) can be expressed in the following form, which shows that $\boldsymbol{x}_{\mathcal{C}_l}(\infty)$ for each $\mathcal{C}_l \in \underline{s}$ lies on the same line in $\mathbb{R}^2$

$$a_1[\boldsymbol{x}_{\mathcal{C}_l}(\infty)]_1 + a_2[\boldsymbol{x}_{\mathcal{C}_l}(\infty)]_2 = b, \tag{7}$$

where $a_1 = [\boldsymbol{\xi}^*]_1, a_2 = [\boldsymbol{\xi}^*]_2, b = \langle \boldsymbol{\xi}^*, \bar{\boldsymbol{x}}(0)\rangle$. Moreover, by combining equations (6) and (7), we can solve the equations and obtain the explicit consensus state for each cluster as

$$\boldsymbol{x}_{\mathcal{C}_l}(\infty) = \begin{bmatrix} \dfrac{[\boldsymbol{\xi}]_1 \langle \boldsymbol{\xi}, \bar{\boldsymbol{x}}_{\mathcal{C}_l}(0)\rangle + [\boldsymbol{\xi}]_2 \langle \boldsymbol{\xi}^*, \bar{\boldsymbol{x}}(0)\rangle}{\|\boldsymbol{\xi}^*\|_2^2} \\ \dfrac{[\boldsymbol{\xi}]_2 \langle \boldsymbol{\xi}, \bar{\boldsymbol{x}}_{\mathcal{C}_l}(0)\rangle + [\boldsymbol{\xi}]_1 \langle \boldsymbol{\xi}^*, \bar{\boldsymbol{x}}(0)\rangle}{\|\boldsymbol{\xi}^*\|_2^2} \end{bmatrix}, l \in \underline{s}. \tag{8}$$

Example 2. We now provide an example to illustrate Theorem 1. Consider a higher-order interaction network shown in Figure 2(a) where $d = 2, n = 6$ and $\boldsymbol{\xi} = [1,1]^T$. Similar to Figure 1, in Figure 2(a), the positive definite edges are highlighted by solid lines, and positive semi-definite edges are highlighted by dashed lines, and we shall continue to use this representation in the upcoming examples. Different colors are used to distinguish nodes inside different clusters according to the node partition $\{\mathcal{C}_l\}_{l=1}^s$ of $\mathcal{G}$, where $\mathcal{C}_1 = \{1,6\}, \mathcal{C}_2 = \{2,3,5\}, \mathcal{C}_3 = \{4\}$. The positive definite matrix weights between connected clusters are matrix weights between connected clusters are

$$A_{16} = \begin{bmatrix} 30 & -30 \\ -30 & 60 \end{bmatrix}, A_{23} = \begin{bmatrix} 6 & -3 \\ -3 & 15 \end{bmatrix},$$

and

$$A_{25} = \begin{bmatrix} 19 & -1 \\ -1 & 31 \end{bmatrix}.$$

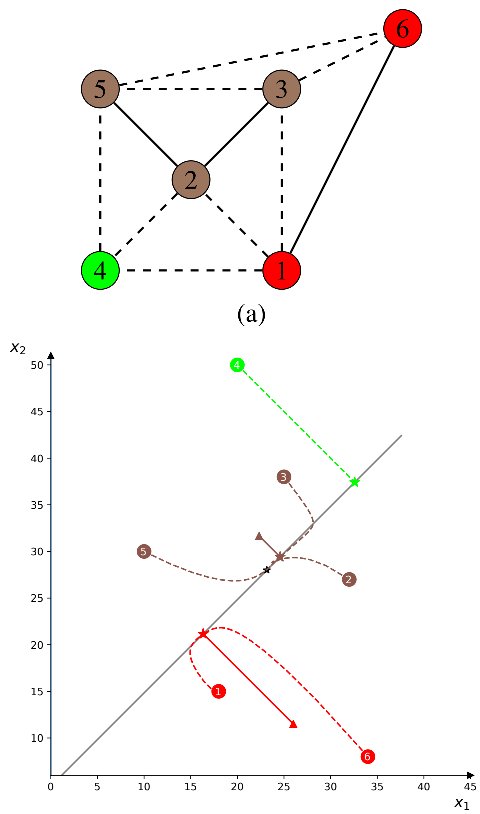
The positive semi-definite matrix weights are

$$A_{12} = \begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix}, A_{13} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix},$$

$$A_{14} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}, A_{24} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix},$$

$$A_{35} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}, A_{36} = \begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix},$$

$$A_{45} = \begin{bmatrix} 35 & -35 \\ -35 & 35 \end{bmatrix}, A_{56} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}.$$



(b)

Fig. 2. (a) The network structure in 2. (b) The corresponding trajectories of agents' states.

The initial states for nodes are

$$
\begin{aligned}
x_1 &= \begin{bmatrix}18 & 15\end{bmatrix}^T, \\
x_2 &= \begin{bmatrix}32 & 27\end{bmatrix}^T, \\
x_3 &= \begin{bmatrix}25 & 38\end{bmatrix}^T, \\
x_4 &= \begin{bmatrix}20 & 50\end{bmatrix}^T, \\
x_5 &= \begin{bmatrix}10 & 30\end{bmatrix}^T, \\
x_6 &= \begin{bmatrix}34 & 8\end{bmatrix}^T.
\end{aligned}
$$

Figure 1(b) illustrates the simulation trajectory result for the 6 nodes. The dashed lines represent the trajectory of nodes, whose equilibrium state is denoted by a small star mark with the same color. Solid lines that start from triangle marks denote the trajectories of $\bar{\boldsymbol{x}}_{\mathcal{C}_l}$. The black star mark denotes the average initial state of all nodes, i.e., $\bar{\boldsymbol{x}}(0)$. The initial and consensus states for all clusters are

$$
\begin{aligned}
\boldsymbol{x}_{\mathcal{C}_1}(0) &= \begin{bmatrix}26 & 11.5\end{bmatrix}^T, \\
\boldsymbol{x}_{\mathcal{C}_1}(\infty) &= \begin{bmatrix}16.33 & 21.17\end{bmatrix}^T, \\
\boldsymbol{x}_{\mathcal{C}_2}(0) &= \begin{bmatrix}22.33 & 31.67\end{bmatrix}^T, \\
\boldsymbol{x}_{\mathcal{C}_2}(\infty) &= \begin{bmatrix}24.58 & 29.42\end{bmatrix}^T, \\
\boldsymbol{x}_{\mathcal{C}_3}(0) &= \begin{bmatrix}20 & 50\end{bmatrix}^T, \\
\boldsymbol{x}_{\mathcal{C}_3}(\infty) &= \begin{bmatrix}32.58 & 37.42\end{bmatrix}^T.
\end{aligned}
$$

Noted that the initial state of the overall system, $\bar{\boldsymbol{x}}(0)$, and the equilibrium states of individual clusters, $\{\bar{\boldsymbol{x}}_{\mathcal{C}_l}(\infty)\}_{l=1}^{s}$, lie on the same line with the slope of $k = 1 = \frac{[\boldsymbol{\xi}]_2}{[\boldsymbol{\xi}]_1}$. Therefore, (5) holds for all clusters, and (6) is also confirmed as the average states of each cluster move orthogonally towards the line. Furthermore, (7) and (8) can be easily verified through straightforward computation.

Remark 2. In Example 2, we choose $d = 2$ to illustrate Theorem 1. However, with the dimension restriction of $d = 2$, there is only one degree of freedom to design the connection weight matrix between clusters based on Assumption 2. To gain further insights into Theorem 1, we provide another simulation result with $d = 3$.

Example 3. Consider a higher-order interaction network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ where $n = 6$, $d = 3$, and $\boldsymbol{\xi} = (1, 2, 3)^T$. The connection matrix weights for edges are arranged according to Assumption 2, while the initial states are chosen randomly. As shown in Figure 3(a), the node partition $\{\mathcal{C}_l\}_{l=1}^{s}$ of $\mathcal{G}$ is $\mathcal{C}_1 = \{1\}, \mathcal{C}_2 = \{2, 6\}, \mathcal{C}_3 = \{3, 4, 5\}$. The equilibrium states of clusters $\{\boldsymbol{x}_{\mathcal{C}_l}(\infty)\}_{l=1}^{s}$ and $\bar{\boldsymbol{x}}(0)$ fall on the same gray line, whose direction vector is $[1, 2, 3]^T$. Thus, it is evident that Equation (5) still holds for $d = 3$. According to 6, the mean derivation direction inside clusters $\dot{\bar{\boldsymbol{x}}}_{\mathcal{C}_l}$ must be orthogonal to $\boldsymbol{\xi}$, which implies that the trajectory of $\bar{\boldsymbol{x}}_{\mathcal{C}_l}$ moves on planes that are perpendicular to $\boldsymbol{\xi} = [1, 2, 3]^T$. We plot the planes with the corresponding color for each cluster respectively in Figure 3(b) to illustrate this point. Therefore, it is still easy to predict the equilibrium states for all nodes in higher dimensions, given the connection states and initial states.

V. Cluster Consensus on Higher-order Interaction Network: 2-dimensional Case

In this section, we proceed to discuss the case where the dimension of the nullspace of semi-definite matrix-valued weight is equal to 2, which is dictated in the following assumption.

Assumption 3. There exists two linearly independent nonzero vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$ $(d \geq 2)$ such that $\mathbf{null}(A_{ij}) = \mathbf{span}\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$ for all $A_{ij} \geq 0$ where $(i, j) \in \mathcal{E}$.

Then, we have an extension of Theorem (1) from the 1-dimensional to the 2-dimensional case, as stated in the following theorem.

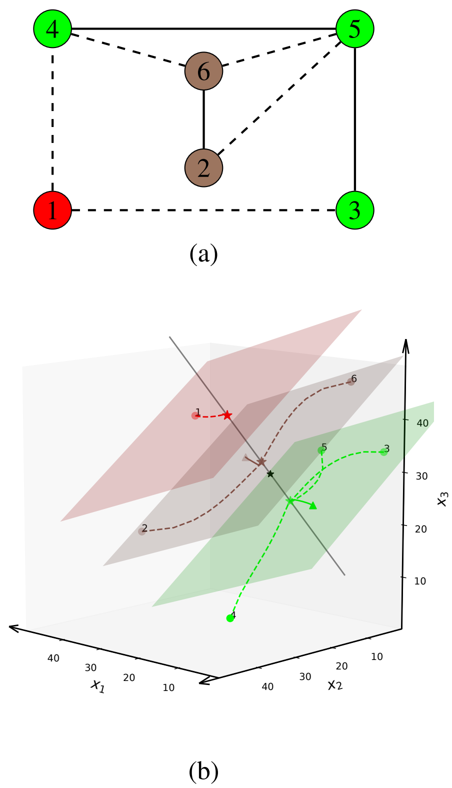


Fig. 3. (a) The network structure in Example 3. (b) The corresponding trajectories of agents' states.

Theorem 2. *Let $\mathcal{G}$ be a higher-order interaction network satisfying Assumptions 1 and 3. Then the consensus state of agents in cluster $\mathcal{C}_l$ ($l \in \underline{s}$) satisfies*

$$\boldsymbol{x}_{\mathcal{C}_l}(\infty) - \bar{\boldsymbol{x}}(0) \in \textbf{span}\left\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\right\}. \tag{9}$$

Moreover, the derivative of the average state of agents in cluster $\mathcal{C}_l$ ($l \in \underline{s}$) is perpendicular to $\textbf{span}\left\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\right\}$, *namely,*

$$\dot{\bar{\boldsymbol{x}}}_{\mathcal{C}_l}(t) \in \textbf{span}\left\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\right\}^{\perp}, \textit{ for all } t \geq 0. \tag{10}$$

Remark 3. To better understand the geometric interpretation of Theorem 2, we will examine the case where $d = 3$ in detail. The vector orthogonal to $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ is denoted by $\boldsymbol{\xi}^* = \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2$, namely, $\boldsymbol{\xi}^* \in \textbf{span}\left\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\right\}^{\perp}$. Note that

$$\textbf{dim}\left\{\textbf{span}\left\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\right\}\right\} = 2,$$

and

$$\textbf{dim}\left\{\textbf{span}\left\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\right\}^{\perp}\right\} = 1,$$

which implies that $\boldsymbol{x}_{\mathcal{C}_l}(\infty) - \bar{\boldsymbol{x}}(0)$ lies in a plane spanned by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Similarly, the trajectory of $\bar{\boldsymbol{x}}_{\mathcal{C}_l}(t)$ is constrained to move on the line with the same direction of $\boldsymbol{\xi}^*$. In fact, the equation (5) can be expressed in the following form, which shows that $\boldsymbol{x}_{\mathcal{C}_l}(\infty)$ ($l \in \underline{s}$) lies on the same plane in $\mathbb{R}^3$

$$a_1[\boldsymbol{x}_{\mathcal{C}_l}(\infty)]_1 + a_2[\boldsymbol{x}_{\mathcal{C}_l}(\infty)]_2 + a_3[\boldsymbol{x}_{\mathcal{C}_l}(\infty)]_3 = b, \tag{11}$$

where $a_1 = [\boldsymbol{\xi}^*]_1, a_2 = [\boldsymbol{\xi}^*]_2, a_3 = [\boldsymbol{\xi}^*]_3$ and $b = \langle \boldsymbol{\xi}^*, \bar{\boldsymbol{x}}(0)\rangle$. Moreover, combining equations (11) and (10), we can obtain the explicit expression of consensus state for each cluster as

$$\boldsymbol{x}_{\mathcal{C}_l}(\infty) = \begin{bmatrix} \dfrac{[\boldsymbol{\xi}^*]_1 \langle \boldsymbol{\xi}^*, \bar{\boldsymbol{x}}(0) - \boldsymbol{x}_{\mathcal{C}_l}(0)\rangle}{\|\boldsymbol{\xi}^*\|_2^2} + [\boldsymbol{x}_{\mathcal{C}_l}(0)]_1 \\ \dfrac{[\boldsymbol{\xi}^*]_2 \langle \boldsymbol{\xi}^*, \bar{\boldsymbol{x}}(0) - \boldsymbol{x}_{\mathcal{C}_l}(0)\rangle}{\|\boldsymbol{\xi}^*\|_2^2} + [\boldsymbol{x}_{\mathcal{C}_l}(0)]_2 \\ \dfrac{[\boldsymbol{\xi}^*]_3 \langle \boldsymbol{\xi}^*, \bar{\boldsymbol{x}}(0) - \boldsymbol{x}_{\mathcal{C}_l}(0)\rangle}{\|\boldsymbol{\xi}^*\|_2^2} + [\boldsymbol{x}_{\mathcal{C}_l}(0)]_3 \end{bmatrix}, l \in \underline{s}. \tag{12}$$

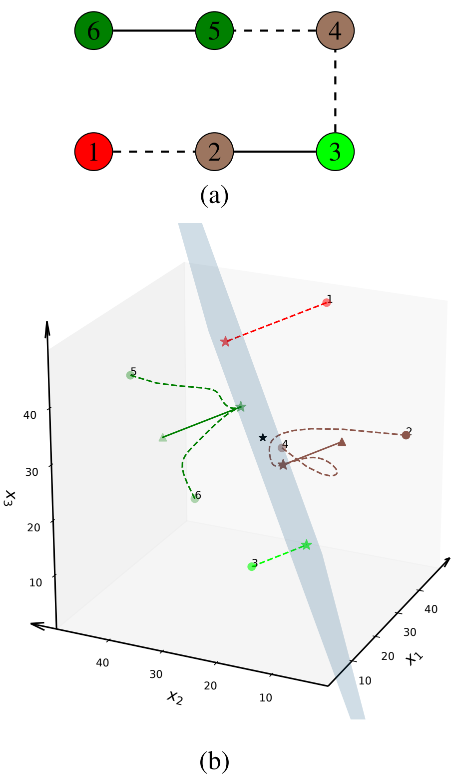


Fig. 4. (a) The network structure. (b) The trajectories of agents' states.

Example 4. We provide the following simulation example to illustrate Theorem 2. We choose $\boldsymbol{\xi}_1 = [1, 2, 3]^T$ and $\boldsymbol{\xi}_2 = [2, 2, 1]^T$. Consider a higher-order interaction network in Figure 4(a) whose node partition according to Assumption 1 is $\mathcal{C}_1 = \{1\}, \mathcal{C}_2 = \{2, 4\}, \mathcal{C}_3 = \{3\}$ and $\mathcal{C}_4 = \{5, 6\}$. The positive definite matrix-valued weights connecting different clusters are

$$A_{24} = \begin{bmatrix} 46 & 30 & 25 \\ 30 & 29 & 20 \\ 25 & 20 & 24 \end{bmatrix},$$

and

$$A_{56} = \begin{bmatrix} 29 & 18 & 14 \\ 18 & 31 & 13 \\ 14 & 13 & 14 \end{bmatrix}.$$

The positive semi-definite matrix-valued weights are

$$A_{14} = \begin{bmatrix} 16 & -20 & 8 \\ -20 & 25 & -10 \\ 8 & -10 & 4 \end{bmatrix},$$

and

$$A_{23} = A_{35} = \begin{bmatrix} 32 & -40 & 16 \\ -40 & 50 & -20 \\ 16 & -20 & 8 \end{bmatrix}.$$

The initial states for agents are

$$\begin{aligned} \boldsymbol{x}_1 &= \begin{bmatrix} 48.81 & 21.89 & 46.96 \end{bmatrix}^T, \\ \boldsymbol{x}_2 &= \begin{bmatrix} 10.27 & 6.59 & 37.09 \end{bmatrix}^T, \\ \boldsymbol{x}_3 &= \begin{bmatrix} 23.12 & 24.34 & 6.88 \end{bmatrix}^T, \\ \boldsymbol{x}_4 &= \begin{bmatrix} 36.88 & 24.91 & 23.66 \end{bmatrix}^T, \\ \boldsymbol{x}_5 &= \begin{bmatrix} 49.04 & 45.44 & 31.55 \end{bmatrix}^T, \\ \boldsymbol{x}_6 &= \begin{bmatrix} 49.95 & 33.37 & 9.54 \end{bmatrix}^T. \end{aligned}$$

The initial and consensus states for all clusters are

$$\begin{aligned} \boldsymbol{x}_{\mathcal{C}_1}(0) &= \begin{bmatrix} 48.81 & 21.89 & 46.96 \end{bmatrix}^T, \\ \boldsymbol{x}_{\mathcal{C}_1}(\infty) &= \begin{bmatrix} 38.78 & 34.43 & 41.94 \end{bmatrix}^T, \\ \boldsymbol{x}_{\mathcal{C}_2}(0) &= \begin{bmatrix} 23.57 & 15.75 & 30.38 \end{bmatrix}^T, \\ \boldsymbol{x}_{\mathcal{C}_2}(\infty) &= \begin{bmatrix} 22.73 & 16.80 & 29.96 \end{bmatrix}^T, \\ \boldsymbol{x}_{\mathcal{C}_3}(0) &= \begin{bmatrix} 23.12 & 24.34 & 6.88 \end{bmatrix}^T, \\ \boldsymbol{x}_{\mathcal{C}_3}(\infty) &= \begin{bmatrix} 30.43 & 15.20 & 10.54 \end{bmatrix}^T, \\ \boldsymbol{x}_{\mathcal{C}_4}(0) &= \begin{bmatrix} 49.49 & 39.41 & 20.55 \end{bmatrix}^T, \\ \boldsymbol{x}_{\mathcal{C}_4}(\infty) &= \begin{bmatrix} 51.70 & 36.65 & 21.65 \end{bmatrix}^T. \end{aligned}$$

Different from the Example 2, the consensus states of each cluster and $\bar{\boldsymbol{x}}(0)$ lie in the same plane instead of a single line, as shown in Figure 4(b). Therefore, equation (9) holds for all clusters. Note that the normal vector of the plane is $\boldsymbol{\xi}^*$. The average state for each cluster, $\bar{\boldsymbol{x}}_{\mathcal{C}_l}$, will move on a line that is perpendicular to this plane, thus confirming (10). Note that equations (11) and (12) can also be verified by a straightforward computation.

VI. Conclusion

This paper explored the cluster consensus design problem on higher-order interaction networks. We proposed a novel interaction protocol design paradigm that manipulates the collective behaviors of higher-order interaction networks by designing the nullspace of specific matrix-valued weights associated with edges in the network. Two cases are investigated, that is, the nullspace of semi-positive matrix-valued edges is spanned by one $d-$dimensional nonzero vector and by two linearly independent $d-$dimensional vectors, respectively. The main results can be immediately extended to the general case where the positive semi-definite edges are spanned by q linearly independent $d-$dimensional vectors where $d-1 \geq q > 2$.

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