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Minimal Control Placement of Turing's Model Using Symmetries

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Abstract— **In this paper, the minimal control placement problem for Turing's reaction-diffusion model is studied. Turing's model describes the process of morphogens diffusing and reacting with each other and is considered as one of the most fundamental models to explain pattern formation in a developing embryo. Controlling pattern formation artificially has gained increasing attention in the field of development biology, which motivates us to investigate this problem mathematically. In this work, the two-dimensional Turing's reaction-diffusion model is discretized into square grids. The minimal control placement problem for the diffusion system is investigated first. The symmetric control sets are defined based on the symmetry of the network structure. A necessary condition is provided to guarantee controllability. Under certain circumstances, we prove that this condition is also sufficient. Then we show that the necessary condition can also be applied to the reaction-diffusion system by means of suitable extension of the symmetric control sets. Under similar circumstances, a sufficient condition is given to place the minimal control for the reaction-diffusion system.**

I. INTRODUCTION

The Turing's reaction-diffusion (RD) system [1] has been widely used to explain the fundamental question about generation of spatial patterns in organisms. The core idea of Turing's model is that patterns can be formed by morphogens, diffusing and reacting in a field of identical cells. Convincing experimental studies [2]–[5] that involve the Turing mechanism in system biology have been conducted. Though other models [6]–[8] with nonlinear reaction terms have been proposed to investigate the mystery of spatial patterns, the idea of Turing's model still plays an important role in the theoretical investigation of pattern formations.

Different properties of Turing's model have been studied and one of the active aspects is controllability. It is believed that understanding controllability of Turing's model is a useful first step in achieving the ultimate goal of understanding the self-organization mechanisms that generate so diversified patterns as are observed in nature. Besides it has been shown in [9] that some of the behaviors in the model can not be observed unless certain control is applied. Furthermore, in its own right controlling the Turing's RD system can contribute to real applications, not only in biology [10] but also in the field of processing units, sensors and memory [11]. A spatially discretized RD system can be considered as a network of cells (agents) in which we have the possibility to inject a control signal to cells of our choice. In order to fully control the given network, a critical issue is to find the suitable number and placement of controls. Various methods have been proposed to locate the minimal number of control (we hereafter refer to this as *minimal control* for brevity). Structural controllability [12] of the network has been widely studied and it is ideal for network systems in which only the underlying topology is known. A ground-breaking contribution is made in [13]. The minimum inputs theory is developed to map the structural controllability problem of a directed network into a graph-theoretical problem.

Based on the Popov-Belevitch-Hautus (PBH) test and geometric multiplicity, a universal tool is provided in [14] to assess state controllability of (directed and undirected) networks. To analyze state controllability from the graph-theoretic perspective, the symmetry of the network structure is a useful property. In [15], the network with a single leader is proved to be uncontrollable if it is leader symmetric. The relationship between the symmetry structure of the network and its controllability is shown in [16] and network equitable partitions are introduced as a tool to extend the conclusion to the multi-leader systems. In [17], the eigenvalues of the system matrix are classified by their geometric multiplicity and the structure of the eigenvectors is characterized by means of suitable decomposition of the network. Simple routines are also given to choose controls to guarantee controllability. However, among the existing results of minimal control problems, most papers focus on providing routines to choose a set of minimal control. The more general rules of minimal control placement problems remain to be investigated.

In our previous work [18], we provide the minimal control that ensures the spatially discretized RD system to be controllable and examples are given to illustrate its effectiveness. In this work, the minimal control placement problems for the discretized RD system are studied in a more general setup. The contribution of this paper is three-fold. Firstly, we identify the relationship of the eigenvectors of system matrices from the diffusion system and the reaction-diffusion system. Then we characterize the structure of the eigenvectors of the diffusion matrix. Secondly, to investigate generality of the minimal control placement problem, symmetric control sets are defined based on the properties of the eigenspaces and we provide accordingly the necessary condition to place the minimal control for the diffusion system. Furthermore, under certain circumstances, this condition is also proved to be sufficient to guarantee controllability. Thirdly, based on the results of the diffusion system, the minimal control

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placement problem for the reaction-diffusion model is studied. We prove that the necessary condition can be applied to the reaction-diffusion model by means of suitable extension of the symmetric control sets. Under similar circumstances, an effective sufficient condition is also given.

Notations: Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote natural numbers, real numbers and complex numbers respectively. For $i \in \mathbb{N}$, let e_i denote the i-th element of the canonical basis. We denote $\otimes$ and $\oplus$ as the Kronecker product and the Kronecker sum respectively. We let I and $\mathbf{0}$ denote the identity matrix and the zero matrix whose dimension are inferred from the context respectively. For $\rho \in \mathbb{R}^n$, we denote $\mathrm{diag}(\rho)$ as an $n \times n$ diagonal matrix with $\mathrm{diag}(\rho)_{ii} = \rho_i$ for all $i = 1, \ldots, n$. For $A \in \mathbb{R}^{n\times n}$, let $\sigma(A)$ denote the set of eigenvalues of A.

II. Problem Formulation

In this section, the Turing's reaction-diffusion model of two morphogens is introduced and the minimal control placement problems are formulated.

The Turing's RD model describes the process of reaction and diffusion between two morphogens, denoted as X and Y, with their concentration denoted by x and y respectively. In this paper, the RD model in two dimensions which is spatially discretized into $N \times N$ grids is considered, where $N \in \mathbb{N}$ and $N \geq 3$. The nodes in the outermost layer are naturally considered as control candidates, except for the corner nodes, since they do not interact with the inner nodes. The set of control candidates on morphogen X is denoted as $U_x = \{u_x(1, i_1), u_x(N, i_2), u_x(i_3, 1), u_x(i_4, N),\ 2 \leq i_1, i_2, i_3, i_4 \leq N-1\}$ and U_y is defined similarly. The inner nodes $x(i,j)$ and $y(i,j)$ are considered as the states, where $2 \leq i, j \leq N-1$. The illustration of morphogen X is shown in Fig. 1 and morphogen Y is of the same structure.

The diffusion process is one of the most fundamental phenomena in the real world. We follow the natural mechanism of the diffusion process that morphogens will pass from the nodes with higher concentration to the neighboring nodes with lower concentration. The diffusion parameters for morphogen X and Y are denoted as p and q respectively. Considering the phenomena of self-decay, the diffusion system of morphogen X can be expressed as

$$\frac{\mathrm{d}x(i,j)}{\mathrm{d}t} = ax(i,j) + p(\sum_{(m,n)\in\Omega_{ij}} x(m,n) - 4x(i,j)), \quad (1)$$

where $\Omega_{ij} = \{(i+1,j), (i-1,j), (i,j+1), (i,j-1)\}$ denotes the adjacent node set of the node (i,j), $2 \leq i, j \leq N-1$.

The state x is built by rows $x = \left[x_{r2}^\top, x_{r3}^\top, \ldots, x_{rN-1}^\top\right]^\top$, where $x_{ri}^\top = \left[x(i,2), \ldots, x(i,N-1)\right]$ consists of the nodes in i-th row. The input vector u_x is formed by choosing k controls in U_x and the diffusion system can be rewritten as

$$\dot{x} = A_{xx}x + B_{xx}u_x, \quad (2)$$

where A_{xx} is the diffusion matrix and the input matrix B_{xx} consists of the columns $pe_{l_1}, \ldots, pe_{l_k}$, where l_m is equal to $(i_1 - 1)$, $(N-2)(N-3) + (i_2 - 1)$, $(i_3 - 2)(N-2) + 1$ or $(i_4 - 1)(N-2)$ when the control is from $u_x(1, i_1)$, $u_x(N, i_2)$, $u_x(i_3, 1)$ or $u_x(i_4, N)$ respectively, where $1 \leq m \leq k$ and $2 \leq i_1, i_2, i_3, i_4 \leq N-1$.

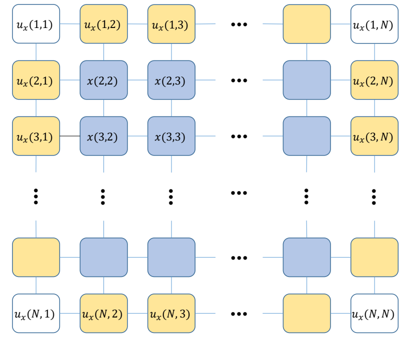

Fig. 1. Illustration of RD model discretizing into $N \times N$ grids using morphogen X as an example. The yellow nodes in the outermost layer are considered as control candidates and the blue nodes are considered as states.

For the reaction process, the concentration of morphogen X and Y will increase at the rate of $f_r(x,y)$ and $g_r(x,y)$ respectively. We follow the assumption in [1] that the concentration of morphogens in the cells is supposed to be small enough that the terms in high powers of x and y will have little effect, which leads us to the following assumption.

Assumption 1. *$f_r(x,y)$ and $g_r(x,y)$ are linear functions,*

$$\begin{aligned} f_r(x,y) &= ax(i,j) + by(i,j) \\ g_r(x,y) &= cx(i,j) + dy(i,j), \end{aligned} \quad (3)$$

where a, b, c, d are reaction parameters and $b^2 + c^2 \neq 0$ since morphogen X and Y will interact with each other.

With the reaction terms, the Turing's reaction-diffusion system of two morphogens can be expressed as (4).

$$\begin{aligned} \frac{\mathrm{d}x(i,j)}{\mathrm{d}t} &= f_r(x,y) + p(\sum_{(m,n)\in\Omega_{ij}} x(m,n) - 4x(i,j)) \\ \frac{\mathrm{d}y(i,j)}{\mathrm{d}t} &= g_r(x,y) + q(\sum_{(m,n)\in\Omega_{ij}} y(m,n) - 4y(i,j)), \end{aligned} \quad (4)$$

where Ω_{ij} is the adjacent node set of the node (i,j) defined before, $2 \leq i, j \leq N-1$.

The state vector y has the same structure as the state vector x. With u_x and u_y formed by choosing k_1 and k_2 controls from U_x and U_y respectively, the reaction-diffusion system (4) can be rewritten as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_{xx} & bI \\ cI & A_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B_{xx} & \mathbf{0} \\ \mathbf{0} & B_{yy} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad (5)$$

where A_{xx}, A_{yy} are the diffusion matrices and B_{xx}, B_{yy} can be derived by the controls in u_x, u_y respectively.

We have shown in our previous work [18] that for the diffusion system (2) and the reaction-diffusion system (5), the minimal control is both $N-2$ and an intuitive example to locate $N-2$ controls is given as choosing the nodes in x_{r1}, i.e., $u_x(1,2), \ldots, u_x(1, N-1)$. But this is not the only way to locate minimal control to guarantee controllability. In this paper, minimal control placement problems are investigated in a more general setup. Our aim is to: (a) provide a universal tool to identify the uncontrollable choices from the graph-theoretic perspective; (b) introduce a systematic method to locate the minimal control to ensure controllability.

III. MAIN RESULTS

In this section, we present the main results of the minimal control placement problem. In Section III-A, some useful properties of the eigenspace of the system matrices from the diffusion system (2) and the reaction-diffusion system (5) are introduced first. In Section III-B, the minimal control placement problem for the diffusion system is investigated. We provide a necessary condition based on the symmetric control sets, which are defined naturally by the symmetry of the topology. Under certain circumstances, we prove that this condition is also sufficient. In Section III-C, we show that the necessary condition can also be applied to the reaction-diffusion system with the extension of the symmetric control sets. Under similar circumstances, the sufficient condition is also proved to be effective for the reaction-diffusion system.

A. Some Properties of Eigenspace

Before investigating the control placement problems, some useful properties of the eigenspace of the system matrices of (2) and (5) are provided in this subsection.

Eigenvalues and eigenvectors of the system matrix in (5) are first investigated. If $b = 0$ or $c = 0$, the system matrix turns into a triangular block matrix, whose eigenvalues and eigenvectors are easy to compute. The situation of $b \neq 0$ and $c \neq 0$ is discussed in Proposition 1.

Proposition 1. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be real symmetric matrices and share the same eigenvectors. Let b and c be non-zero constants. If $w_i^\top A_1 = \lambda_{1,i} w_i^\top$, $w_i^\top A_2 = \lambda_{2,i} w_i^\top$, then $\left[w_i^\top, \alpha_{ij} w_i^\top\right]$ is a left eigenvector of $A = \begin{bmatrix} A_1 & bI_n \\ cI_n & A_2 \end{bmatrix}$, where α_{ij} satisfies $c\alpha_{ij}^2 + \alpha_{ij}(\lambda_{1,i} - \lambda_{2,i}) - b = 0$, $i = 1, \ldots, n$, $j = 1, 2$.*

Proof. Since A_1 and A_2 are symmetric and share the same eigenvectors, they can be diagonalized as $A_1 = Q\Lambda_1 Q^\top$ and $A_2 = Q\Lambda_2 Q^\top$, where $\Lambda_1 = \mathrm{diag}\{\lambda_{1,1}, \ldots, \lambda_{1,n}\}$ and $\Lambda_2 = \mathrm{diag}\{\lambda_{2,1}, \ldots, \lambda_{2,n}\}$, $\lambda_{1,i}$ and $\lambda_{2,i}$ are the i-th eigenvalues of A_1 and A_2 respectively, the i-th column of Q denoted by w_i is the corresponding i-th common eigenvector, $i = 1, \ldots, n$.

The eigenvalues of A are determined by solving

$$\begin{aligned} 0 &= \det(A - \lambda I_{2n}) = \det((A_1 - \lambda I_n)(A_2 - \lambda I_n) - bcI_n) \\ &= \det(Q((\Lambda_1 - \lambda I_n)(\Lambda_2 - \lambda I_n) - bcI_n)Q^\top), \end{aligned}$$

which yields $\lambda_i^2 - (\lambda_{1,i} + \lambda_{2,i})\lambda_i + (\lambda_{1,i}\lambda_{2,i} - bc) = 0$, where $i = 1, 2, \cdots, n$. The solutions of i-th equation are given by

$$\lambda_{ij} = \frac{(\lambda_{1,i} + \lambda_{2,i}) \pm \sqrt{(\lambda_{1,i} - \lambda_{2,i})^2 + 4bc}}{2}, \qquad (6)$$

which can be written as $\lambda_{ij} = \lambda_{1,i} + c\alpha_{ij}$, where α_{ij} satisfies $c\alpha_{ij}^2 + (\lambda_{1,i} - \lambda_{2,i})\alpha_{ij} - b = 0$, $j = 1, 2$. For $\lambda_{ij} = \lambda_{1,i} + c\alpha_{ij}$, we can obtain that $\left[w_i^\top, \alpha_{ij} w_i^\top\right](A - \lambda_{ij} I_{2n}) = 0$.

Thus, all the eigenvalues of A can be written in the form of $\lambda_{1,i} + c\alpha_{ij}$ and $\left[w_i^\top, \alpha_{ij} w_i^\top\right]$ is a corresponding left eigenvector, $i = 1, 2, \ldots, n$ and $j = 1, 2$. ■

Proposition 1 reveals that eigenvalues and eigenvectors of A can be expressed by those of A_1 and A_2. In the RD system (5), the diffusion process is denoted as A_{xx} and A_{yy}, which are related to tridiagonal Toeplitz matrices and the Kronecker sum. We state eigenvalues and eigenvectors of tridiagonal Toeplitz matrices [19] in Lemma 1 and some properties of the Kronecker sum [20] are stated in Lemma 2.

Lemma 1. *The tridiagonal Toeplitz matrix is denoted by $T = (n; \gamma, \delta, \tau)$, where $T \in \mathbb{C}^{n\times n}$ and the elements on the diagonal, superdiagonal and subdiagonal of T are δ, τ and γ respectively, while the other elements are zero. The eigenvalues of T are given by $\lambda_h = \delta + 2\sqrt{\gamma\tau}\cos\frac{h\pi}{n+1}$, $h = 1, \cdots, n$. When $\gamma\tau \neq 0$, the components of the right eigenvector $v_h = \left[v_{h,1}, v_{h,2}, \cdots, v_{h,n}\right]^\top$ corresponding with λ_h are given by $v_{h,k} = (\frac{\gamma}{\tau})^{\frac{k}{2}} \sin\frac{hk\pi}{n+1}$, $h, k = 1, \cdots, n$.*

Lemma 2. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ be given. If $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$ and x, y are the corresponding eigenvectors respectively, then $\lambda + \mu$ is an eigenvalue of $A \oplus B$ and $y \otimes x$ is a corresponding eigenvector. Furthermore, every eigenvalue of the Kronecker sum can be expressed as a sum of some eigenvalues of A and B respectively.*

Next, the symmetry of the eigenspaces corresponding with simple eigenvalues and the eigenvalues of maximum geometric multiplicity are shown in the following proposition.

Proposition 2. *Let $T = (n; \tau, \delta, \tau)$ be a tridiagonal Toeplitz matrix and $\tau \neq 0$. Then $T \oplus T$ has 2δ as an eigenvalue of maximum geometric multiplicity n, and it also has simple eigenvalues, namely eigenvalues of multiplicity one. Let w be an eigenvector of $T \oplus T$ corresponding to either 2δ or simple eigenvalues, then the elements in w satisfy*

$$|w_i| = |w_{n(i-1)+1}| = |w_{n^2+1-i}| = |w_{n(n-i+1)}|, \qquad (7)$$

where $i = 1, 2, \cdots, n$.

Proof. By Lemma 1 and Lemma 2, the eigenvalues of $T \oplus T$ and the corresponding eigenvectors are given by

$$\begin{aligned} \lambda_{(h_1,h_2)} &= \lambda_{h_1} + \lambda_{h_2} \\ &= 2\delta + 2|\tau|\cos\frac{h_1\pi}{n+1} + 2|\tau|\cos\frac{h_2\pi}{n+1} \qquad (8) \\ w_{(h_1,h_2)} &= v_{h_2} \otimes v_{h_1}, \end{aligned}$$

where λ_{h_1}, λ_{h_2} are the eigenvalues of T and v_{h_1}, v_{h_2} are the corresponding eigenvectors, $h_1, h_2 = 1, \cdots, n$.

The i-th, $(n(i-1)+1)$-th, (n^2+1-i)-th and $(n(n-i+1))$-th elements in $w_{(h_1,h_2)}$ are shown below

$$\begin{aligned} w_i &= \sin\frac{h_2\pi}{n+1}\sin\frac{ih_1\pi}{n+1} \\ w_{n(i-1)+1} &= \sin\frac{ih_2\pi}{n+1}\sin\frac{h_1\pi}{n+1} \\ w_{n^2+1-i} &= \sin\frac{nh_2\pi}{n+1}\sin\frac{(n+1-i)h_1\pi}{n+1} \\ w_{n(n-i+1)} &= \sin\frac{(n+1-i)h_2\pi}{n+1}\sin\frac{nh_1\pi}{n+1}. \end{aligned} \qquad (9)$$

For the simple eigenvalues, it only holds for $\lambda_{(k,k)}$ with the eigenvector $w_{(k,k)}$, where $2k \neq n+1$. Then the eigenvectors are $w^{'} = \alpha_1 w_{(k,k)}$, where $\alpha_1 \neq 0$. For $i = 1, \cdots, n$, the elements in $w^{'}$ satisfy

$$\begin{aligned} |w_i^{'}| &= |w_{n(i-1)+1}^{'}| = |w_{n^2+1-i}^{'}| = |w_{n(n-i+1)}^{'}| \\ &= |\alpha_1||\sin\frac{k\pi}{n+1}\sin\frac{ik\pi}{n+1}|. \end{aligned} \tag{10}$$

By (8), $\lambda_{(h_1,h_2)} = 2\delta$ if and only if $h_1 + h_2 = n+1$. The corresponding eigenvector of $\lambda_{(h_1,n+1-h_1)}$ is $w_{(h_1,n+1-h_1)}$. The eigenvectors $v_1, \ldots, v_n$ are linearly independent since T is a real symmetric matrix. Then $w_{(h_1,n+1-h_1)}$, $h_1 = 1, \ldots, n$, are also linearly independent. The geometric multiplicity of 2δ is n. The eigenvectors corresponding with 2δ are $w^{''} = \sum_{h_1=1}^{n} \alpha_{h_1} w_{(h_1,n+1-h_1)}$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not all zero. For $i = 1, \ldots, n$, the elements in $w^{''}$ satisfy

$$\begin{aligned} |w_i^{''}| &= |w_{n(i-1)+1}^{''}| = |w_{n^2+1-i}^{''}| = |w_{n(n-i+1)}^{''}| \\ &= |\sum_{h_1=1}^{n} \alpha_{h_1} \sin\frac{h_1\pi}{n+1}\sin\frac{ih_1\pi}{n+1}|. \end{aligned} \tag{11}$$

Then we prove that for the Kronecker sum $T \oplus T$, if the geometric multiplicity of the eigenvalue is 1 or n, the elements in the corresponding eigenvectors satisfy (7). ■

B. Minimal Control Placement for the Diffusion System

In this part, the minimal control placement problem of the diffusion system (2) is investigated. To place the minimal control, the following definition is first introduced to characterize control candidates by symmetry.

Definition 1. *For $i = 2, \cdots, N-1$, the symmetric control set T_i consists of the control candidates*

$$\{u_x(1,i), u_x(i,1), u_x(N-i+1,N), u_x(N,N-i+1)\}. \tag{12}$$

Controllability of the linear system can be investigated by PBH test [21] and we state it as Lemma 3.

Lemma 3. *The linear system $\dot{x}(t) = Ax(t) + Bu(t)$ is uncontrollable if and only if there exists a left eigenvector v of A, i.e., $v^\top A = \lambda v^\top$ for some λ, such that $v^\top B = 0$.*

Based on the symmetric control sets, we are now ready to provide the necessary condition for the minimal control placement problem for the diffusion system (2).

Theorem 1. *For the diffusion system (2), it is controllable only if the control set is formed by choosing at least one control from every symmetric control set T_i, $i = 2, 3, \cdots, N-1$.*

Proof. In (2), A_{xx} can be expressed as

$$A_{xx} = (a-4p)I_{(N-2)^2} + P \otimes I_{(N-2)} + I_{(N-2)} \otimes P, \tag{13}$$

where $P = (N-2; p, 0, p)$ is a tridiagonal Toeplitz matrix. The diffusion matrix A_{xx} shares the eigenvectors with $P \oplus P$.

To prove necessity, the null space of $P \oplus P$ is considered. By Proposition 2, 0 is an eigenvalue of $P \oplus P$ with the geometric multiplicity of $N-2$ and $w_{(1,N-2)}, \cdots, w_{(N-2,1)}$ form a basis of the corresponding eigenspace, which is denoted as $W = \left[w_{(1,N-2)}, w_{(2,N-3)}, \cdots, w_{(N-2,1)}\right]$.

Let $\bar{W}$ denote the first $N-2$ rows of W, i.e.,

$$\bar{W} = \left[v_{N-2,1}v_1, v_{N-3,1}v_2, \cdots, v_{1,1}v_{N-2}\right], \tag{14}$$

where v_i denotes i-th eigenvector of P and $v_{i,j}$ denotes the j-th element in v_i, $1 \leq i, j \leq N-2$.

The matrix $\bar{W}$ is non-singular since $v_1, \cdots, v_{N-2}$ are the linearly independent eigenvectors of P and the first elements $v_{i,1} \neq 0$, according to Lemma 1. Thus, there always exists a non-zero vector $\bar{\alpha} = \left[\alpha_1, \alpha_2, \cdots, \alpha_{N-2}\right]$ such that $\bar{W}\bar{\alpha} = e_i$, for all $1 \leq i \leq N-2$.

The eigenvector of A_{xx} corresponding with $a - 4p$ is $\bar{w} = W\bar{\alpha}$. In the first $N-2$ elements of $\bar{w}$, the only non-zero element is $\bar{w}_i$. By Proposition 2, none of the elements $\bar{w}_{(i-1)(N-2)+1}$, $\bar{w}_{(N-2)^2+1-i}$ and $\bar{w}_{(N-2)(N-1-i)}$ is zero. These four elements correspond to control candidates $u_x(1, i+1)$, $u_x(i+1, 1)$, $u_x(N, N-i)$ and $u_x(N-i, N)$, which form the symmetric control set T_{i+1}. All the other elements corresponding to the other control candidates in the outermost layer are zero.

If the system (2) is controllable and the control set does not have any control from the symmetric control set T_k, $2 \leq k \leq N-1$, there exists another non-zero vector $\bar{\alpha}^{'} = \left[\alpha_1^{'}, \ldots, \alpha_{N-2}^{'}\right]$ such that $\bar{w}_{k-1}^{'} \neq 0$ is the only non-zero elements in the first $N-2$ elements of $\bar{w}^{'} = W\bar{\alpha}^{'}$. Then $(\bar{w}^{'})^\top B_{xx} = 0$, which contradicts with the assumption. ■

Under certain circumstances, we prove in the next theorem that it is also sufficient to guarantee (2) controllable.

Theorem 2. *For the diffusion system (2), when all the eigenvalues of A_{xx} have the geometric multiplicity 1, 2 or $N-2$, it is controllable if the control set is formed by choosing at least one control from every symmetric control set T_i, $i = 2, 3, \cdots, N-1$.*

Proof. The eigenvalues and the corresponding eigenvectors of A_{xx} in (13) can be expressed as

$$\begin{aligned} \lambda_{(h_1,h_2)} &= a - 4p + \lambda_{h_1} + \lambda_{h_2} \\ w_{(h_1,h_2)} &= v_{h_2} \otimes v_{h_1}, \end{aligned} \tag{15}$$

where λ_{h_1}, λ_{h_2} are the eigenvalues of P and v_{h_1}, v_{h_2} are the corresponding eigenvectors, $h_1, h_2 = 1, \ldots, N-2$.

Theorem 2 is proved by characterizing eigenvalues by their geometric multiplicity. For simple eigenvalues, it holds when the eigenvalues are $\lambda_{(k,k)}$, where $2k \neq N-1$. By Proposition 2, the eigenvectors can be expressed as $w^{'} = \alpha w_{(k,k)}$, $\alpha \neq 0$. Its first element is $w_1^{'} = \alpha \sin\frac{k\pi}{N-1}\sin\frac{k\pi}{N-1} \neq 0$, then $|w_1^{'}| = |w_{(N-2)^2}^{'}| \neq 0$. There is at least one control chosen from T_2, which means that at least one of e_1 and $e_{(N-2)^2}$ is the column of B_{xx}. Then $(w^{'})^\top B_{xx} \neq 0$.

For eigenvalues with geometric multiplicity of 2, it holds for $\lambda_{(k_1,k_2)}$ when $k_1 \neq k_2$ and $k_1 + k_2 \neq N-1$. With α_1, α_2 not both zero, the eigenvectors can be expressed as $w^{''} = \alpha_1 w_{(k_1,k_2)} + \alpha_2 w_{(k_2,k_1)}$. Its first element is $w_1^{''} = (\alpha_1 + \alpha_2)\sin\frac{k_1\pi}{N-1}\sin\frac{k_2\pi}{N-1}$. If $|w_1^{''}| \neq 0$, then $|w_{(N-2)^2}^{''}| = |w_1^{''}| \neq 0$, which means $(w^{''})^\top B_{xx} \neq 0$. Otherwise, we can obtain that $\alpha_1 + \alpha_2 = 0$. Its second element $w_2^{''} = \alpha_1(\sin\frac{k_2\pi}{N-1}\sin\frac{2k_1\pi}{N-1} - \sin\frac{k_1\pi}{N-1}\sin\frac{2k_2\pi}{N-1}) \neq 0$.

Then $w''_{(N-2)+1}$ can be computed as $w''_{(N-2)+1} = -w''_2 \neq 0$. One can check $|w''_{(N-3)(N-2)}| = |w''_{(N-2)^2-1}| = |w''_2| \neq 0$. Since at least one control chosen from T_3, at least one column from e_2, $e_{(N-2)+1}$, $e_{(N-3)(N-2)}$ and $e_{(N-2)^2-1}$ is the column of B_{xx}. Then $(w'')^\top B_{xx} \neq 0$.

For eigenvalues $a-4p$ with geometric multiplicity of $N-2$, the corresponding eigenvectors are generally expressed as $\bar{w} = \sum_{j=1}^{N-2} \alpha_j w_{(j,N-1-j)}$, where $\alpha_1, \dots, \alpha_{N-2}$ are not all zero. Since $\bar{W}$ in (14) is non-singular, there is at least one non-zero elements in the first $N-2$ elements. We can assume that $\bar{w}_i \neq 0$, $1 \leq i \leq N-2$, then $\bar{w}_{(i-1)(N-2)+1}$, $\bar{w}_{(N-2)^2+1-i}$ and $\bar{w}_{(N-2)(N-1-i)}$ are all not zero, which means $\bar{w}^\top B_{xx} \neq 0$.

Thus, we prove that if the geometric multiplicity of all the eigenvalues of A_{xx} can only be 1, 2 or $N-2$, the system (2) is controllable if the control set is formed by choosing at least one control from all T_i, $i = 2, 3, \cdots, N-1$. ■

Remark 1. *As the eigenvalues and the corresponding eigenvectors of A_{xx} are given in (15), the geometric multiplicity of $\lambda_{(h_1,h_2)}$ can only be 1, 2 or $N-2$ if and only if when $h_1 + h_2 \neq N-1$, there does not exist h_3 and h_4 such that*

$$\lambda_{(h_1,h_2)} = \lambda_{(h_3,h_4)}, \tag{16}$$

where $1 \leq h_3, h_4 \leq N-2$, $h_3 \neq h_1$ and $h_3 \neq h_2$.

The solutions of (16) are discussed in [22]. Based on the results, for the most of $N \in \mathbb{N}$ and $N \geq 3$, the circumstance described in Theorem 2 is satisfied.

C. Minimal Control Placement for the RD System

The minimal control placement problem for the reaction-diffusion system (5) is investigated in this part. When two morphogens are considered, the symmetric control sets are dependent on the reaction parameters b and c. When $b = 0$ and $c \neq 0$, the RD system becomes

$$\begin{aligned} \dot{x} &= A_{xx}x + B_{xx}u_x \\ \dot{y} &= A_{yy}y + cx. \end{aligned} \tag{17}$$

Morphogen X is controllable by choosing at least one control from every symmetric control set defined in (12). Morphogen Y is also controllable by using feedback control $x = \frac{1}{c}(-A_{yy}y + z)$, where z may be a function of time.

When $b \neq 0$ and $c = 0$, the symmetric control set can be similarly defined on morphogen Y. For $i = 2, \dots, N-1$, T_i consists of the control candidates

$$\{u_y(1,i), u_y(i,1), u_y(N-i+1,N), u_y(N,N-i+1)\}.$$

When $b \neq 0$ and $c \neq 0$, according to Proposition 1, the necessary condition for the diffusion system still holds for the reaction-diffusion system with some extension of the symmetric control sets and we state it in the Theorem 3.

Theorem 3. *Consider the RD system (5), when $b \neq 0$ and $c \neq 0$. For $i = 2, \dots, N-1$, the symmetric control set T_i consists of the control candidates*

$$\begin{aligned} &\{u_x(1,i), u_x(i,1), u_x(N-i+1,N), u_x(N,N-i+1), \\ &u_y(1,i), u_y(i,1), u_y(N-i+1,N), u_y(N,N-i+1)\}. \end{aligned}$$

Then the RD system is controllable only if the control set is formed by choosing at least one control from every symmetric control set T_i, $i = 2, 3, \cdots, N-1$.

Proof. Similar to (13), A_{yy} can be written as

$$A_{yy} = (d-4q)I_{(N-2)^2} + R \otimes I_{(N-2)} + I_{(N-2)} \otimes R, \tag{18}$$

where $R = (N-2; q, 0, q)$ whose eigenvectors are the same as those of P. Then A_{xx} and A_{yy} share the eigenvectors.

In the necessity proof in the Theorem 1, we show that $a-4p$ is the eigenvalue of A_{xx} and $\bar{w}$ is a corresponding left eigenvector, where $\bar{w}_i \neq 0$ is the only non-zero elements in the first N-2 elements, $1 \leq i \leq N-2$. By Proposition 1, the vector $\bar{w}$ is a left eigenvector of A_{xx} and A_{yy} of eigenvalue $a-4p$ and $d-4q$ respectively, then $(\bar{w}^\top, \alpha_k \bar{w}^\top)$ is a left eigenvector of the system matrix of (5) corresponding with the eigenvalue λ_k, where $\lambda_k = a-4p+c\alpha_k$, and α_k satisfies $c\alpha_k^2 + (a-4p-d+4q)\alpha_k - b = 0$, $k = 1, 2$.

If the RD system (5) is controllable, we can obtain that

$$(\bar{w}^\top, \alpha_k \bar{w}^\top) \begin{bmatrix} B_{xx} & 0 \\ 0 & B_{yy} \end{bmatrix} = (\bar{w}^\top B_{xx}, \alpha_k \bar{w}^\top B_{yy}) \neq 0.$$

If the control set is not formed by choosing at least one control from every T_i, we assume that the control set does not have any control from T_k, $2 \leq k \leq N-1$. Then there exists a left eigenvector $\bar{w}$, where $\bar{w}_{k-1} \neq 0$ and $\bar{w}_j = 0$ for all $1 \leq j \leq N-2$ and $j \neq k-1$, such that $\bar{w}^\top B_{xx} = \bar{w}^\top B_{yy} = 0$, which contradicts with our assumption. Hence, we prove that it is a necessary condition. ■

Following (13) and (18), the eigenvalues of A_{xx}, A_{yy} and the corresponding common eigenvectors are given below

$$\begin{aligned} \lambda^x_{(h_1,h_2)} &= a - 4p + p(\lambda_{h_1} + \lambda_{h_2}) \\ \lambda^y_{(h_1,h_2)} &= d - 4q + q(\lambda_{h_1} + \lambda_{h_2}) \\ w_{(h_1,h_2)} &= v_{h_2} \otimes v_{h_1}, \end{aligned} \tag{19}$$

where $\lambda_{h_1}, \lambda_{h_2}$ are the eigenvalues of $(N-2; 1, 0, 1)$ and v_{h_1}, v_{h_2} are the corresponding eigenvectors, $h_1, h_2 = 1, \dots, N-2$.

By Proposition 1, the eigenvalues of the system matrix in (5) can be expressed as $\lambda_{(h_1,h_2),j} = \lambda^x_{(h_1,h_2)} + c\alpha_{(h_1,h_2),j}$, where $\alpha_{(h_1,h_2),j}$, $j = 1, 2$, satisfies

$$c\alpha^2_{(h_1,h_2),j} + (\lambda^x_{(h_1,h_2)} - \lambda^y_{(h_1,h_2)})\alpha_{(h_1,h_2),j} - b = 0. \tag{20}$$

The discriminant of (20) is shown as

$$\begin{aligned} \Delta_{(h_1,h_2)} &= (\lambda^x_{(h_1,h_2)} - \lambda^y_{(h_1,h_2)})^2 + 4bc \\ &= [(a-d) + (p-q)(\lambda_{h_1} + \lambda_{h_2} - 4)]^2 + 4bc. \end{aligned} \tag{21}$$

Based on the Theorem 2, a sufficient condition for minimal control placement problem for the reaction-diffusion system (5) is given to ensure controllability.

Corollary 1. *For the reaction-diffusion system (5), assuming the following conditions are satisfied*

(1) all the eigenvalues of A_{xx} and A_{yy} have the geometric multiplicity 1, 2 or $N-2$;
(2) $\Delta_{(h_1,h_2)} \neq 0$;
(3) if $\lambda^x_{(h_1,h_2)} \neq \lambda^x_{(h_3,h_4)}$, $\lambda_{(h_1,h_2),j_1} \neq \lambda_{(h_3,h_4),j_2}$;

where $h_1, h_2, h_3, h_4 = 1, 2, \ldots, N-2,\ j_1, j_2 \in \{1, 2\}$.

Then system (5) is controllable if the control set is formed by choosing at least one control from every symmetric control set T_i, $i = 2, \ldots, N-1$, as defined in (12).

Proof. With the condition (2) satisfied, there are two different solutions $\alpha_{(h_1,h_2),j}$, $j = 1, 2$ in (20) for all $h_1, h_2 = 1, 2, \ldots, N-2$. With the condition (3) satisfied, the eigenvalues $\lambda_{(h_1,h_2),j}$ have the same geometric multiplicity as $\lambda^x_{(h_1,h_2)}$ of A_{xx}.

Similarly, we characterize eigenvalues $\lambda_{(h_1,h_2),j}$ by their geometric multiplicity. For simple eigenvalues $\lambda_{(h_1,h_1),j}$, where $2h_1 \neq N-1$, the left eigenvectors $(w^{'}_{(h_1,h_1),j})^\top$ are

$$(w^{'}_{(h_1,h_1),j})^\top = \beta \left[w^\top_{(h_1,h_1)}, \alpha_{(h_1,h_1),j} w^\top_{(h_1,h_1)} \right],$$

where $\beta w^\top_{(h_1,h_1)}$ is a left eigenvector of A_{xx}.

For eigenvalues $\lambda^x_{(h_1,h_2)}$, $h_1 \neq h_2$ and $h_1 + h_2 \neq N-1$, the geometric multiplicity of $\lambda_{(h_1,h_2),j}$ is 2. With β_1, β_2 not both zero, the left eigenvectors $(w^{''}_{(h_1,h_2),j})^\top$ are

$$\begin{aligned}(w^{''}_{(h_1,h_2),j})^\top =& \beta_1 \left[w^\top_{(h_1,h_2)}, \alpha_{(h_1,h_2),j} w^\top_{(h_1,h_2)} \right] \\ &+ \beta_2 \left[w^\top_{(h_2,h_1)}, \alpha_{(h_1,h_2),j} w^\top_{(h_2,h_1)} \right],\end{aligned}$$

where $\beta_1 w^\top_{(h_1,h_2)} + \beta_2 w^\top_{(h_2,h_1)}$ is a left eigenvector of A_{xx}.

For eigenvalues $\lambda^x_{(h_1,N-1-h_1)}$, the geometric multiplicity of $\lambda_{(h_1,N-1-h_1),j}$ is $N-2$. With $\beta_1, \ldots, \beta_{N-2}$ not all zero, the left eigenvectors $(\bar{w}_j)^\top$ are

$$\sum_{h_1=1}^{N-2} \beta_{h_1} \left[w^\top_{(h_1,N-1-h_1)}, \alpha_{(h_1,N-1-h_1),j} w^\top_{(h_1,N-1-h_1)} \right],$$

where $\sum_{h_1=1}^{N-2} \beta_{h_1} w^\top_{(h_1,N-1-h_1)}$ is a left eigenvector of A_{xx}.

The matrix $B_{yy} = \mathbf{0}$ since controls are all from morphogen x. By the results in Theorem 2, we can obtain that $(w^{'}_{(h_1,h_1),j})^\top B \neq 0$, $(w^{''}_{(h_1,h_2),j})^\top B \neq 0$ and $\bar{w}_j^\top B \neq 0$, where B is the input matrix of (5). Then the reaction-diffusion system (5) is controllable. ∎

Remark 2. *Similar sufficient conditions can be derived by choosing the controls only from morphogen Y.*

IV. CONCLUSIONS AND FUTURE WORKS

In this paper, the minimal control placement problem for the Turing's reaction-diffusion system is investigated. The two-dimensional RD system is discretized into square grids and the nodes in the outermost layer are considered as control candidates. Symmetric control sets are defined naturally using the property of symmetry of the network structure. The minimal control placement problem for the diffusion system is investigated first. The necessary condition is provided based on the idea of symmetric control sets. Then we prove that this condition is also sufficient to ensure controllability when the multiplicity of eigenvalues satisfies certain conditions. We show further that symmetric control sets can be extended and prove that the necessary condition can also be applied to the reaction-diffusion system, i.e., the Turing's model. The sufficient condition is proved to be effective for the reaction-diffusion system under similar circumstances. Our conclusion can also be useful for other multi-agent systems with the same topology. Directions of future study include nonlinear reaction terms and time-varying systems.

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