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Beyond the 'Enemy-of-my-Enemy' Alliances: Coalitions in Networked Contest Games

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***Abstract*— Competitive resource allocation describes scenarios where multiple agents compete by spending their limited resources. For these settings, contest games offer a game-theoretic framework to analyze how players can efficiently invest their assets. Moreover, for this family of games, the resulting behavior can be modified through external interactions among the players. For instance, players could be able to make coalitions that allow budgetary transfers among them, trying to improve their outcomes. In this work, we study budgetary transfers in contest games played over networks. In particular, we aim to characterize the networks and players that guarantee that a transfer is beneficial for all players in the coalition. For this, we provide conditions for the existence of beneficial transfers. In addition, we provide a construction that guarantees that the benefit of making coalitions is independent of the graph structure and the chosen player to make an alliance.**

***Index Terms*— Coalitions, contest games, networks, resource allocation.**

I. Introduction

Multi-agent resource allocation analysis aims to describe how agents should expend their limited resources. The inherently competitive nature of these scenarios makes them suitable to applications in security deployment along multiple infrastructures [1], [2], protection in cyber-physical systems [3], multi-robot task allocation [4], among others. Contest games present models where multiple players strategically compete for a finite set of items by spending their resources on them; as such, we will use contest games as an appropriate game-theoretic framework for adversarial resource allocation problems.

Contest games formulations include Colonel Blotto games [5], [6] and models in advertising campaigns [7], [8]. While both models present multi-item contests, the main differences reside in the definition of the winning rule for each item. In those settings, modeling interconnections between players and the contested items is relevant to characterize the resulting behavior in presence of exogenous interactions. To represent those interactions, contest games consider that items are enclosed in a network structure. For instance, we can consider that the items represent physical locations. Then, the network structure models the possible paths between the contested items that the players desire to preserve by investing resources. This setting is appropriate for security games where the players want to secure paths between locations. For this model, equilibrium payoffs and strategies have been analyzed for two-player General Lotto Games, a variation of Colonel Blotto Games [9]. In addition, it could be possible to consider multi-player contests games. However, players may not compete for all the available items nor value them equally. These scenarios can be modeled through an undirected weighted graph where the edges represent a bilateral conflict of two players for a particular item. Existence and uniqueness of Nash equilibrium for this model have been presented for a family of winning rules similar to the Tullock contest success function [10].

To manipulate the resulting behavior in contest games, strategic opportunities can be offered to the players such as changes in the revealed information [11], [12], division of the players' assets [13] or alliances with budgetary transfers [14], [15]. In these alliances, players are able to transfer a portion of their assets to another player seeking to increase their own payoffs. Intuitively, giving away resources to others players can be considered a harmful decision. However, it has been verified that it could potentially improve the players' payoffs by making its enemies weaker. However, the effect of such alliances has not been considered for contest games on complex networks.

In this work, we analyze coalitions with budgetary transfers for network contest games. In particular, we tackle the questions: *Under which network structures there exists a beneficial coalition?* and *Which players are worth making an alliance with?* With this in mind, we list the main contributions of this document as follows. First, we present the model for budget-constrained network contest games. Second, we provide an equivalent formulation using per unit cost for the assets that allow us to define the equilibrium strategies for all players. With them, we can recover the equilibrium payoffs for the budget constrained formulation and the equivalent budgets for the players. Then, we present sufficient conditions to ensure the existence of a transfer that benefits both players in the coalition. Moreover, we offer a construction for any contest game that ensure the existence of such transfer with any player that is not contesting the same item. Therefore, the existence of beneficial coalitions is independent of the network and the chosen player to make the alliance. Finally, we use our results to build instances where there exists a beneficial transfer for the player who gives part of its budget.

The remainder of the paper is organized as follows. In Section II we present the model for the budget-constrained networked contest game. Section III presents the per unit cost parametrization of the game and the resulting equilibrium

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behavior. In Section IV we present conditions to assert the existence of a beneficial transfer independently of the network structure of the game. Then, in Section V, we show numerical simulations to verify existence of beneficial transfers. The proofs of our results are provided in the Appendix.

II. Model

A weighted undirected graph is defined as a tuple $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ where $\mathcal{P} = \{1, \cdots, n\}$ is the set of nodes and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set that describes the interaction between nodes. In a networked contest game $\mathcal{P}$ represent the set of players, each one with a budget of $B_i > 0$ resources to spend. Similarly, each edge $(i,j) \in \mathcal{E}$ represents an item contested by two different players with value $v_{i,j} = v_{j,i} \geq 0$ associated to it. For every player, the set of available actions is,

$$\mathcal{A}_i := \left\{ x_i \in \mathbb{R}^{n_i}_{\geq 0} \ : \ \sum_{j \in \mathcal{N}_i} x_{i,j} = B_i \right\}.$$

where $\mathcal{N}_i := \{j \, : \, v_{i,j} > 0\}$ is the neighbors set of the player i and $n_i = |\mathcal{N}_i|$. Thus, $x_i = (x_{i,1}, \cdots, x_{i,n}) \in \mathcal{A}_i$ denotes a possible allocation of resources for player i among all its contested items with the constraint that the total spent resources does not exceed its budget B_i. For any strategy profile $x = (x_1, \cdots, x_n)$ with $x_i \in \mathcal{A}_i$ the player i's payoff is given by,

$$U_i(x) = \sum_{j \in \mathcal{N}_i} v_{i,j} \alpha \left(x_{i,j}, x_{j,i} \right), \tag{1}$$

where $x_{i,j}$ defines the resources spent by player i int the item constested with player j and $\alpha\left(x_{i,j}, x_{j,i}\right) \in [0,1]$ defines the probability of player i to win the item (i,j). Therefore, the utility in Equation (1) can be interpreted as the expected value obtained along all contested items for player i given all players' efforts x. In this work, we focus on the Tullock winning rule [7], [10], [16], described as,

$$\alpha \left(x_{i,j}, x_{j,i} \right) = \frac{x_{i,j}}{x_{i,j} + x_{j,i}}. \tag{2}$$

To avoid an ill definition of the winning rule in Equation (2) we need to extend its definition using a tie-braking rule such as $\alpha(0,0) \in (0,1)$. However, this extension does not change the results presented in this paper.

For these games, we are interested in the emergent behavior in competitive environments. In particular, we consider strategies that are stable under other player's decisions. For this, we focus our analysis on the Nash equilibrium strategies. Those are strategy profiles $x^* = (x_1^*, \cdots, x_n^*)$ such that,

$$U_i(x_i^*, x_{-i}^*) \geq U_i(x_i, x_{-i}^*) \quad \forall x_i \in \mathcal{A}_i \text{ and } i \in \mathcal{P},$$

where $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. This means that there is no incentive for any player to unilaterally deviate from x^*. Note that the emerging behavior, characterized by the equilibrium allocation x^*, depends on the parameters of the contest game: the budgets of the players B_i for $i \in \mathcal{P}$ and the values of the items $v_{i,j}$ for $(i,j) \in \mathcal{E}$.

Given this dependence, the players may consider to make coalitions to affect the resulting equilibrium strategies. In particular, we define coalitions as budget transfers between players. That is, for $(a,b) \in \mathcal{P}^2$ we can define a $\tau \in [0, B_a)$ such that $\tilde{B_a} = B_a - \tau$ and $\tilde{B_b} = B_b + \tau$. With the new set of budgets $\tilde{B}$, a new equilibrium allocation $\tilde{x}^*$ will emerge, affecting the obtained payoffs for all the players in the network. We say that a transfer τ is *mutually beneficial* if the new equilibrium payoff is better than the originally obtained with budgets B for players a and b, i.e., $U_a\left(\tilde{x}^*\right) > U_a(x^*)$ and $U_b\left(\tilde{x}^*\right) > U_b(x^*)$.

Intuitively, an increased budget should be beneficial to the receiving player. However, it is not clear if the payoff of the player who gives resources can increase. With this in mind, let us present the following numerical example with a mutually beneficial transfer.

Let us consider the 3-player contest game with $(B_1, B_2, B_3) = (6, 6, 1)$, $v_{1,2} = 2$ and $v_{2,3} = 10$ as shown in Figure 1a. With these values of B and v the players can define their equilibrium strategy x^* and receive their corresponding payoff $U_1(x^*) \approx 1.7657$ and $U_3(x^*) \approx 1.6119$. Now, consider the budget transfer $\tau = 1$ from player 1 to player 3, as in Figure 1b. As expected, player 3 significantly increases its equilibrium payoff since it double its budget, obtaining a payoff $U_3(x^*) \approx 2.6263$. More importantly, the increased budget of player 3 alters others players behavior. From this change, player 1 also receives a higher payoff $U_1(x^*) \approx 1.8571$. Therefore, there exists a mutually beneficial transfer for the setting in Figure 1.

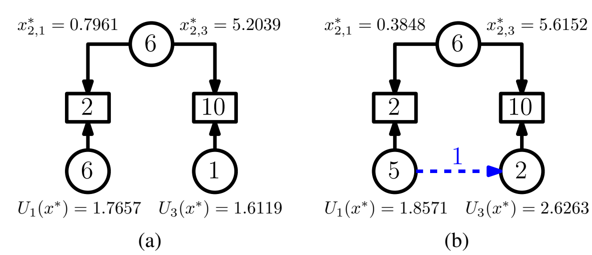

Fig. 1. Numerical Example of Contest Game in a Network. (a) Without transfer between players. (b) With a mutually beneficial transfer.

This presented example suggests there exists cases when giving some assets to another player could potentially increase the giving player's payoff. Therefore, two players can be motivated to form a coalition that allows them to transfer resources between them. However, it is not known if situations where beneficial transfers for both players will ever present themselves. With this in mind, we devote the rest of the paper to determine if there exists items valuations $v_{i,j}$ and budgets B_i for a given networked contest game $\mathcal{G}$ such that there is a mutually beneficial transfer between players a and b.

III. From Budget Constrained to Per Unit Cost Contest Games

In this section we aim to characterize the equilibrium strategies of players in a networked contest game and their respective payoffs. Let us start by analyzing the best response for each player defined as,

$$BR(x_{-i}) := \underset{x_i \in \mathcal{A}_i}{\arg\max} \quad U_i(x_i, x_{-i}).$$

The equality constraint included in $\mathcal{A}_i$ could be removed through the use of a Lagrange multiplier λ_i as follows,

$$\begin{aligned} BR(x_{-i}) &= \underset{x_i \in \mathcal{A}_i}{\arg\max} \quad U_i(x_i, x_{-i}), \\ &= \underset{x_i \in R_{\geq 0}^{n_i}}{\arg\max} \quad U_i(x_i, x_{-i}) - \lambda_i \sum_{j \in \mathcal{N}_i} x_{i,j} + \lambda_i B_i, \\ &= \underset{x_i \in R_{\geq 0}^{n_i}}{\arg\max} \quad U_i(x_i, x_{-i}) - \lambda_i \sum_{j \in \mathcal{N}_i} x_{i,j}. \end{aligned}$$

Therefore, we can reformulate the networked contest game in a setting where the players, instead of having a limited budget B_i, have an unlimited budget but they are charged $\lambda_i > 0$ per unit of resources spent. With this in mind, let us define the alternative payoffs,

$$\hat{U}_i(x_i, x_{-i}) = U_i(x_i, x_{-i}) - \lambda_i \sum_{j \in \mathcal{N}_i} x_{i,j}. \tag{3}$$

where λ_i is the cost of allocating resources for player i. We called the formulation with the payoffs in Equation (3) the *per unit cost parametrization* of the networked contest game. While both formulations have the same best response, for the per unit cost parameterization we can find the Nash equilibrium, the equilibrium payoff and the equivalent budget for each player.

Lemma 1 *For the per unit cost contest game with costs λ_i and payoffs defined as in Equation (3) the Nash equilibrium is described by the allocations,*

$$x_{i,j}^* = v_{i,j} \frac{\lambda_j}{(\lambda_i + \lambda_j)^2}.$$

Moreover, the equilibrium payoffs for the equivalent budget constrained contest game are,

$$U_i^* := U_i(x^*) = \sum_{j \in \mathcal{N}_i} v_{i,j} \frac{\lambda_j}{\lambda_i + \lambda_j}, \tag{4}$$

with equivalent budgets,

$$B_i = \sum_{j \in \mathcal{N}_i} v_{i,j} \frac{\lambda_j}{(\lambda_i + \lambda_j)^2}. \tag{5}$$

Lemma 1 presents two important results: we obtain an entire characterization of the Nash equilibrium for per unit cost contest games and we find an equation that find budgets using per unit costs. Now, in order to establish equivalence between the two games we need to ensure that we can find the costs λ_i from the budgets B_i.

Remark 1 *The mapping $B : \mathbb{R}_{>0}^n \to \mathbb{R}_{>0}^n$ defined by Equation (5) is locally invertible everywhere in its domain.*

With Remark 1 we proved equivalence between budget constrained contest games and per unit cost contest games. While the budgetary transfers are defined in the budget constraint model, without losing generality, we will analyze those budgetary transfers using the per unit cost parameterization.

IV. Existence of Positive Transfer in Networked Contest Games

In this section, we analyze how budgetary transfers τ between two players in networked contest games could potentially be beneficial for the players involved. Moreover, we can ensure that the existence of a beneficial transfer is independent of the conflict network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that support the contest game. This means that there exist strategic opportunities to increase the payoff for a particular player through coalitions with players that are beyond its local network. With that in mind, let us present our main result as follows,

Theorem 1 *For any networked contest game supported in a connected graph $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ and any pair of players $(a, b) \in \mathcal{P}^2$ such that $(a, b) \notin \mathcal{E}$ there exist valuations $v_{i,j}$ for $(i, j) \in \mathcal{E}$ and per unit costs λ_i for $i \in \mathcal{P}$ such that transferring resources from player a to player b is beneficial to player a.*

In order to verify Theorem 1 we will characterize conditions to ensure that the equilibrium payoff for a player i strictly increases for a given transfer τ. Then, for a given graph, we build an instance of a networked contest game such that a transfer is beneficial to the giving player.

Without losing generality, let us assume that the budget transfer happens from player 1 to player n and $(1, n) \notin \mathcal{E}$. Then, for any transfer τ, we can ensure that the new budgets $\tilde{B}$ are infinitesimally changing according to the vector,

$$\frac{\partial \tilde{B}}{\partial \tau} = (-1, 0, \cdots, 0, 1), \tag{6}$$

at $\tau = 0$. Using this, we can state conditions on the valuations and per unit cost that guarantee the existence of a positive transfer as follows,

Lemma 2 *For any player $i \in \mathcal{P}$, there exists a budget transfer $\tau > 0$, with vector $\frac{\partial \tilde{B}}{\partial \tau}$ as in Equation (6), that is beneficial to player i if,*

$$\frac{\partial U_i^*}{\partial \tau} = \nabla_\lambda^\top U_i^* \quad \mathbb{J}_{B/\lambda}^{-1}(\lambda) \quad \frac{\partial \tilde{B}}{\partial \tau} > 0 \tag{7}$$

where $\nabla_\lambda U_i^*$ is the gradient of U_i^*, as in Equation (4), with respect to λ and $\mathbb{J}_{B/\lambda}(\lambda)$ is the Jacobian matrix of the map $B : \mathbb{R}_{>0}^n \to \mathbb{R}_{>0}^n$ defined in Equation (5).

Remark 2 *Since we are fixing the vector $\frac{\partial \tilde{B}}{\partial \tau}$ as in Equation (6) then, condition in Equation (7) can be written as $q_n > q_1$ where q_k is the kth entry of the vector that solves the equation $\mathbb{J}_{B/\lambda}^\top(\lambda) q = \nabla_\lambda U_i^*$.*

Using the condition presented in Lemma 2 we can build an instance where a beneficial transfer exists for an arbitrary connected graph. Note that for any connected graph there exists a path between player 1 and player n. Therefore, it is sufficient to guarantee existence of a beneficial transfer form player 1 to player n for the line subgraph that connects them. For line graphs, condition in Lemma 2 can be simplified as follows,

Lemma 3 *For a n-line graph with $n > 2$, a transfer from player 1 to player n is beneficial to player 1 if,*

$$-(\lambda_1 + \lambda_2) \prod_{i=2}^{n-1} (\lambda_i - \lambda_{i+1}) > \\ 2\lambda_{n-1} \prod_{i=2}^{n-1} 2 \left[\lambda_{i-1} + \frac{v_{i,i+1}}{v_{i-1,i}} \frac{(\lambda_{i-1} + \lambda_i)^3}{(\lambda_i + \lambda_{i+1})^3} \lambda_{i+1} \right], \quad (8)$$

with $3\lambda_1 \geq \lambda_2 \geq \lambda_1$.

Even if the condition for the existence of a positive transfer is simplified in Lemma 3, it is not clear if there exist a mutually beneficial transfer exist for any graph structure. However, we can ensure the existence of an instance for any line subgraph.

Remark 3 *For any n-line graph with $n > 2$, there is at least one instance that achieve conditions stated in Lemma 3.*

Hence, using results in Lemma 3 and Remark 3, we can assert result in Theorem 1. This means that coalitions, in terms of budgetary transfers, represent an strategic opportunity for every player to improve their payoff. Moreover, the existence of such mutually beneficial transfer is independent of the graph $\mathcal{G}$ and the chosen player to make the coalition.

V. Simulations

In this section, we present numerical simulations to highlight the behavior presented in Section IV. First, let us revisit the scenario presented in Figure 1a. For this set of parameter we already show a value of $\tau > 0$ that increases player's 1 payoff in Figure 1b. Now, we are going to use result in Lemma 2 to verify the existence of such beneficial transfer. In Figure 2 we show how the player's 1 equilibrium payoff $U_1(x^*)$ evolves with respect to τ.

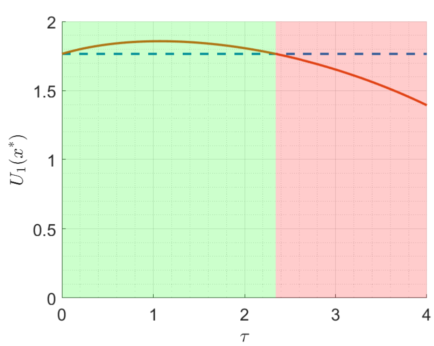

Fig. 2. Equilibrium payoff for Player 1 as a function of the budgetary transfer τ for a 3-line graph. The dashed line represent the equilibrium payoff with $\tau = 0$. The region where a transfer is beneficial for Player 1 is highlighted in green, while the red region highlights the values of τ where is not beneficial.

Notice that the value of $U_1(x^*)$ increases for small values of τ. This is equivalent to ensure that the rate of change of $U_1(x^*)$ is strictly positive with respect to τ when τ is close enough to 0. Therefore, the condition presented in Equation (7) is a sufficient condition to guarantee the existence of beneficial budgetary transfer.

Now, we use the result presented in Lemma 3 and Remark 3 to build an instance of a networked contest game supported in a 4-line graph where a beneficial transfer exist. With this in mind, we fix different parameters such that conditions are satisfied. Using the procedure described in the proof of Remark 3 we obtain the values in Table I.

TABLE I

Parameters for a 4-line graph.

Agent i	β_i	λ_i	K_i	$v_{i,i+1}$
1	-	λ_0	-	v_0
2	3.9	$3\lambda_0$	0.09	$0.09v_0$
3	2.5	$11\lambda_0$	0.12	$0.0108v_0$
4	0.9	λ_0	-	-

Note that in Table I the values for the per unit cost and the valuations are parameterized by λ_0 and v_0 respectively. Therefore, values in Table I offer a family of instances where the existence of a beneficial transfer is guaranteed. Similar to previous numerical example, we plot the player's 1 equilibrium payoff $U_1(x^*)$ for different values of τ. For $\lambda_0 = 0.005$ and $v_0 = 50$ we observe the expected increase for the giving player payoff as shown in Figure 3.

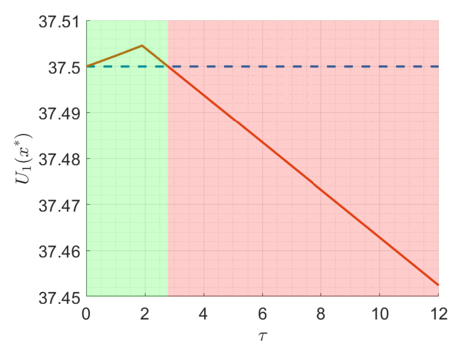

Fig. 3. Equilibrium payoff for Player 1 as a function of the budgetary transfer τ for a 4-line graph. The dashed line represent the equilibrium payoff with $\tau = 0$. The region where a transfer is beneficial for Player 1 is highlighted in green, while the red region highlights the values of τ where is not beneficial.

VI. Conclusions

In this work we study the effect of coalitions, in terms of budgetary transfers, for networked contest games. Using an equivalent game formulation, we provide sufficient conditions to check if a potential transfer is beneficial for both players involved in the coalition. With these conditions, we were able to provide a game construction were a beneficial transfer exists for any two players who are not direct enemies. With it, we were able to assert that the existence of beneficial transfers is independent to the network where the game is played but the budgets of the players and the value of the items they are fighting for. This means that, for every player in a networked contest game, there are potentially strategic opportunities to create alliances with players that are significantly beyond its local network.

While we have characterized the existence of beneficial transfer for networked contest games, there still are a broad range of questions to address in this area. For instance, algorithms to design budgetary transfers that guarantees the best performance in equilibrium conditions can be considered. In addition, more altruistic alliance models can be considered for future analysis.

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APPENDIX

A. Proof of Lemma 1

For any $(i,j) \in \mathcal{E}$ note that the optimality conditions $\frac{\partial \hat{U}_i}{\partial x_{i,j}} = 0$ and $\frac{\partial \hat{U}_j}{\partial x_{j,i}} = 0$ imply that,

$$\frac{v_{i,j}}{\lambda_i} x^*_{j,i} = \left(x^*_{i,j} + x^*_{j,i}\right)^2 = \frac{v_{i,j}}{\lambda_j} x^*_{i,j}.$$

Therefore, solving for $x^*_{i,j}$ we obtain,

$$x^*_{i,j} = v_{i,j} \frac{\lambda_j}{\left(\lambda_i + \lambda_j\right)^2},$$

for any $(i,j) \in \mathcal{E}$. Replacing the definition of $x^*_{i,j}$ in Equation (1) and the budget constraint we obtain,

$$U_i(x^*) = \sum_{j \in \mathcal{N}_i} v_{i,j} \frac{x^*_{i,j}}{x^*_{i,j} + x^*_{ji}} = \sum_{j \in \mathcal{N}_i} v_{i,j} \frac{\lambda_j}{\lambda_i + \lambda_j}.$$

$$B_i = \sum_{j \in \mathcal{N}_i} x^*_{i,j} = \sum_{j \in \mathcal{N}_i} v_{i,j} \frac{\lambda_j}{\left(\lambda_i + \lambda_j\right)^2},$$

which matches the expressions in Lemma 1.

B. Proof of Remark 1

In order to verify that the map described in Equation (5) is locally invertible in its domain we build its Jacobian matrix $\mathbb{J}_{B/\lambda}(\lambda)$ as,

$$\left[\mathbb{J}_{B/\lambda}(\lambda)\right]_{i,i} = \frac{\partial B_i}{\partial \lambda_i} = -2 \sum_{j \in \mathcal{N}_i} v_{i,j} \frac{\lambda_j}{\left(\lambda_i + \lambda_j\right)^3},$$

and,

$$\left[\mathbb{J}_{B/\lambda}(\lambda)\right]_{i,j} = \frac{\partial B_i}{\partial \lambda_j} = v_{i,j} \frac{(\lambda_i - \lambda_j)}{\left(\lambda_i + \lambda_j\right)^3} = -\left[\mathbb{J}_{B/\lambda}(\lambda)\right]_{ji}.$$

Note that for any $q \in \mathbb{R}^n$,

$$\begin{aligned} q^\top \mathbb{J}_{B/\lambda}(\lambda) q &= \frac{1}{2} q^\top \left(\mathbb{J}_{B/\lambda}(\lambda) + \mathbb{J}^\top_{B/\lambda}(\lambda)\right) q \\ &= q^\top \operatorname{diag}\left(\left[\mathbb{J}_{B/\lambda}(\lambda)\right]_{ii}\right) q < 0, \end{aligned}$$

for any $\lambda \in \mathbb{R}^n_{>0}$. Therefore, the Jacobian matrix is negative definite and non-singular for any $\lambda \in \mathbb{R}^n_{>0}$. By the inverse function theorem, we can ensure that $B : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ as defined by Equation (5) is locally invertible everywhere in its domain.

C. Proof of Lemma 2

Note that if $\frac{\partial U^*_i}{\partial \tau} > 0$ at $\tau = 0$ then there exists a $\tau > 0$ that increases the value of U^*_i. Using the chain rule,

$$\left.\frac{\partial U^*_i}{\partial \tau}\right|_{\tau=0} = \sum_k \left.\frac{\partial U^*_i}{\partial \tilde{B}_k}\right|_{\tilde{B}_k = B_k} \left.\frac{\partial \tilde{B}_k}{\partial \tau}\right|_{\tau=0} = \nabla^\top_B U^*_i \frac{\partial \tilde{B}}{\partial \tau} \tag{9}$$

with $\nabla_B U^*_i$ is the gradient of U^*_i with respect to B. Using the definition of the Jacobian matrix we have,

$$\begin{aligned} &\nabla_\lambda U^*_i = \mathbb{J}^\top_{B/\lambda}(\lambda) \nabla_B U^*_i \\ \iff &\nabla_B U^*_i = \left[\mathbb{J}^\top_{B/\lambda}(\lambda)\right]^{-1} \nabla_\lambda U^*_i. \end{aligned} \tag{10}$$

Therefore, using Equations (9) and (10) we obtain,

$$\frac{\partial U^*_i}{\partial \tau} > 0 \iff \nabla^\top_\lambda U^*_i \;\; \mathbb{J}^{-1}_{B/\lambda}(\lambda) \;\; \frac{\partial \tilde{B}}{\partial \tau} > 0.$$

D. Proof of Lemma 3

From Remark 2 we know that a transfer is beneficial to player 1 if $q_n > q_1$ with,

$$\mathbb{J}_{B/\lambda}^{\top}(\lambda)q = \nabla_\lambda U_1^* \iff -\mathbb{J}_{B/\lambda}^{\top}(\lambda)q = -\nabla_\lambda U_1^*.$$

Let us define $A := -\mathbb{J}_{B/\lambda}^{\top}(\lambda)$ and $b := -\nabla_\lambda U_1^*$.. Then, q is the solution of the linear system $Aq = b$. Using Cramer's rule and noticing that $A \succ 0$ we obtain,

$$\begin{aligned} q_n > q_1 &\iff \frac{\det(A_n)}{\det(A)} > \frac{\det(A_1)}{\det(A)} \\ &\iff \frac{2}{(\lambda_1+\lambda_2)}\det(A_n) > \frac{2}{(\lambda_1+\lambda_2)}\det(A_1) \\ &\iff \det(\hat{A}_n) > \det(\hat{A}_1) \end{aligned}$$

where A_i is obtained by replacing the ith column of matrix A with vector b and $\hat{A}_i$ is obtained by replacing the ith column of matrix A with vector $\frac{2}{(\lambda_1+\lambda_2)}b$. The value of $\det(\hat{A}_n)$ can be obtained using the Laplace expansion,

$$\begin{aligned} \det(\hat{A}_n) &= (-1)^{n+1}\left[\frac{-2}{(\lambda_1+\lambda_2)}\frac{\partial U_1^*}{\partial\lambda_1}\right]\left(-\frac{\partial B_1}{\partial\lambda_2}\right)\prod_{i=2}^{n-1}\left(-\frac{\partial B_i}{\partial\lambda_{i+1}}\right) \\ &+ (-1)^{n+2}\left[\frac{-2}{(\lambda_1+\lambda_2)}\frac{\partial U_1^*}{\partial\lambda_2}\right]\left(-\frac{\partial B_1}{\partial\lambda_1}\right)\prod_{i=2}^{n-1}\left(-\frac{\partial B_i}{\partial\lambda_{i+1}}\right) \\ &= \left[\frac{2(-1)^n}{(\lambda_1+\lambda_2)}\prod_{i=2}^{n-1}\left(-\frac{\partial B_i}{\partial\lambda_{i+1}}\right)\right]\left[\frac{\partial U_1^*}{\partial\lambda_2}\frac{\partial B_1}{\partial\lambda_1} - \frac{\partial U_1^*}{\partial\lambda_1}\frac{\partial B_1}{\partial\lambda_2}\right] \\ &= \frac{2}{(\lambda_1+\lambda_2)}\left[\prod_{i=2}^{n-1}\left(\frac{\partial B_i}{\partial\lambda_{i+1}}\right)\right]\left[\frac{\partial U_1^*}{\partial\lambda_2}\frac{\partial B_1}{\partial\lambda_1} - \frac{\partial U_1^*}{\partial\lambda_1}\frac{\partial B_1}{\partial\lambda_2}\right]. \end{aligned}$$

Note that,

$$\begin{aligned} &\frac{\partial U_1^*}{\partial\lambda_2}\frac{\partial B_1}{\partial\lambda_1} - \frac{\partial U_1^*}{\partial\lambda_1}\frac{\partial B_1}{\partial\lambda_2} \\ &= \left[\frac{v_{1,2}\lambda_1}{(\lambda_1+\lambda_2)^2}\right]\left[\frac{-2v_{1,2}\lambda_2}{(\lambda_1+\lambda_2)^3}\right] - \left[\frac{-v_{1,2}\lambda_2}{(\lambda_1+\lambda_2)^2}\right]\left[\frac{v_{1,2}(\lambda_1-\lambda_2)}{(\lambda_1+\lambda_2)^3}\right] \\ &= \frac{v_{1,2}^2\lambda_2}{(\lambda_1+\lambda_2)^5}\left[-2\lambda_1 + (\lambda_1-\lambda_2)\right] = -\frac{v_{1,2}^2\lambda_2}{(\lambda_1+\lambda_2)^4}. \end{aligned}$$

Then,

$$\det(\hat{A}_n) = -\frac{2v_{1,2}^2\lambda_2}{(\lambda_1+\lambda_2)^5}\prod_{i=2}^{n-1} v_{i,i+1}\frac{(\lambda_i-\lambda_{i+1})}{(\lambda_i+\lambda_{i+1})^3}$$

On the other hand, note that if $3\lambda_1 \geq \lambda_2 \geq \lambda_1$ then $\hat{A}_1$ is diagonally dominant with positive diagonal entries. Thus, we can upper bound $\det(\hat{A}_1)$ using Hadamard's inequality,

$$\begin{aligned} \det(\hat{A}_1) &\leq \prod_{i=1}^{n}\left[\hat{A}_1\right]_{i,i} = \left[\frac{2v_{1,2}\lambda_2}{(\lambda_1+\lambda_2)^3}\right]\prod_{i=2}^{n}\left(-\frac{\partial B_i}{\partial\lambda_i}\right) \\ &= \left[\frac{2v_{1,2}\lambda_2}{(\lambda_1+\lambda_2)^3}\right]\left[\frac{2v_{n-1,n}\lambda_{n-1}}{(\lambda_{n-1}+\lambda_n)^3}\right]\prod_{i=2}^{n-1}2\left(\frac{v_{i-1,i}\lambda_{i-1}}{(\lambda_{i-1}+\lambda_i)^3} + \frac{v_{i,i+1}\lambda_{i+1}}{(\lambda_i+\lambda_{i+1})^3}\right) \end{aligned}$$

Therefore, we can guarantee $\det(\hat{A}_n) > \det(\hat{A}_1)$ if,

$$\begin{aligned} &\frac{-v_{1,2}}{(\lambda_1+\lambda_2)^2}\prod_{i=2}^{n-1} v_{i,i+1}\frac{(\lambda_i-\lambda_{i+1})}{(\lambda_i+\lambda_{i+1})^3} > \\ &\frac{2v_{n-1,n}\lambda_{n-1}}{(\lambda_{n-1}+\lambda_n)^3}\prod_{i=2}^{n-1}2\left(\frac{v_{i-1,i}\lambda_{i-1}}{(\lambda_{i-1}+\lambda_i)^3} + \frac{v_{i,i+1}\lambda_{i+1}}{(\lambda_i+\lambda_{i+1})^3}\right) \\ \iff &\frac{-v_{1,2}}{(\lambda_1+\lambda_2)^2}\prod_{i=2}^{n-1}(\lambda_i-\lambda_{i+1}) > \\ &2\lambda_{n-1}\left[\frac{v_{1,2}}{(\lambda_1+\lambda_2)^3}\right]\prod_{i=2}^{n-1}2\left[\lambda_{i-1} + \frac{v_{i,i+1}}{v_{i-1,i}}\frac{(\lambda_{i-1}+\lambda_i)^3}{(\lambda_i+\lambda_{i+1})^3}\lambda_{i+1}\right] \\ \iff &-(\lambda_1+\lambda_2)\prod_{i=2}^{n-1}(\lambda_i-\lambda_{i+1}) > \\ &2\lambda_{n-1}\prod_{i=2}^{n-1}2\left[\lambda_{i-1} + \frac{v_{i,i+1}}{v_{i-1,i}}\frac{(\lambda_{i-1}+\lambda_i)^3}{(\lambda_i+\lambda_{i+1})^3}\lambda_{i+1}\right] \end{aligned}$$

which matches the expression in Equation (8).

E. Proof of Remark 3

Note that we can decompose the inequality in Equation (8) as follows,

$$\lambda_1+\lambda_2 > \beta_2\left[\lambda_1 + K_2\frac{(\lambda_1+\lambda_2)^3}{(\lambda_2+\lambda_3)^3}\lambda_3\right], \tag{11}$$

$$\lambda_3-\lambda_2 > \beta_3\left[\lambda_2 + K_3\frac{(\lambda_2+\lambda_3)^3}{(\lambda_3+\lambda_4)^3}\lambda_4\right], \tag{12}$$

$$\lambda_{i-1}-\lambda_i > \beta_i\left[\lambda_{i-1} + K_i\frac{(\lambda_{i-1}+\lambda_i)^3}{(\lambda_i+\lambda_{i+1})^3}\lambda_{i+1}\right], \quad \forall i \in \{4,\cdots,n-1\}, \tag{13}$$

$$\lambda_{n-1}-\lambda_n > \beta_n\lambda_{n-1}. \tag{14}$$

where $K_i = \frac{v_{i,i+1}}{v_{i-1,i}}$ and $\prod_{i=2}^{n}\beta_i > 2^{n-1}$. For Equations (11), (12) and (13) we can obtain conditions on K_i as follows,

$$\begin{aligned} K_2 &< \frac{1}{\beta_2\lambda_3}\frac{(\lambda_2+\lambda_3)^3}{(\lambda_1+\lambda_2)^3}\left[(1-\beta_2)\lambda_1+\lambda_2\right], \\ K_3 &< \frac{1}{\beta_3\lambda_4}\frac{(\lambda_3+\lambda_4)^3}{(\lambda_2+\lambda_3)^3}\left[\lambda_3-(1+\beta_3)\lambda_2\right], \\ K_i &< \frac{1}{\beta_i\lambda_{i+1}}\frac{(\lambda_i+\lambda_{i+1})^3}{(\lambda_{i-1}+\lambda_i)^3}\left[(1-\beta_i)\lambda_{i-1}-\lambda_i\right], \\ &\quad \forall i \in \{4,\cdots,n-1\}. \end{aligned}$$

Thus, we can build the values $v_{i,j}$ using the β_i, λ_i and the conditions on K_i. Positivity of K_i and assumptions in Lemma 3 give us the following conditions for the β_i,

$$\begin{aligned} \lambda_2 &> (\beta_2-1)\lambda_1, \\ 3\lambda_1 &\geq \lambda_2 \geq \lambda_1, \\ \lambda_3 &> (1+\beta_3)\lambda_2, \\ (1-\beta_i)\lambda_{i-1} &> \lambda_i, \qquad \forall i \in \{4,\cdots,n\}, \\ \prod_{i=2}^{n}\beta_i &> 2^{n-1}. \end{aligned}$$

Therefore, we can pick a sequence of β_i and then define the per unit costs λ_i from it. For instance, the sequence $\beta_2 = 2$, $\beta_3 > 2^{2n-5}$ and $\beta_i = \frac{1}{2}$ for $i \in \{4,\cdots,n\}$ satisfies the conditions above.