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A New Quasi-Finite-Rank Approximation of Compression Operators with Application to the $\mathcal{L}_1$ Discretization for Sampled-Data Systems

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Abstract— **This paper develops a new $\mathcal{L}_1$ discretization procedure for sampled-data systems, in which the minimization of the $\mathcal{L}_\infty$-induced norm of sampled-data systems is concerned with. This discretization is based on developing a new quasi-finite-rank approximation (QFRA) of compression operators occurring from the lifting-based approach to sampled-data systems. More precisely, we develop a more sophisticated method for the QFRA of compression operators by using the idea of piecewise linear kernel approximation (PLKA) approach, rather than the conventional method based on the fast-sample/fast-hold (FSFH) approach. The minimization problem for the corresponding QFRA error is shown to be solved through a linear programming (LP) problem. Furthermore, the theoretical effectiveness for the QFRA is established by deriving the associated convergence rate of $1/M$, where M is the corresponding approximation parameter. Finally, this QFRA of compression operators leads to a new $\mathcal{L}_1$ discretization procedure for sampled-data systems.**

I. Introduction

By noting the fact that sampled-data systems take into account their inter-sample behavior, there have been a number of studies on the disturbance rejection control for sampled-data systems to deal with various types of practical problems. Depending on the different types of control objectives and the nature of disturbances, one could define various system norms for sampled-data systems. For instance, the $\mathcal{H}_\infty$ norm has been considered in [1]–[6] to deal with the maximum energy of the output for the worst disturbance with a unit energy, while the $\mathcal{H}_2$ norm has been tackled in [5]–[10] to take the effects of the impulse disturbances. The main idea for solving the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control problems of sampled-data systems in [1]–[10] is to discretize the continuous-time plants in an equivalent fashion without changing these norms.

With respect to dealing with some practical problems of suppressing the maximum magnitude of a signal such as the collision avoidance of robotic manipulators, on the other hand, the $\mathcal{L}_1$ control problem of sampled-data systems has been discussed in the literature. More precisely, this problem aims at designing an optimal controller for minimizing the $\mathcal{L}_\infty$-induced norm of sampled-data systems, and this induced norm corresponds to the peak value of the output for the worst disturbance with a unit magnitude. Because the $\mathcal{L}_\infty$-induced norm for a continuous-time single output system coincides with the $\mathcal{L}_1$ norm of the impulse response, the control problem of the $\mathcal{L}_\infty$-induced norm has been called the $\mathcal{L}_1$ control problem [11]–[15]. In contrast to the cases of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control problems of sampled-data systems, it is quite difficult to derive an equivalent discretization of the continuous-time plants in the $\mathcal{L}_1$ control problem of sampled-data systems. This is because the Banach space $\mathcal{L}_\infty$ is taken as the underlying function space for the problem. Hence, various approximate discretizations for the $\mathcal{L}_1$ control problem of sampled-data systems have been discussed, and they could be classified by the input approximation (IA; [16], [17]) and the kernel approximation (KA; [18]) approaches. In other words, these approaches lead to discretizing the continuous-time plant by which the l_∞-induced norm of the resulting approximate discrete-time closed-loop system is shown to converge to the $\mathcal{L}_\infty$-induced norm of the original sampled-data system as the corresponding discretization parameter L becomes larger. However, the arguments in [16]–[18] might lead to some conservative results because the $\mathcal{L}_1$ discretization procedures could be interpreted as ignoring the compression operator occurring from the operator-theoretic representation of sampled-data systems [1], [3], [19] due to its infinite-rank property.

To resolve the difficulties from the infinite-rank property of the compression operator defined on $\mathcal{L}_\infty[0,h)$ with the sampling period h, a method of quasi-finite-rank approximation (QFRA) was developed in [20]. More precisely, this method is based on the fast-sample/fast-hold (FSFH) technique in [21], and the minimization problem of the corresponding approximation error is formulated by a linear programming (LP) problem. Even though the theoretical effectiveness of the QFRA is validated by deriving the associated convergence order of $1/M$ with the relevant approximation parameter M, it was also observed from the numerical study in [20] that a quite large value of M is usually required for reducing the approximation error in a desired level.

In connection with this, we aim at developing a new QFRA method of compression operators, for which a required value of M to reduce the approximation error in a certain level is smaller than that used for the QFRA in [20], and providing a new framework for the $\mathcal{L}_1$ discretization procedure for sampled-data systems by using the new QFRA of compression operators. To do this, we first employ the idea of the piecewise linear kernel approximation (PLKA) [22] and introduce a new QFRA of compression operators on

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TABLE I: Table of Notations

Notations	Definitions
$\mathbb{N}$	The set of positive integers
$\mathbb{R}^\nu$	The set of ν-dimensional real vectors
$\mathcal{K}_\nu$	The Banach space $(\mathcal{L}_\infty[0,h))^\nu$
$\mathcal{K}'_\nu$	The Banach space $(\mathcal{L}_\infty[0,h'))^\nu$
$\lvert\cdot\rvert$	The ∞-norm of a matrix of a vector
$\lVert\cdot\rVert$	The $\mathcal{L}_\infty[0,h)$-induced norm of an operator The $\mathcal{L}_\infty[0,h')$-induced norm of an operator
$\lVert\cdot\rVert_1$	The $\mathcal{L}_1[0,h)$-induced norm of an operator The $\mathcal{L}_1[0,h')$-induced norm of an operator
X_*	Pre-dual space of the Banach space X
$\mathbf{T}_*: Y_* \to X_*$	Pre-adjoint of a linear operator $\mathbf{T}: X \to Y$
$R(\cdot)$	Range of the operator $(\cdot)$

$\mathcal{L}_\infty[0,h)$. We next show that the problem of minimizing the corresponding approximation error can be also converted to an LP problem, similarly to the results in [20], in an asymptotically equivalent fashion. In other words, the theoretical validity of the PLKA-based QFRA is established by deriving the associated convergence rate of $1/M$. We then introduce a new $\mathcal{L}_1$ discretization procedure for sampled-data systems by modifying the conventional discretization procedure [16], [17] in terms of taking some advantages of the new PLKA-based QFRA of compression operators. Finally, we validate the theoretical effectiveness of this new $\mathcal{L}_1$ discretization by deriving the corresponding convergence order of $1/L$.

Note that the mathematical notations used in this paper are given in Table I.

II. Operator-Theoretic Representation of Sampled-Data Systems via Lifting

Consider the sampled-data system $\Sigma_{\rm SD}$ depicted in Fig. 1, where the continuous-time linear time-invariant (LTI) plant P and the discrete-time LTI controller Ψ are interconnected by the ideal sampler $\mathcal{S}$ and the zero-order hold $\mathcal{H}$ with the sampling period h in a synchronous fashion. Assume that P and Ψ are described by

$$P: \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x \end{cases} \tag{1}$$

$$\Psi: \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^{n_z}$, $y(t) \in \mathbb{R}^{n_y}$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $\psi(t) \in \mathbb{R}^{n_\psi}$, $y_k = y(kh)$, and $u(t) = u_k$ for $kh \le t < (k+1)h$. With respect to dealing with the linear periodically time-varying (LPTV) nature of $\Sigma_{\rm SD}$, it is generally considered to take the lifting technique [1], [3], [19], in which a given continuous-time signal $f(\cdot) \in \mathcal{L}_\infty^\nu$ is viewed as its lifted form denoted by $\{\hat{f}_k\}_{k=0}^\infty$ with $\hat{f}_k(\cdot) \in \mathcal{K}_\nu$, where

$$\hat{f}_k(\theta) = f(kh+\theta) \qquad (0 \le \theta < h) \tag{3}$$

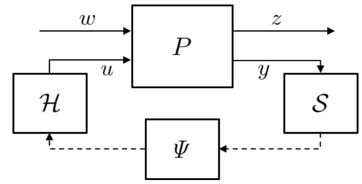

Fig. 1: Sampled-data system $\Sigma_{\rm SD}$.

Applying the lifting technique to $w \in \mathcal{L}_\infty^{n_w}$ and $z \in \mathcal{L}_\infty^{n_z}$ together with discretizing x by $x_k := x(kh)$ lead to the lifted representation of P described by

$$\widehat{P}: \begin{cases} x_{k+1} = A_d x_k + \mathbf{B}_1 \hat{w}_k + B_{2d} u_k \\ \hat{z}_k = \mathbf{C}_1 x_k + \mathbf{D}_{11} \hat{w}_k + \mathbf{D}_{12} u_k \\ y_k = C_{2d} x_k \end{cases} \tag{4}$$

with the matrices

$$A_d = e^{Ah}, \quad B_{2d} = \int_0^h e^{A\theta} B_2 d\theta, \quad C_{2d} = C_2 \tag{5}$$

and the operators

$$\mathbf{B}_1 w = \int_0^h e^{A(h-\theta)} B_1 w(\theta) d\theta \tag{6}$$

$$(\mathbf{C}_1 x)(\theta) = C_1 e^{A\theta} x \tag{7}$$

$$(\mathbf{D}_{11} w)(\theta) = \int_0^\theta C_1 e^{A(\theta-\tau)} B_1 w(\tau) d\tau + D_{11} w(\theta) \tag{8}$$

$$(\mathbf{D}_{12} u_k)(\theta) = \int_0^\theta C_1 e^{A(\theta-\tau)} B_2 d\tau u_k + D_{12} u_k \tag{9}$$

Here, it should be remarked that $\mathbf{D}_{11}$ is called the compression operator associated with the sampled-data system $\Sigma_{\rm SD}$, and it has an infinite-rank while the others are finite-rank operators. This infinite-rank property often makes it difficult to tackle the problems of performance analysis and controller synthesis with respect to $\Sigma_{\rm SD}$ in an efficient fashion. In connection with this, the following section is devoted to providing a new method of quasi-finite-rank approximation (QFRA) for $\mathbf{D}_{11}$ on $\mathcal{L}_\infty[0,h)$, by which some difficulties in the $\mathcal{L}_1$ controller synthesis for $\Sigma_{\rm SD}$ could be alleviated.

III. Quasi-Finite-Rank Approximation of Compression Operators on $\mathcal{L}_\infty[0,h)$

This section develops a new method of QFRA for $\mathbf{D}_{11}$ on its underlying function space $\mathcal{L}_\infty[0,h)$ tailored to the context of the $\mathcal{L}_1$ controller synthesis for $\Sigma_{\rm SD}$. More precisely, we employ the idea of the piecewise linear kernel approximation (PLKA) [22] for more improved accuracy in terms of the associated QFRA, compared to the conventional method in [20] based on the fast-sample/fast-hold (FSFH) approximation [21].

A. PLKA-based QFRA of compression opeartors

By noting the fact that the matrix D_{11} is an infinite-rank operator when it is viewed as a multiplication operator, we first decompose $\mathbf{D}_{11}$ as $\mathbf{D}_{11} = \mathbf{D}_{110} + D_{11}$, where

$$(\mathbf{D}_{110}w)(\theta) = \int_0^{\theta} C_1 e^{A(\theta-\tau)} B_1 w(\tau) d\tau \qquad (10)$$

Then, the QFRA of $\mathbf{D}_{11}$ is to approximate $\mathbf{D}_{110}$ by

$$\mathbf{C}_1 X \mathbf{B}_1 \qquad (11)$$

with an adequate matrix X, and thus $\mathbf{D}_{11}$ is approximated ultimately by

$$\mathbf{C}_1 X \mathbf{B}_1 + D_{11} \qquad (12)$$

Note that $\mathbf{C}_1 X \mathbf{B}_1$ is a finite-rank operator while D_{11} is an infinite-rank operator.

We next consider a method for determining a suitable X with respect to the $\mathcal{L}_1$ controller synthesis problem of Σ_{SD}. Because the controller synthesis problem is formulated by taking $\mathcal{L}_\infty[0,h)$ as the underlying function space relevant to $\hat{P}$, the matrix-valued parameter X could be taken for minimizing the $\mathcal{L}_\infty[0,h)$-induced norm of $\mathbf{E}_X$ defined as

$$\mathbf{E}_X := \mathbf{D}_{11} - (\mathbf{C}_1 X \mathbf{B}_1 + D_{11}) = \mathbf{D}_{110} - \mathbf{C}_1 X \mathbf{B}_1 \quad (13)$$

However, it is quite difficult to find an optimal X for minimizing $\|\mathbf{E}_X\|$ rigorously since $\mathbf{E}_X$ is defined on the Banach space $\mathcal{L}_\infty[0,h)$ as discussed in [20]. Hence, we develop a new framework for the QFRA of $\mathbf{D}_{11}$ in an approximate fashion by using the idea of PLKA [22].

As a preliminary step to employ the above ideas, we first introduce the fast-lifted representation of $\mathbf{E}_X$. With the corresponding parameter $M \in \mathbb{N}$ and $h' := h/M$, fast-lifting of a signal $\hat{f} \in \mathcal{K}_\nu$ leads to $\check{f} := [(\hat{f}^{(1)})^T \ \cdots \ (\hat{f}^{(M)})^T]^T$ where

$$\hat{f}^{(i)}(\theta') = \hat{f}((i-1)h' + \theta') \qquad (0 \le \theta' < h') \qquad (14)$$

and it is denoted by $\check{f} = \mathbf{L}_M \hat{f}$. Then, the fast-lifted representation of $\mathbf{E}_X$ (denoted by $\mathbf{E}_{MX}$) is described by

$$\begin{aligned}\mathbf{E}_{MX} &:= \mathbf{L}_M \mathbf{E}_X \mathbf{L}_M^{-1} = \mathbf{L}_M(\mathbf{D}_{110} - \mathbf{C}_1 X \mathbf{B}_1)\mathbf{L}_M^{-1} \\ &= \overline{\mathbf{C}_1'}(\Delta_{M0} - C_{dM}' X B_{dM}')\overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{110}'} \end{aligned} \qquad (15)$$

where the operators $\mathbf{C}_1'$, $\mathbf{B}_1'$, and $\mathbf{D}_{110}'$ are defined equivalently as $\mathbf{C}_1$, $\mathbf{B}_1$, and $\mathbf{D}_{110}$ by replacing h with h', and the matrices are given by

$$B_{dM}' := \begin{bmatrix}(A_d')^{M-1} & \cdots & I\end{bmatrix}, \quad C_{dM}' := \begin{bmatrix} I \\ \vdots \\ (A_d')^{M-1}\end{bmatrix}$$

$$\Delta_{M0} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A_d')^{M-2} & \cdots & I & 0 \end{bmatrix} \qquad (16)$$

with $A_d' := e^{Ah'}$. Defining $\Delta_{MX} := \Delta_{M0} - C_{dM}' X B_{dM}'$ further admits the representation

$$\mathbf{E}_{MX} = \overline{\mathbf{C}_1'} \Delta_{MX} \overline{\mathbf{B}_1'} + \overline{\mathbf{D}_{110}'} \qquad (17)$$

and we are in a position to replace the operators $\mathbf{C}_1'$ and $\mathbf{B}_1'$ with more tractable ones used for computing the $\mathcal{L}_\infty$-induced norm of LTI systems as in [22].

More precisely, we first introduce the approximation operators $\mathbf{B}_{\mathrm{k}1}'$ and $\mathbf{C}_{\mathrm{a}1}'$ defined respectively as

$$\mathbf{B}_{\mathrm{k}1}' w := \int_0^{h'} e^{Ah'}(I - A\theta') B_1 w(\theta') d\theta' \qquad (18)$$

$$(\mathbf{C}_{\mathrm{a}1}' x)(\theta') := C_1(I + A\theta')x \quad (0 \le \theta' < h') \qquad (19)$$

With these operators, we take replacing $\mathbf{B}_1'$ and $\mathbf{C}_1'$ with $\mathbf{B}_{\mathrm{k}1}'$ and $\mathbf{C}_{\mathrm{a}1}'$, respectively, and ignoring $\mathbf{D}_{110}'$ in $\mathbf{E}_{MX}$ (given by (17)), i.e., introducing

$$\mathbf{E}_{MX\mathrm{a}} := \overline{\mathbf{C}_{\mathrm{a}1}'} \Delta_{MX} \overline{\mathbf{B}_{\mathrm{k}1}'} \qquad (20)$$

To put it another way, we aim at obtaining X minimizing $\|\mathbf{E}_{MX\mathrm{a}}\|$ instead of minimizing $\|\mathbf{E}_{MX}\|$ because the former is relatively simpler than the latter, and such an optimal X with respect to minimizing $\|\mathbf{E}_{MX\mathrm{a}}\|$ can be derived by solving a linear programming (LP) problem. For this purpose, we see from (18) that $\mathbf{B}_{\mathrm{k}1}'$ admits the representation

$$\mathbf{B}_{\mathrm{k}1}' w = B_{\mathrm{k}h}' \phi \qquad (21)$$

where

$$B_{\mathrm{k}h}' := \begin{bmatrix} h' A_d' B_1 & -\frac{(h')^2}{2} A_d' A B_1 \end{bmatrix} \qquad (22)$$

$$\phi := \begin{bmatrix}\phi_0^T & \phi_1^T\end{bmatrix}^T \qquad (23)$$

$$\phi_0 := \frac{1}{h'}\int_0^{h'} w(\tau')d\tau', \quad \phi_1 := \frac{2}{(h')^2}\int_0^{h'} \tau' w(\tau')d\tau' \quad (24)$$

Here, the set of all ϕ with respect to w in the unit ball on $\mathcal{K}_{n_w}'$ is denoted by Φ_M. Because it is quite difficult to describe the area of Φ_M by a finite-dimensional matrix in the $\mathcal{L}_\infty$ sense, its outer-approximation $\Phi_M^{[N]}$ with the corresponding approximation parameter $N \in \mathbb{N}$ was developed in [23]. The area relevant to $\Phi_M^{[N]}$ converges to that relevant to Φ_M as N becomes larger, with an arbitrary degree of accuracy, and we can obtain a matrix $\Omega^{[N]} \in \mathbb{R}^{2n_w \times Nn_w}$ defined as

$$\Phi_M \subset \Phi_M^{[N]} := \{\Omega^{[N]} w_d \mid |w_d| \le 1\} \qquad (25)$$

where $\Omega^{[N]}$ is exactly the same as that introduced in [23]. Then, the operator $\mathbf{B}_{\mathrm{k}1}'$ can be replaced by the matrix $B_{1d}^{[N]'}$ defined as

$$B_{1d}^{[N]'} := B_{\mathrm{k}h}' \Omega^{[N]} \in \mathbb{R}^{n \times Nn_w} \qquad (26)$$

with (approximately) holding the range of $\mathbf{B}_{\mathrm{k}1}'$ for the inputs in the unit ball relevant to $\mathcal{K}_{n_w}'$. On the other hand, if we note that the output of $\mathbf{C}_{\mathrm{a}1}'$ has its maximum value on $\theta' = 0$ or $\theta' = h'$, the operator $\mathbf{C}_{\mathrm{a}1}'$ can be replaced by

$$C_{1d}' := \begin{bmatrix} C_1 \\ C_1(I + Ah') \end{bmatrix} \in \mathbb{R}^{2n_z \times n} \qquad (27)$$

without changing $\|\mathbf{E}_{MX\mathrm{a}}\|$. To summarize, the problem of obtaining an optimal X for minimizing the $\mathcal{L}_\infty[0,h)$-induced norm of $\mathbf{E}_{MX\mathrm{a}}$ is replaced by that for minimizing the matrix ∞-norm of $E^{[N]}_{MX\mathrm{a}}$ defined as

$$E^{[N]}_{MX\mathrm{a}} := \overline{C'_{1d}}\Delta_{MX}\overline{B^{[N]'}_{1d}} = \overline{C'_{1d}}(\Delta_{M0} - C'_{dM}XB'_{dM})\overline{B^{[N]'}_{1d}} \tag{28}$$

The problem to find the optimal X for minimizing the matrix ∞-norm $|E^{[N]}_{MX\mathrm{a}}|$ can be easily converted to an LP problem by following the essentially equivalent procedure in [20]. Detailed procedure for such conversion is omitted because it is quite technical.

B. Error analysis for PLKA-based QFRA

This subsection aims at establishing a theoretical validity for taking an optimal X that minimizes $\|\mathbf{E}_{MX\mathrm{a}}\|$ by deriving the convergence rate corresponding to the gap between $\mathbf{E}_{MX}$ and $\mathbf{E}_{MX\mathrm{a}}$. To do this, we first introduce the following lemmas.

Lemma 1 ([24]): There exists a constant K_D such that

$$\|\mathbf{D}'_{110}\| \leq \frac{K_D}{M} \tag{29}$$

Lemma 2 ([23], [25]): Suppose that (A, B_1) is controllable and (C_1, A) is observable, and define the operators $\mathbf{J}'_{\mathrm{k}1} : \mathcal{K}'_{n_w} \to \mathcal{K}'_{n_w}$ and $\mathbf{H}'_{\mathrm{a}1} : \mathcal{K}'_{n_z} \to \mathcal{K}'_{n_z}$ respectively as $\mathbf{B}'_{\mathrm{k}1} =: \mathbf{B}'_1\mathbf{J}'_{\mathrm{k}1}$ and $\mathbf{C}'_{\mathrm{a}1} =: \mathbf{H}'_{\mathrm{a}1}\mathbf{C}'_1$, i.e.,

$$(\mathbf{J}'_{\mathrm{k}1}w)(\theta') = B_1^T e^{A^T(h'-\theta')} W_{h'}^{-1} \int_0^{h'} A'_d(I - A\tau')B_1 w(\tau')d\tau'$$

$$(\mathbf{H}'_{\mathrm{a}1}x)(\theta') = z(0) + \theta' \frac{dz(\tau')}{d\tau'}|_{\tau'=0}$$

with the controllability Grammian defined as

$$W_{h'} := \int_0^{h'} e^{A(h'-\theta')}B_1B_1^T e^{A^T(h'-\theta')}d\theta' \tag{30}$$

Then, the following assertions hold in terms of the pre-adjoints $\mathbf{J}'_{\mathrm{k}1*}$ and $\mathbf{B}'_{1*}$ (of $\mathbf{J}'_{\mathrm{k}1}$ and $\mathbf{B}'_1$, respectively) and the operators $\mathbf{H}'_{\mathrm{a}1}$ and $\mathbf{C}'_1$.

a) There exists a constant K_B such that

$$\|(I - \mathbf{J}'_{\mathrm{k}1*})|_{R(\mathbf{B}'_{1*})}\|_1 \leq \frac{K_B}{M^2} \tag{31}$$

b) There exists a constant K_C such that

$$\|(I - \mathbf{H}'_{\mathrm{a}1})|_{R(\mathbf{C}'_1)}\| \leq \frac{K_C}{M^2} \tag{32}$$

Remark 1: The controllability and observability assumptions in Lemma 2 on the pairs (A, B_1) and (C_1, A), respectively, are just for the ease of the proof of this lemma. They can be actually removed because those pairs can always be replaced with some controllable and observable ones without changing the ranges $R(\mathbf{B}_{1*})$ and $R(\mathbf{C}'_1)$.

From Lemma 2, we can derive the following proposition.

Proposition 1: There exists a constant K_{CB} independent of X such that

$$\|\overline{\mathbf{C}'_1}\Delta_{MX}\overline{\mathbf{B}'_1} - \overline{\mathbf{C}'_{\mathrm{a}1}}\Delta_{MX}\overline{\mathbf{B}'_{\mathrm{k}1}}\| \leq \frac{K_{CB}}{M^2}\|\overline{\mathbf{C}'_{\mathrm{a}1}}\Delta_{MX}\overline{\mathbf{B}'_{\mathrm{k}1}}\| \tag{33}$$

From Lemma 1 and Proposition 1, we have the following theorem relevant to the convergence rate between $\mathbf{E}_{MX}$ and $\mathbf{E}_{MX\mathrm{a}}$.

Theorem 1: The following inequality holds.

$$\left(1 - \frac{K_{CB}}{M^2}\right)\|\mathbf{E}_{MX\mathrm{a}}\| - \frac{K_D}{M} \leq \|\mathbf{E}_{MX}\| \leq \left(1 + \frac{K_{CB}}{M^2}\right)\|\mathbf{E}_{MX\mathrm{a}}\| + \frac{K_D}{M} \tag{34}$$

If we note that $\|\mathbf{E}_X\| = \|\mathbf{E}_{MX}\|$, Theorem 1 clearly implies that an optimal X minimizing $\|\mathbf{E}_{MX\mathrm{a}}\|$ leads to (approximately) minimizing also $\|\mathbf{E}_X\|$ with the convergence order of $1/M$.

IV. $\mathcal{L}_1$ Discretization Procedure for Σ_{SD} with PLKA-based QFRA of Compression Operators

This section provides an $\mathcal{L}_1$ discretization procedure for the sampled-data system Σ_{SD} with taking the PLKA-based QFRA developed in this paper, and discuss its comparison to the existing $\mathcal{L}_1$ discretization procedure in [16], [17].

Let us first introduce the existing $\mathcal{L}_1$ discretization procedure [16], [17], in which both the input w and the output z in P are approximated by piecewise constant signals. If we denote the corresponding discretization parameter by L and redefine $h' := h/L$, then this piecewise constant approximation can be interpreted as discretizing the operators $\mathbf{B}_1$, $\mathbf{C}_1$, and $\mathbf{D}_{12}$ by the matrices $B_{1dL} \in \mathbb{R}^{n \times Ln_w}$, $C_{1dL} \in \mathbb{R}^{Ln_z \times n}$, and $D_{12dL} \in \mathbb{R}^{Ln_z \times n_u}$ defined respectively as

$$B_{1dL} := B'_{dL}\overline{B'_{0d}}, \quad [C_{1dL}\ D_{12dL}] := \overline{[C_1\ D_{12}]}A'_{2dL} \tag{35}$$

where

$$A'_{2dL} := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{L-1} \end{bmatrix},\ B'_{dL} := \begin{bmatrix} (A'_d)^{L-1} & \cdots & I \end{bmatrix},$$

$$B'_{0d} := \int_0^{h'} e^{A(h'-\tau')}B_1 d\tau',\ A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \tag{36}$$

with redefining $A'_d := e^{Ah'} (= e^{Ah/L})$ and $A'_{2d} := e^{A_2h'} (= e^{A_2h/L})$. On the other hand, with respect to the treatment of $\mathbf{D}_{11}$ in [16], [17], if we note its fast-lifted representation given by

$$\mathbf{L}_L\mathbf{D}_{11}\mathbf{L}_L^{-1} = \overline{\mathbf{C}'_1}\Delta_{L0}\overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} \tag{37}$$

where Δ_{L0} is defined equivalently as Δ_{M0} in (16) with redefining A'_d and M by the above fashion, then the discretization of $\mathbf{D}_{11}$ in that study corresponds to ignoring the operator $\mathbf{D}'_{110}$ and replacing the operators $\mathbf{C}'_1$, $\mathbf{B}'_1$ with C_1 and B'_{0d}, respectively, i.e., the matrix D_{11dL} defined as

$$D_{11dL} = \overline{C_1}\Delta_{L0}\overline{B'_{0d}} + \overline{D_{11}} \tag{38}$$

To summarize, the $\mathcal{L}_1$ discretization in [16], [17] is to derive the discrete-time plant P_{dL} described by

$$P_{dL} : \begin{cases} x_{k+1} = A_d x_k + B_{1dL}w_k + B_{2d}u_k \\ z_k = C_{1dL}x_k + D_{11dL}w_k + D_{12dL}u_k \\ y_k = C_{2d}x_k \end{cases} \tag{39}$$

We denote the discrete-time closed-loop system consisting of the discretized plant P_{dL} and the controller Ψ by Σ_{dL}. Then, we have the following lemma relevant with the $\mathcal{L}_1$ performance deterioration arising from the $\mathcal{L}_1$ discretization procedure in [16], [17].

Lemma 3 ([17]): There exist constants T_0 and T_1 independent of the controller Ψ such that the following ineqaulity holds:

$$\|\Sigma_{dL}\| \leq \|\Sigma_{\rm SD}\| \leq \left(1+\frac{T_0}{L}\right)\|\Sigma_{dL}\| + \frac{T_1}{L} \tag{40}$$

Lemma 3 implies that $\|\Sigma_{dL}\|$ converges to $\|\Sigma_{\rm SD}\|$ with the convergence order of $1/L$. We would like to note that the lower bound readily follows by the fact that taking piecewise constant approximations for the input and output signals as in [16], [17] is norm-contractive. For the upper bound in (40), more importantly, T_0 is independent of the approximation of $\mathbf{D}_{11}$ while T_1 depends on the approximation. Hence, the $\mathcal{L}_\infty[0,h')$-induced norm of $\mathbf{D}'_{110}$ is directly reflected to the value of T_1 since P_{dL} is obtained without considering the effects of $\mathbf{D}'_{110}$. Thus, if we do not ignore $\mathbf{D}'_{110}$ and treat it in a more sophisticated fashion by which the corresponding gap between $\mathbf{D}'_{110}$ and its new treatment is smaller than $\|\mathbf{D}'_{110}\|$, we can lead to a smaller value of T_1.

With this in mind, we are in a position to treat $\mathbf{D}'_{110}$ in (37) more rigorously tailored to the context of the $\mathcal{L}_1$ synthesis problem. More precisely, we first approximate $\mathbf{D}'_{110}$ by $\mathbf{C}'_1 X \mathbf{B}'_1$ through the arguments in the preceding section, and it is also discretized by the matrix $C_1 X B'_{0d}$, while the other discretization procedures for the operators $\mathbf{B}_1, \mathbf{C}_1$ and $\mathbf{D}_{12}$ are the same as those in [16], [17]. In other words, this is equivalent to replacing the matrix D_{11dL} in (39) with

$$D_{11dL}^{[X]} = \overline{C_1}(\Delta_{L0} + \overline{X})\overline{B'_{0d}} + \overline{D_{11}} \tag{41}$$

and we obtain ultimately the discretized plant $P_{dL}^{[X]}$ given by

$$P_{dL}^{[X]} : \begin{cases} x_{k+1} = A_d x_k + B_{1dL} w_k + B_{2d} u_k \\ z_k = C_{1dL} x_k + D_{11dL}^{[X]} w_k + D_{12dL} u_k \\ y_k = C_{2d} x_k \end{cases} \tag{42}$$

The discrete-time closed-loop system obtained by connecting $P_{dL}^{[X]}$ and Ψ is denoted by $\Sigma_{dL}^{[X]}$.

Next, if we note the fact that X is obtained through the optimization problem of minimizing $\|\mathbf{D}'_{110} - \mathbf{C}'_1 X \mathbf{B}'_1\|$, it immediately leads to

$$\|\mathbf{D}'_{11} - (\mathbf{C}'_1 X \mathbf{B}'_1 + D_{11})\| = \|\mathbf{D}'_{110} - \mathbf{C}'_1 X \mathbf{B}'_1\| \leq \|\mathbf{D}'_{110}\| \tag{43}$$

This together with the arguments after Lemma 3 and the essentially equivalent arguments for deriving Lemmas 1 and 2 lead to the following theorem relevant to the convergence order of the developed $\mathcal{L}_1$ discretization.

Theorem 2: The inequality

$$\|\Sigma_{dL}^{[X]}\| - \frac{\tilde{T}_1}{L} \leq \|\Sigma_{\rm SD}\| \leq \left(1+\frac{T_0}{L}\right)\|\Sigma_{dL}^{[X]}\| + \frac{T_1^{[X]}}{L} \tag{44}$$

holds, where $T_0, \tilde{T}_1$ are independent of X and the controller Ψ, and $T_1^{[X]}$ is not larger than T_1 given in (40).

Theorem 2 clearly shows that the $\mathcal{L}_1$ discretization developed in this paper has the convergence rate of $1/L$ with respect to the $\mathcal{L}_1$ optimal controller synthesis for sampled-data systems.

V. Conclusion

This paper introduced a new framework for the $\mathcal{L}_1$ discretization of sampled-data systems by using approximated treatment of compression operators on the Banach space $\mathcal{L}_\infty[0,h)$. To this end, we first developed a new quasi-finite-rank approximation (QFRA) of compression operators by employing the piecewise linear kernel approximation (PLKA) [22]. The problem for minimizing the relevant approximation error was also shown to be reduced to (approximately) a linear programming (LP) problem. The theoretical effectiveness of taking such an LP problem was validated by deriving the corresponding convergence rate of $1/M$, where M is the associated approximation parameter. Based on this new QFRA, we obtained a new approximate $\mathcal{L}_1$ discretization procedure for sampled-data systems, which has the convergence order of $1/L$ in the sense of minimizing the $\mathcal{L}_\infty$-induced norm of sampled-data systems, where L is the discretization parameter.

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