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Algebraic prescribed-time KKL observer for continuous-time autonomous nonlinear systems

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***Abstract*— Designing observers for nonlinear systems is challenging, especially when prescribed convergence is required. Such a convergence is crucial for some applications, such as tactical missile guidance, communication networks, and robot assembly lines. The nonlinear prescribed time observers reported in the literature focus on specific classes of nonlinear systems with mainly linear outputs and rely on a scaling function or a time-varying gain that goes to infinity as the time approaches the prescribed convergence time, rendering the observer highly sensitive to measurement noise. This paper proposes an algebraic prescribed time observer for a general class of nonlinear systems that does not require any scaling function or exploding gain. The observer design relies on the KKL (Kazantzis-Kravaris/Luenberger) transformation that writes the system in a linear form in another set of coordinates. Then modulating functions, combined with an integral operator are applied over a window specified by the desired convergence time to provide a closed-form solution of the state estimate at the prescribed time. Moreover, we study the performance of the proposed algebraic prescribed time observer to guarantee a disturbance attenuation level in the presence of measurement noise. The effectiveness of the proposed observer is evaluated in numerical simulations and its performance is further assessed in the presence of measurement noise.**

I. INTRODUCTION

State estimation for dynamical systems is crucial for control design, monitoring, and fault detection. Algorithms called observers are designed to estimate the unmeasured states of a given system. However, observer design for nonlinear systems is challenging and most of the existing methods target specific classes of nonlinear systems or require strong assumptions. Only a few observers suitable for a general class of nonlinear systems exist in the literature, such as the Extended Kalman Filter [1] which relies on the linearization of the system providing, therefore, only local convergence guarantees. Furthermore, nonlinear observer design becomes significantly more challenging when additional performances are required, such as non-asymptotic convergence. Depending on the nature of the convergence we distinguish asymptotic and non-asymptotic observers.

Among the existing nonlinear asymptotic observers, Kazantzis-Kravaris-Luenberger (KKL) observer is increasingly gaining popularity, as it is one of the most powerful observers that offers a general estimation scheme for a wide class of nonlinear systems satisfying weak observability conditions [2]. The KKL observer resulted from the extension of the original method of the Luenberger observer [3] to nonlinear systems [4]. The observer relies on finding an injective nonlinear map of the states that transforms the nonlinear system into a stable linear system in another set of coordinates of possibly higher dimensions, where an observer already exists. Thanks to the injectivity of the transformation, the state estimate in the original coordinates is obtained by computing the left-inverse. The conditions of existence and injectivity of the transformation were first established locally around the equilibrium point in [4]–[6]. Then, the local nature of the development was dropped in [7] where global results were proposed under the strong observability condition of finite complexity. The conditions were further relaxed [8] by introducing the mild observability condition of *backward distinguishability*. Moreover, exponential convergence and tunability of the observer can be achieved under additional conditions [9].

The literature abounds of non-asymptotic or finite time observers, such as higher-order sliding mode observers [10]–[12], homogeneity-based observers [13]–[15], and algebraic estimators such as the so-called Modulating Function Method (MFM) [16]–[20]. The MFM is of particular interest due to its robustness features and its independence from any initial condition. Through an integral operator, the MFM transforms the estimation problem from differential equations into solving a set of algebraic equations. Therefore, instead of solving the direct problem, a closed-form solution of the variable of interest is derived without requiring any initial condition. The MFM was first introduced for parameter identification of linear systems in the late fifties in [21], and was then generalized to various classes of systems [22]–[26]. Only a decade ago, was the MFM proposed for state estimation. It was first extended for state estimation of linear ODEs [27] and nonlinear ODE where the nonlinearity depends solely on the output [16]. Then it was extended to linear and nonlinear PDEs [28], [29], and linear descriptor systems [30]. Recently, a MFM has been designed for triangular nonlinear systems subject to disturbance [18]. Additionally, a modulating function-based coordinate transformation that transforms the system into an observer canonical was proposed in [31]. Nevertheless, the MFM has not been extended yet to a general class of nonlinear systems.

In a similar line of research, finite time observers that converge in prescribed time independently of the initial condi-

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tion have recently attracted the attention of the control community. Prescribed time convergence is crucial in several applications, such as tactical missile guidance, communication systems, and robot assembly lines, where initial conditions are uncertain or unknown and accurate state estimation must be achieved within a predefined finite time [32]. Inspired by [33], where a prescribed time stabilization was achieved by using feedback with a time-varying gain that tends to infinity as the time approaches the prescribed convergence time, a new approach to prescribed time observers was proposed in [34] for linear time-invariant systems in an observer canonical form. The observer was then extended to nonlinear triangular systems in [35], where a prescribed time high gain observer was proposed. Furthermore, a prescribed-time sliding mode observer for triangular systems was proposed in [36]. Recently, a prescribed-time safety filter was proposed and implemented experimentally on a seven-degrees-of-freedom robot [37]. The prescribed-time observer was also extended to multi-agent systems [38] and linear time-invariant systems subject to input or output delay [39], [40]. The advantage of the aforementioned observers is their ease of implementation and the ability to define the convergence time independently from the initial condition. However, they rely on a time-varying grain that becomes unbounded as time approaches the prescribed convergence time which makes the observer highly sensitive to measurement noise, which alters the estimation accuracy. Moreover, to the best of our knowledge, nonlinear prescribed time observers mostly consider a specific class of nonlinear systems such as triangular nonlinear systems, and have not been extended yet to a general class of nonlinear systems.

Motivated by the above-mentioned methods, we propose in this paper an algebraic prescribed-time KKL observer for autonomous nonlinear systems that does not require any scaling function or exploding gain design. Instead, it relies on the modulating function method which provides a closed-form solution of the state at a user-prescribed time. Leveraging the power of the KKL observer with the properties of the modulating function method, the proposed prescribed time observer can be applied to a general class of nonlinear systems with possibly nonlinear outputs. Moreover, measurement noise is attenuated thanks to the integral operator involved in the MFM (modulation operator). Additionally, the proposed observer is easy to implement, and the prescribed time convergence is a natural consequence of the modulation operator applied on an integration window whose length is specified by the prescribed convergence time independently of any parameter or initial condition.

The present paper is organized as follows: Section II provides some background on the KKL observer and the MFM. Section III presents the proposed algebraic prescribed-time KKL observer for autonomous nonlinear systems. The disturbance attenuation analysis of the proposed observer in the presence of measurement noise is provided in section IV and numerical simulations are performed in section V. Finally, concluding remarks and future work directions are provided in section VI.

II. Preliminaries

This section gives background on the KKL observer design for autonomous nonlinear systems and the Modulating Functions Method.

A. KKL observer for autonomous nonlinear systems

Consider the following nonlinear autonomous system

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ y(t) = h(x(t)) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are smooth vector fields. The solution of (1) at time t, initialized with $x(0) = x_0$ is denoted by $X(t, x_0)$.

The KKL observer design relies on finding an injective map T that transforms the nonlinear system (1) into a linear form in another set of coordinates z, where an observer is designed. Finally, using the left-inverse T^{-1}, one obtains the state estimate $\hat{x}(t)$.

In what follows, we briefly provide the conditions of existence and injectivity of the transformation T by considering the following assumptions [8]:

Assumption 1: System (1) is forward invariant within $\mathcal{X}$ i.e. there exist a compact set $\mathcal{X} \subset \mathbb{R}^n$, such that for all initial conditions $x(0) \in \mathcal{X}_0 \subset \mathcal{X}$ and all $t > 0, X(t, x_0) \in \mathcal{X}$.

Assumption 2: System (1) is backward $\mathcal{O}$-distinguishable on $\mathcal{X}$ i.e, Given open set $\mathcal{O} \supset \mathcal{X}$, for any pair of initial condition $(x_a, x_b) \in \mathcal{X}_a \times \mathcal{X}_b$, if $x_a \neq x_b$, then there exists $\tau < 0$ such that $(X(t, x_a), X(t, x_b)) \in \mathcal{O}^2$ is well defined for $t \in [\tau; 0]$ and $h(X(\tau, x_a)) \neq h(X(\tau, x_b))$.

We recall the following theorem from [2] establishing the existence and injectivity of the KKL transformation T.

Theorem 1: Consider system (1) and let Assumptions 1 and 2 hold. Then, for almost any pair $(\tilde{A}, \tilde{B}) \in \left(\mathbb{R}^{(2n+1)\times(2n+1)}, \mathbb{R}^{(2n+1)\times 1}\right) \backslash \mathcal{J}$ such that $\tilde{A} + \delta I_{(2n+1)}$ is Hurwitz for some $\delta > 0$, and $\mathcal{J} \subset \mathbb{R}^{(2n+1)\times(2n+1)} \times \mathbb{R}^{(2n+1)\times 1}$ is a set of zero Lebesgue measure, there exists a uniformly injective map $T : \mathcal{X} \to \mathbb{R}^{(2n+1)p}$ satisfying

$$\frac{\partial T}{\partial x}(x) f(x) = AT(x) + Bh(x), \quad \forall x \in \mathcal{X} \tag{2}$$

with $A = I_p \otimes \tilde{A} \in \mathbb{R}^{(2n+1)p\times(2n+1)p}$ and $B = I_p \otimes \tilde{B} \in \mathbb{R}^{(2n+1)p\times p}$.

Based on the result of Theorem 1, system (1) is transformed into a linear system in the new coordinates $z = T(x) \in \mathbb{R}^{(2n+1)p}$ given by

$$\dot{z} = Az + By. \tag{3}$$

Where A and B are as in Theorem 1, where $\tilde{A}$ is a diagonalizable matrix and $(\tilde{A}, \tilde{B})$ controllable, which is satisfied for almost any a pair $(\tilde{A}, \tilde{B}) \in \left(\mathbb{R}^{(2n+1)\times(2n+1)}, \mathbb{R}^{(2n+1)\times 1}\right)$ [41]. Furthermore, we consider the following assumption

Assumption 3: The left-inverse T^{-1} of the map T satisfying (2) is locally Lipschitz

$$\left\|T^{-1}(z) - T^{-1}(\hat{z})\right\| \leqslant l \|z(k) - \hat{z}(k)\|,$$

where $l > 0$ is the Lipschitz constant.

Typically, the KKL observer in the new coordinates is a copy of (3), where the Hurwitz and controllability of the design matrices ensure the exponential convergence of the KKL observer in the z-coordinates. Additionally, if Assumption 3 holds, the KKL observer converges exponentially in the original coordinates [9]. Moreover, if the observer is initialized at time $t_0 \geqslant 0$ with $\hat{z}(t_0) = z(t_0)$ then the convergence of the KKL observer is exact for $t \geqslant t_0$.

Based on the above observation, the objective is to design an estimation scheme that can perfectly reconstruct the state z at a given time T_p chosen by the user independently of the initial condition. One way to do this is to combine the KKL observer with the so-called modulating functions approach which allows to obtain a closed-form solution of the state at a given specified time T_p without requiring any initial condition. As a result, an exact convergence of the observer will be obtained for $t \geqslant T_p$. More details about the Modulating Functions Method are given in the next subsection.

B. Modulating Function Method

The Modulating Functions Method is an algebraic method that transforms the model differential equation into an algebraic equation, which provides a closed-form solution of the estimate of the variable of interest. Therefore, no initial condition is needed. In what follows and based on [26], the definition of the modulating functions and their main property are established.

Definition 1: (Modulating function) A non-zero function $\phi(t) : [0, T] \to \mathbb{R}^n$, for a fixed $T > 0$, is said to be a k^{th} order modulating function, with $k \in \mathbb{N}^*$, if it satisfies the following

(P1): $\phi \in \mathcal{C}^k([0, T])$

(P2): $\phi^{(i)}(0) = \phi^{(i)}(T) = 0, \quad i = 0, 1, \ldots, k-1.$

Definition 2: (Modulation operator) Let $y : [0, T] \subset \mathbb{R}^+ \to \mathbb{R}$ be an integrable signal and $\phi(t) \in \mathcal{C}^k([0, T])$ a k^{th} order modulating function. The corresponding modulation operator is given by the following inner product over the interval $I = [0; T]$:

$$\langle \phi, y \rangle_I = \int_0^T \phi(t) y(t) \mathrm{d}t.$$

Property 1: Using integration by parts and the boundary conditions (P2), one can derive the following

$$\begin{aligned} \langle \phi, y^{(i)} \rangle_I &= \int_0^T \phi(t) y(t)^{(i)} \mathrm{d}t \\ &= (-1)^i \int_0^T \phi(t)^{(i)} y(t) \mathrm{d}t = (-1)^i \langle \phi^{(i)}, y \rangle_I. \end{aligned}$$

Property 1 emphasises one of the main advantages of the Modulating Functions Method, which consists of shifting the derivatives from the unknown and possibly noisy signal to the known and smooth modulating function. Moreover, the mitigation of measurement noise on the estimation is achieved thanks to the modulation operator.

III. Algebraic prescribed-time KKL observer

Assume that $z_i(t)$, $k = 1, ..., d$ is a continuous and bounded function of time. Using the Weierstrass approximation theorem [42], we decompose each state on $I = [0; T_p]$ in the space spanned by M known polynomial basis functions $\alpha_{i,j}(t)$, for M large enough

$$z_i(t) = \sum_{j=1}^{M} a_{i,j} \alpha_j(t), \quad \forall i = 1, ..., d. \tag{4}$$

Each state z_i is a linear combination of M known polynomial basis functions where the coefficients $a_{i,j}$ are estimated using the MFM. Considering a copy of system (3) initialized at $\hat{z}(T_p)$ obtained by the MFM, the KKL observer estimation will be exact for $t \geqslant T_p$, which leads to a prescribed time KKL observer.

Remark 1: The boundedness of the state z is ensured by assumption 1, the smoothness of the function $h(x)$, and the Hurwitz property of the matrix A, which imply that the trajectories $z(t, T(x_0)) \in \mathcal{Z} \subset \mathbb{R}^{(2n+1)p}$ are bounded, $\forall t > 0$ and $\forall x_0 \in \mathcal{X}_0 \subset \mathcal{X}$.

Without loss of generality, the following proposition provides the structure of the proposed algebraic prescribed-time KKL observer for system (1) with a diagonal matrix $A = \mathrm{diag}\left(\begin{bmatrix} \lambda_1, & ..., & \lambda_d \end{bmatrix} \right)$, with $d = (2n+1)p$. Moreover, we consider the following notation $B = \begin{bmatrix} b_1^T & .... & b_d^T \end{bmatrix}^T$.

Proposition 1: Consider system (1) and let Assumptions 1 and 2 hold. Denote by $\{\phi_i\}_{i=1}^{i=M}$ a set of modulating functions of order $l \geqslant 1$, T the injective KKL transformation, and T^{-1} its left-inverse satisfying Assumption 3. Then, the following observer is a prescribed-time observer for system (1)

$$\begin{cases} \dot{\hat{z}}(t) = A\hat{z}(t) + By \quad t > T_p \\ \hat{z}(t) = 0; \quad t < T_p \\ \hat{z}(T_p) = \hat{\Theta}\mathcal{M}; \\ \hat{x}(t) = T^{-1}(\hat{z}(t)) \end{cases} \tag{5}$$

where A is Hurwitz, (A, B) is controllable, and T_p is the prescribed convergence time. The coefficient matrix and the basis functions vector denoted by $\hat{\Theta} \in \mathbb{R}^{d \times M}$ and $\mathcal{M} \in \mathbb{R}^{M \times 1}$, respectively, are given by

$$\hat{\Theta} = \begin{bmatrix} - & \hat{\theta}_1^T & - \\ - & \hat{\theta}_2^T & - \\ & \vdots & \\ - & \hat{\theta}_d^T & - \end{bmatrix} \text{ and } \mathcal{M} = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_M(t) \end{bmatrix}. \tag{6}$$

The parameter vectors $\hat{\theta}_i \in \mathbb{R}^{M \times 1}$, $i = 1, ..., d$ are given by the closed-form solution

$$\hat{\theta}_i = -\Phi_i^{-1} \begin{bmatrix} \langle \phi_1(t), b_i y(t) \rangle \\ \vdots \\ \langle \phi_M(t), b_i y(t) \rangle \end{bmatrix}, \tag{7}$$

with Φ_i as in equation (‡).

Proof: Assuming that system (1) satisfies Assumptions

$$\Phi_i = \begin{bmatrix} \langle \dot{\phi}_1(t) + \lambda_i \phi_1(t), \alpha_1(t)\rangle_I \cdots \langle \dot{\phi}_1(t) + \lambda_i \phi_1(t), \alpha_M(t)\rangle_I \\ \vdots \\ \langle \dot{\phi}_M(t) + \lambda_i \phi_M(t), \alpha_1(t)\rangle_I \cdots \langle \dot{\phi}_M(t) + \lambda_i \phi_M(t), \alpha_M(t)\rangle_I \end{bmatrix} \quad (\ddagger)$$

1 and 2, then according to theorem 1, there exists an injective map T transforming the system into the stable linear system (3).

Step1: Non-asymptotic estimation of $z(T_p)$

Exploiting the diagonal structure of the matrix A, the equations of (3) can be decoupled in the z-coordinates, and the steps of the MFM are identical for z_i, $i = 1, ...d$.

Consider the following equation

$$\dot{z}_i(t) = \lambda_i z_i(t) + b_i y(t). \quad (8)$$

Multiplying both sides of (3) with a modulating function $\phi(t)$ of order $l \geq 1$, one obtains

$$\phi(t)\dot{z}_i(t) = \lambda_i \phi(t) z_i(t) + \phi(t) b_i y(t). \quad (9)$$

Applying the modulation operator to the above equation on $I = [0; T_p]$, with T_p representing the prescribed convergence time defined independently of the initial condition, and using Property 1, one can shift the derivative from the unknown state $\dot{z}_i$ to the known modulating function $\phi(t)$.

$$-\langle \dot{\phi}(t), z_i(t)\rangle_I = \langle \lambda_i \phi(t), z_i(t)\rangle_I + \langle \phi(t), b_i y(t)\rangle_I, \quad (10)$$

substituting $z_i(t)$ by (4) for M big enough, one obtains

$$\begin{aligned} \langle \dot{\phi}(t) + \lambda_i \phi(t), z_i(t)\rangle_I &= \sum_{j=1}^{M} a_{i,j} \langle \dot{\phi}(t) + \lambda_i \phi(t), \alpha_j(t)\rangle_I \\ &= -\langle \phi(t), b_i y(t)\rangle_I, \quad (11) \end{aligned}$$

which is equivalent in vector notations to

$$\left[\langle \dot{\phi}(t) + \lambda_i \phi(t), \alpha_1\rangle_I \ldots \langle \dot{\phi}(t) + \lambda_i \phi(t), \alpha_M\rangle_I\right] \begin{bmatrix} a_{i,1} \\ \vdots \\ a_{i,M} \end{bmatrix} = -\langle \phi(t), b_i y(t)\rangle_I. \quad (12)$$

The estimation of the coefficients $a_{i,j}, j = 1, \ldots, M$ is achieved by solving M linearly independent equations (12). Therefore, we use M different modulating functions ϕ_i of order $l \geqslant 1$, leading to the following:

$$\Phi_i \underbrace{\begin{bmatrix} \hat{a}_{i,1} \\ \vdots \\ \hat{a}_{i,M} \end{bmatrix}}_{\hat{\theta}_i} = - \begin{bmatrix} \langle \phi_1(t), b_i y(t)\rangle \\ \vdots \\ \langle \phi_M(t), b_i y(t)\rangle \end{bmatrix} \quad (13)$$

where $\Phi_i \in \mathbb{R}^{M \times M}$ is given by equation ($\ddagger$).

The parameter vector $\hat{\theta}_i$ is obtained by solving equation (13).

The estimated state in the z-coordinates at time T_p is then given by

$$\hat{z}(T_p) = \hat{\Theta}\mathcal{M},$$

with $\hat{\Theta}$ and $\mathcal{M}$ as in (6).

Step 2: Estimation error at $t = T_p$

Let $\eta(t)$ be the polynomial approximation error defined as

$$\eta(t) = z(t) - \hat{z}(t); \quad t \leqslant T_P$$

Given the boundedness and smoothness of $z(t)$ (see Remark 1), there exist a positive finite number η_{max} such that $||\eta(t)|| \leqslant \eta_{max}; \quad t \leqslant T_p$.

For an appropriate choice of a large enough number of polynomial basis functions M over the interval $[0; T_p]$, one obtains $\eta_{max} \to 0$. As a result, we obtain the following estimation error in the z-coordinates

$$||e_z(T_p)|| = ||z(T_p) - \hat{z}(T_p)|| = ||\eta(T_P)|| \leqslant \eta_{max} \to 0 \quad (14)$$

This leads to the prescribed time convergence of the state estimation in z-coordinates at $t = T_P$.

Considering Assumption 3, then the estimation error in the original coordinates satisfies for $t = T_p$

$$\begin{aligned} ||\hat{x}(t) - x(t)|| &= ||T^{-1}(\hat{z}) - T^{-1}(z)|| \quad (15) \\ &\leqslant l||\hat{z}(t) - z(t)|| \end{aligned}$$

which ensures

$$\lim_{t \to T_p} ||\hat{x}(t) - x(t)|| = 0.$$

Step 3: Estimation error for $t \geqslant T_p$

Given the Hurwitz nature of the matrix A, there exist two positive constant α and β such that the observer error for $t > T_p$ satisfies

$$||\hat{z}(t) - z(t)|| \leqslant \alpha e^{-\beta(t - Tp)} \underbrace{||\hat{z}(T_p) - z(T_p)||}_{=0} \quad (16)$$

which is equivalent to

$$||\hat{z}(t) - z(t)|| = 0, \quad t > T_p. \quad (17)$$

Furthermore and provided that Assumption 3 holds, then the estimation error in the original coordinates satisfies for $t \geqslant T_p$

$$\begin{aligned} ||\hat{x}(t) - x(t)|| &= ||T^{-1}(\hat{z}) - T^{-1}(z)|| \leq l||\hat{z}(t) - z(t)|| \quad (18) \\ &\leqslant l\alpha e^{-\beta(t - Tp)}||\hat{z}(T_p) - z(T_p)|| = 0 \end{aligned}$$

As a result, observer (5) converges in prescribed time in the original coordinates. This ends the proof. ∎

Remark 2: The number of basis functions M is a hyper-parameter in the proposed observer that has to be chosen big enough and is often obtained by trial and error. In the case where M is not chosen appropriately and the polynomial approximation error is not negligible, this latter will be mitigated by the Hurwitz property of A as suggested by (16) and (18), obtaining therefore in the worst case scenario a fast convergence.

IV. Disturbance attenuation analysis of the algebraic prescribed time KKL observer to measurement noise

Consider the following autonomous nonlinear dynamical system with a noisy output

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ y_e(t) = y(t) + v(t) = h(x(t)) + v(t) \end{cases} \tag{19}$$

where $v(t) \in \mathbb{R}^p$ is a bounded measurement noise satisfying the following assumption.

Assumption 4: The output measurement noise $v(t)$ possesses a finite bound on its:

(i): $\mathcal{L}_\infty$ $\forall t \geq T_p$: $||v(t)||_{\mathcal{L}_\infty} \leqslant \bar{v}_1$;

(ii): $\mathcal{L}_2$ $\forall t \in [0; T_p]$: $||v(t)||_{\mathcal{L}_2} \leqslant \bar{v}_2$.

In the presence of output measurement noise, the design of the KKL observer presented in section II remains the same for system (19) (see for instance [43]).

Proposition 2: Consider system (19) satisfying assumptions 1 and 2, and consider the observer (5) with the noisy output $y_e(t)$ and T^{-1} satisfying assumption 3. If $v(t)$ satisfies assumption 4, then there exists finite positive constants α, β, μ, and ρ such that the error in the original coordinates attenuates the measurement noise through the modulating function and satisfies

$$||\hat{x}(t) - x(t)|| \leqslant l\alpha\mu e^{-\beta(t-Tp)}\bar{v}_2 + \rho\bar{v}_1, \tag{20}$$

where ρ is the disturbance attenuation level.

Proof: Relying on assumption 3 and the results of Proposition 1, the proof of the error bound of $||\hat{z}(T_p) - z(T_p)||$ follows the same steps as [44], and the rest of the error bound in (20) is derived following the approach of [43]. ∎

V. Numerical Simulations

To evaluate the performance of the proposed observer, we consider the following system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \\ y = x_1^2 - x_2^2 + x_1 + x_2 \end{cases} \tag{21}$$

System (21) is forward complete within $\mathcal{X} = [-1; 1]^2$. Moreover, it has been shown in [45] that the system is weakly differentially observable.

The KKL transformation T is given in [45] for $d = (2n + 1) = 5$ by

$$T_i(x) = x^T \begin{bmatrix} a_i & c_i/2 \\ c_i/2 & b_i \end{bmatrix} x + \begin{bmatrix} d_i & e_i \end{bmatrix} x; \quad i = 1, .., d. \tag{22}$$

with

$$a_i = -\frac{\lambda_i}{4 + \lambda_i^2}, \quad b_i = -a_i, \quad c_i = -\frac{4}{4 + \lambda_i^2}$$
$$d_i = \frac{1 - \lambda_i}{1 + \lambda_i^2}, \quad e_i = -\frac{1 + \lambda_i}{1 + \lambda_i^2}$$

An approximation of the inverse is given in [45] by solving the following system

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ a_1 & c_1 & d_1 & e_1 \\ a_2 & c_2 & d_2 & e_2 \\ a_3 & c_3 & d_3 & e_5 \end{bmatrix} \begin{bmatrix} x_1^2 - x_2^2 \\ x_1 x_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix}. \tag{23}$$

We set the prescribed convergence time as $T_p = 0.8s$ and the design matrices as $A = \text{diag}\left(\begin{bmatrix}-4, & -5, & -6, & -7, & -8,\end{bmatrix}\right)$ and $B = \begin{bmatrix}1, & 1, & 1 & 1, & 1\end{bmatrix}^T$.

As for the MFM part of the observer, we consider polynomial basis functions $\alpha_j(t) = t^{j-1}$, for $j = 1, ..., M$, with $M = 7$, Moreover, polynomial normalized modulating functions are considered [25]

$$\begin{cases} \phi_j(t) = \frac{\varphi_j(t)}{\|\varphi_j(t)\|_{\mathcal{L}_2}} \\ \varphi_j(t) = (T_p - t)^{(q+j)} t^{(q+M+1-j)}; \quad j = 1, .., M. \end{cases}$$

where $q \in \mathbb{N}^*$ is a degree of freedom.

Fig. 1 shows the states in the z-coordinates and their estimated values using the proposed prescribed-time observer. One can see that the estimated state converges to the real one at exactly $T_p = 0.8$s. This observation is further confirmed through the error plot in Fig. 2, where the error in both coordinates converges to the origin at the prescribed convergence time T_p. Moreover, the errors in both coordinates are represented in Fig. 3 in the *log* scale to illustrate the nature of the convergence. One can see a clear prescribed-type convergence in both coordinates.

Further simulations were performed by considering several initial conditions and different prescribed convergence times. Fig. 4 illustrates the error in both coordinates for three different initial conditions, and one can see that all error trajectories converge to the origin at $T_p = 0.8$s independently from the initial condition considered. Moreover, for a given initial condition $x_0 = [0, 5; 1]^T$ Fig. 5 shows that the error trajectories in both coordinates converge to the origin at different chosen prescribed times. Which reaffirms, the independence of the prescribed convergence time from the initial condition.

The performance of the proposed algebraic prescribed-time KKL observer is also assessed in the presence of measurement noise. Figs. 6 and 7 show the estimation error in both coordinates in the presence of 1% and 4% of white Gaussian measurement noise, respectively. One can observe that even in the presence of noise the proposed observer converges and the estimation errors are bounded.

VI. Conclusion

In this work, we proposed an algebraic prescribed observer for a general class of nonlinear systems with possibly nonlinear outputs, relying on the KKL observer design method and the modulating function approach. The prescribed time convergence of the proposed observer lies in the modulating

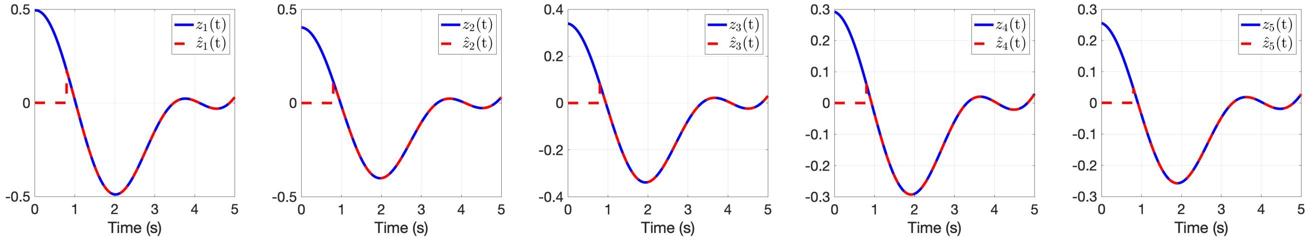

Fig. 1. State estimates in the new coordinates using the proposed algebraic prescribed-time observer for $T_p = 0.8s$ and $x_0 = [1; 0]^T$.

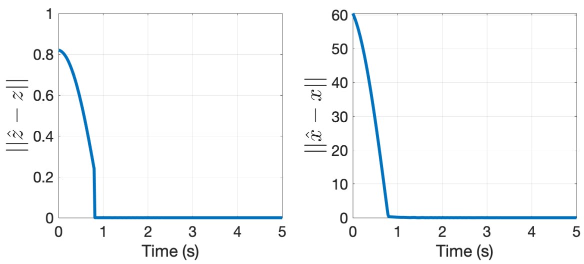

Fig. 2. Estimation error in z and x-coordinates for $T_p = 0.8s$ and $x_0 = [1; 0]^T$.

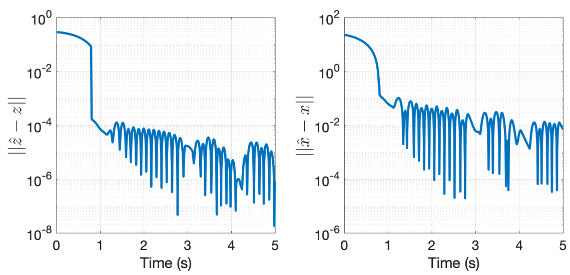

Fig. 3. Estimation error in *log* scale for $T_p = 0.8s$ and $x_0 = [1; 0]^T$.

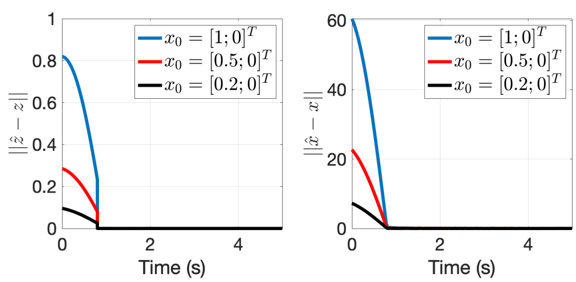

Fig. 4. Estimation error in z and x-coordinates for different initial conditions for T_p=0.8 s

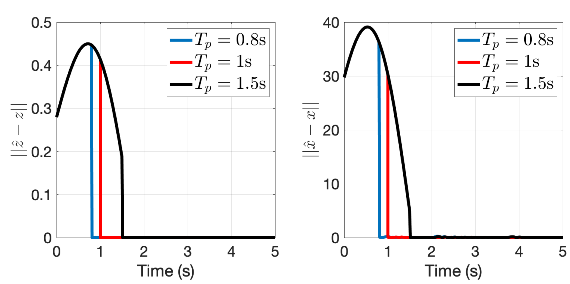

Fig. 5. Estimation error in z and x-coordinates for different prescribed convergence times and $x_0 = [0.5; 1]^T$

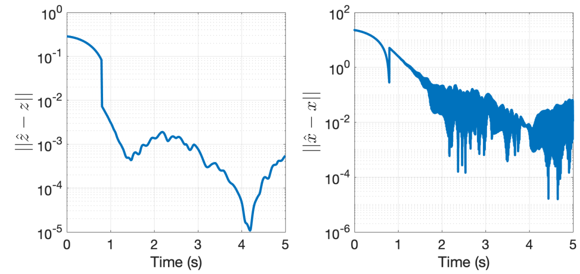

Fig. 6. Estimation error in *log* scale with %1 of measurement noise for $T_p = 0.8s$ and $x_0 = [0.5; 0]^T$.

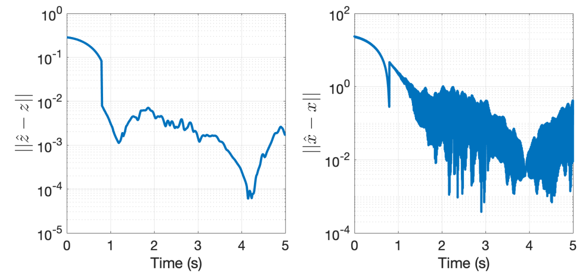

Fig. 7. Estimation error in *log* scale with %4 of measurement noise for $T_p = 0.8s$ and $x_0 = [0.5; 0]^T$.

functions combined with an integral operator applied over a prescribed time window to provide a closed-form solution to the estimated state at that specified time. Moreover, the integral operator involved in the modulating function method combined with the filtering properties of the KKL observer allows the mitigation of the effect of measurement noise on the estimation. Bounds on the estimation error were derived in the case of measurement noise. The performance and noise attenuation of the proposed observer were evaluated on a numerical example with a nonlinear output. Future work will extend the proposed algebraic observer to non-autonomous nonlinear systems and compare it to finite-time observers.

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