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A dynamic population game model of non-monetary bottleneck congestion management under elastic demand using karma

Ezzat Elokda, Carlo Cenedese, Kenan Zhang, Andrea Censi, Saverio Bolognani and Emilio Frazzoli

Abstract— **The morning commute bottleneck congestion problem has classically been modelled as a static game in which commuters act strategically based on their immediate Value of Time (VOT). This has restricted existing congestion mitigation techniques to rely on essentially monetary incentives to affect the static costs of the commuters. In contrast, a dynamic model enables characterizing the strategic trade-off between immediate and future resource access rights and inspires the design of new classes of fair, non-monetary congestion mitigation schemes. In this paper, we show how the recently proposed Dynamic Population Game (DPG) framework can be leveraged to study a non-monetary economy for bottleneck congestion management based on karma, a non-tradable mobility credit. Our DPG model allows to consider an elastic demand of commuters that only travel if congestion is reduced, and we show that a Stationary Nash Equilibrium (SNE) is guaranteed to exist despite of the dynamic participation of these commuters. Through numerical case studies we illustrate how our tools can assist policy makers in taking informed decisions about complex policy outcomes. In particular, we show how the dynamic karma scheme is robust to a potentially detrimental rebound effect that would manifest in a static monetary scheme.**

I. Introduction

In the seminal Vickrey bottleneck model [1]–[3], commuters participate in a static game in which they strategically choose their departure times to trade-off between arriving early/late or facing congestion delays due to a limited bottleneck capacity. Without intervention, the Nash equilibrium of the game is inefficient as it exhibits costly congestion. In theory, congestion can be completely eliminated with an optimal tolling scheme, known as Vickrey's toll [1], which charges higher monetary tolls at the most desired departure times.

Despite of their theoretical efficiency, policies based on Vickrey's toll and similar congestion pricing schemes [4]–[6] are often faced with public dismay [7] as they tend to favor wealthier travelers [8]–[10]. This has led to a plethora of credit-based approaches that attempt to address the fairness issue of classical congestion pricing [11]–[14]. In principle, the scarce supply of mobility credits is meant to incentivize commuters to spend them when they need it the most, i.e., when they have highest Value of Time (VOT), thereby acting as a substitute for money. However, due to the reliance on static extensions of Vickrey's model [15]–[17], existing approaches insist on associating a monetary value to the credits. In practice, this manifests in either the credits being tradable in a monetary market [11], [13], [14] or being used as vouchers to pay for monetary tolls [12]. This fails to fundamentally address the fairness issue of classical congestion pricing: wealthier commuters have a larger capacity to buy credits or bypass the credit system by paying monetary tolls.

Interestingly and in contrast to that, a dynamic formulation of the congestion management problem enables the analysis and design of completely non-monetary mobility credit schemes. In fact, considering their credit budget as a dynamic state, commuters can learn to ration the use of credits over multiple days or periods without having to associate a monetary value to them. However, this is highly non-standard in the existing literature: it is a setting with a large number of commuters featuring dynamic credit states whose evolution depends non-trivially on the commuters' joint actions. A recently proposed framework for dynamic games played in large populations is Dynamic Population Games (DPGs) [18]. DPGs complement similar formulations of anonymous sequential games [19], [20] and mean field games [21]–[23] by showing that they can be reduced to the well studied class of (static) population games [24]. This eases the analysis of these games and provides tractable equilibrium computation algorithms based on evolutionary dynamics [24].

In this paper, we showcase the potential of a dynamic model in devising new classes of fair and efficient congestion mitigation schemes by developing a tractable DPG model of a non-monetary, credit-based bottleneck congestion management scheme. This builds on previous work [25], which develops a DPG model of a general non-monetary economy for resource allocations, called a *karma economy*, and [26], which specializes a karma economy to the bottleneck congestion problem. The bottleneck is split into a regulated *fast lane* that is free of congestion and a general purpose *slow lane* that is subject to congestion. Commuters use a non-monetary credit called *karma* to bid for access to the fast lane, and the karma collected by those who succeed to enter the fast lane gets redistributed at the end of each day. Compared to [26], in this work we depart from previous DPG formulations by considering a dynamic number of game participants (in addition to the player state dynamics). Namely, we consider the presence of an elastic demand of commuters whose participation in the game is a function of the efficiency of its outcomes. This lets us investigate the important problem of *rebound effects* which manifest when efficiency gains are counter-acted by newly induced demand [27]–[29].

All authors are with ETH Zurich, 8092 Zurich, Switzerland. {elokdae,ccenedese,kenzhang,acensi,bsaverio, efrazzoli}@ethz.ch

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In the remainder of the paper, we first introduce the DPG model for karma-based bottleneck congestion management under elastic demand in Section II. We then perform a numerical case study illustrating the benefit of our dynamic formulation in Section III. We finally conclude with a discussion in Section IV.

A. Notation

Let $a, d \in D \subseteq \mathbb{N}$ and let $c \in C \subseteq \mathbb{R}^n$, then for a function $f : D \times C \to \mathbb{R}$, we distinguish discrete and continuous arguments through the notation $f[d](c)$. Alternatively, we write $f : C \to \mathbb{R}^{|D|}$ as the vector-valued function $f(c)$, with $f[d](c)$ denoting its d-th element. Similarly, $g[a \mid d](c)$ denotes the conditional probability of a given d and c. Specifically, $g[d^+ \mid d](c)$ denotes one-step transition probabilities for d. We denote by $p \in \Delta(D) := \left\{ \sigma \in \mathbb{R}_+^{|D|} \middle| \sum_{d \in D} \sigma[d] = 1 \right\}$ a probability distribution over the elements of D, with $p[d]$ denoting the probability of element d. Finally, when considering heterogeneous commuter types, we denote by x_τ a quantity associated to type τ.

II. Dynamic Bottleneck Model with Elastic Demand

In this section, we first briefly recap CARMA, the karma-based bottleneck congestion management scheme proposed in [26], and introduce its extension to the setting with elastic demand (Section II-A). We then detail the dynamic strategic problem that both *fixed* and *elastic* commuters face (Section II-B). The existence of an equilibrium in the resulting DPG is lastly guaranteed (Section II-C).

A. Description of karma-based congestion management scheme

We consider $N_{\texttt{fix}}$ commuters that travel through a bottleneck with total capacity s [veh/min], which is split into a managed *fast lane* with capacity s^{fast} and an unmanaged *slow lane* with capacity $s^{\text{slow}} = s - s^{\text{fast}}$. Commuters not participating in the karma scheme can only use the slow lane. On the other hand, commuters participating in the karma scheme are given an initial endowment of karma which they can use to bid for access to the fast lane. In particular, we discretize the feasible departure times into T intervals of length Δt [min], and every morning commuters make a decision on their departure time $t \in \mathcal{T} = \{1, \ldots, T\}$, as well as, on a karma bid to enter the fast lane $b \in \mathcal{B}[k] = \{0, \ldots, k\}$, where $k \in \mathbb{N}$ denotes their current karma budget. Consequently, the highest $s^{\text{fast}} \Delta t$ bidders departing at t are allowed to enter the fast lane (settling ties randomly), while all others have to use the slow lane, as illustrated in Figure 1. At the end of each day, karma is collected from the fast lane commuters and redistributed to the population according to some protocol to be specified by policy makers (discussed in Section II-B.2), and the process repeats thereafter.

In addition to the $N_{\texttt{fix}}$ commuters that must travel daily, we consider the presence of $N_{\texttt{elast}}$ commuters that travel only if it is convenient. In particular, in the absence of any policy intervention (i.e., if $s^{\text{fast}} = 0$), those *elastic commuters* prefer a less costly alternative to travelling under the resulting congestion, e.g., working from home, moving closer to work, or taking a different transit mode.

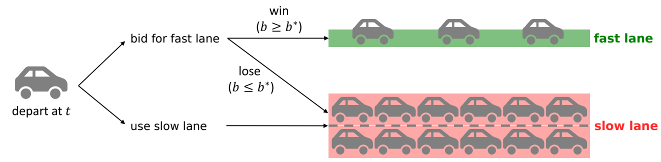

Fig. 1: Karma auction to enter the fast lane.

B. Commuters' strategic model

The standard DPG model considers that all players participate in the game during every time period [18], [25], [26]. To incorporate the presence of elastic commuters, who do not always travel through the bottleneck, we enrich the model proposed in [26] by including these commuters as a new "type" who have *do not travel* (denoted by $\neg$) as a feasible action. Therefore, the commuters are heterogeneous with type $\tau \in \Gamma = \{\texttt{fix}, \texttt{elast}\}$, hereafter referred to as fixed or elastic, respectively. The distribution of types in the population is $g \in \Delta(\Gamma)$ with $g_{\texttt{fix}} = \frac{N_{\texttt{fix}}}{N}$, and $N = N_{\texttt{fix}} + N_{\texttt{elast}}$. Both types of commuters could have a different *VOT process* (denoted by $\phi_\tau[u^+ \mid u]$), that is, the exogenous Markov chain describing the evolution of their daily VOT u. For example, this allows to model the case in which the fixed and elastic commuters belong to different income classes, see Section III. With minimal loss of generality, we assume a finite number of VOT levels $u \in \mathcal{U} = \{u_1, \ldots, u_{n_u}\}$. Each day the commuters feature a dynamic state $x = [u, k] \in \mathcal{U} \times \mathbb{N}$ which consists of their current VOT u and karma budget k. The time-varying joint distribution of types and states in the population is denoted by $d \in \mathcal{D} = \{d \in \mathbb{R}^{|\Gamma| \times |\mathcal{U}| \times \infty} \mid \forall \tau \in \Gamma, \sum_{u,k} d_\tau[u, k] = g_\tau\}$.

On each day, fixed commuters must choose an action $a = [t, b] \in \mathcal{A}_{\texttt{fix}}[k] = \mathcal{T} \times \mathcal{B}[k]$ which consists of their departure time t and the karma bid to enter the fast lane b. In contrast, elastic commuters can choose an action $a \in \mathcal{A}_{\texttt{elast}}[k] = \mathcal{A}_{\texttt{fix}}[k] \cup \{\neg\}$, where $a = \neg$ denotes their choice to not travel through the bottleneck. The policy followed by the population is denoted by $\pi \in \Pi$, with $\pi_\tau[a \mid u, k] \in [0, 1]$ denoting the probability that commuters of type τ and state $[u, k]$ choose action $a \in \mathcal{A}_\tau[k]$.

The *social state* is $(d, \pi) \in \mathcal{D} \times \Pi$, which gives the distribution of commuter types and states as well as their actions, thereby providing a macroscopic description of the competitive landscape. Namely, the individual commuter faces a δ-discounted Markov decision process (MDP) that is coupled to others through (d, π). In what follows, we specify the key elements of this MDP: the immediate reward function $\zeta_\tau(d, \pi)$, and the karma transition function $\kappa_\tau(d, \pi)$.

1) Immediate reward function $\zeta_\tau[u, a](d, \pi)$: Following the classical bottleneck model [1], we divide the immediate reward for commuters travelling through the bottleneck in two parts: queuing delay t^{q}, and early or late schedule delay denoted by t^{e} or t^{l}, respectively. Given the departure time and bid, how much delay each commuter endures depends on the outcome of the karma auction. Let $\psi[o \mid a](d,\pi)$ denote the probability of an ego commuter having outcome $o \in \mathcal{O} = \{\text{fast}, \text{slow}, \text{other}\}$, given its action a and the other commuters' actions that are a function of the social state (d,π). It holds that $\psi[o = \text{other} \mid a = \neg] = 1$, and $\psi[o = \text{other} \mid a = t, b] = 0$ for all $[t,b] \in \mathcal{A}_{\texttt{fix}}$. Then, the immediate reward for $a = [t,b]$ can be written as

$$\zeta_\tau[u,t,b](d,\pi) = -u \sum_{o \in \mathcal{O}} \psi[o \mid t,b] \left(\alpha\, t^{\mathrm{q}} + \beta\, t^{\mathrm{e}} + \gamma\, t^{\mathrm{l}}\right), \tag{1}$$

where $\beta < \alpha < \gamma$ denote the sensitivity to the different delays, and t^{q}, t^{e} and t^{l} are given by

$$t^{\mathrm{q}}[t,o](d,\pi) = \begin{cases} \frac{q[t](d,\pi)}{s_{\text{slow}}}, & o = \text{slow}, \\ 0, & \text{otherwise}, \end{cases} \tag{2a}$$

$$t^{\mathrm{e}}[t,o](d,\pi) = \max\{0,\ t^* - t - t^{\mathrm{q}}[t,o](d,\pi)\}, \tag{2b}$$

$$t^{\mathrm{l}}[t,o](d,\pi) = \max\{0,\ t + t^{\mathrm{q}}[t,o](d,\pi) - t^*\}, \tag{2c}$$

where $q[t](d,\pi)$ is the queue length on the slow lane at time t (see [26] for derivation) and t^* is the commuters' desired arrival time. The immediate reward for $a = \neg$ (by elastic commuters) is given by

$$\zeta_{\texttt{elast}}[u,\neg](d,\pi) = -u\, c^{\text{other}}, \tag{3}$$

where c^{other} denotes the cost of the alternative to travelling for the elastic commuters. To model that the elastic commuters would choose not to travel under no policy intervention, c^{other} must satisfy

$$c^{\text{other}} \leq c^{\text{NOM}} = \frac{\beta\,\gamma}{\beta + \gamma} \frac{N_{\texttt{fix}}}{s}, \tag{4}$$

where c^{NOM} is the equilibrium travel cost of the unmanaged bottleneck (i.e., $s^{\text{fast}} = 0$) with $N_{\texttt{fix}}$ commuters [30].

To complete the definition of (1), we now derive $\psi[o \mid t,b](d,\pi)$. We define a threshold bid $b^*[t]$ such that

- if $b > b^*[t]$, the commuter enters the fast lane for sure, i.e., $o = \text{fast}$;
- if $b < b^*[t]$, the commuter enters the slow lane for sure, i.e., $o = \text{slow}$;
- if $b = b^*[t]$, the commuter ties with others and enters the fast lane via a random draw on the remaining capacity.

Let $\nu[t,b](d,\pi)$ be the mass of commuters departing at t and bidding b, i.e.,

$$\nu[t,b](d,\pi) = \sum_{\tau,u,k} d_\tau[u,k]\, \pi_\tau[t,b \mid u,k]. \tag{5}$$

Then, the threshold bid is given by

$$b^*[t](d,\pi) = \max\left\{ b \in \mathbb{N} \;\middle|\; \sum_{b' \geq b} \nu[t,b'] \geq \frac{s^{\text{fast}}}{N} \right\}. \tag{6}$$

Note that the ratio s^{fast}/N in (6) is taken with respect to all players N (including those that are not travelling) since ν is defined relative to the total population mass. Accordingly, the probability of entering the fast lane is derived as

$$\psi[o = \text{fast} \mid t,b](d,\pi) = \begin{cases} 1, & b > b^*, \\ 0, & b < b^*, \\ \frac{s_{\text{fast}}/N - \sum_{b' > b^*} \nu[t,b']}{\nu[t,b]}, & b = b^*. \end{cases} \tag{7}$$

Note that $\psi[o = \text{fast} \mid t,b](d,\pi)$ is continuous in (d,π) except where $\nu[t,b](d,\pi) = 0$. To guarantee the existence of a SNE, see Section II-C, it can be approximated with a function that is continuous in (d,π) everywhere, given by

$$\begin{aligned} &\psi^\epsilon[o = \text{fast} \mid t,b](d,\pi) \\ &= \begin{cases} 1, & \sum_{b' > b} \nu[t,b'] \leq \frac{s^{\text{fast}}}{N} - \nu[t,b] - \epsilon, \\ 0, & \sum_{b' > b} \nu[t,b'] \geq \frac{s^{\text{fast}}}{N}, \\ \frac{s^{\text{fast}}/N - \sum_{b' > b} \nu[t,b']}{\nu[t,b] + \epsilon}, & \text{otherwise}, \end{cases} \end{aligned} \tag{8}$$

where $\epsilon > 0$ is an arbitrarily small approximation parameter.

2) Karma transition function $\kappa_\tau[k^+ \mid k,a](d,\pi)$: The karma transition function encodes the protocol by which karma is exchanged among the commuters. It gives the probability that a commuter of type τ transitions from karma level k to k^+ after playing action a, as a function of the social state (d,π). Notably, there is a considerable degree of freedom for policy makers to design the karma exchange protocol, as long as it preserves the average amount of karma in the system, see Section II-C. The effect of the karma exchange protocol on the equilibrium traffic allocation is complex, and our modelling framework is flexible enough to investigate a plethora of possibilities. For the sake of exposition, in this paper we consider a simple scheme where all commuters entering the fast lane pay their bids, and, at the end of each day, the total payments are uniformly redistributed to all users in the system (including the elastic commuters that did not travel). Then, the *average payment* to be redistributed to all commuters is given by

$$\bar{p}(d,\pi) = \sum_{t,b} \nu[t,b]\, \psi^\epsilon[o = \text{fast} \mid t,b]\, b. \tag{9}$$

To preserve the integer value of karma, $\lceil \bar{p}(d,\pi) \rceil$ is randomly distributed to a fraction of $f(d,\pi) = \bar{p}(d,\pi) - \lfloor \bar{p}(d,\pi) \rfloor$ of the commuters, and $\lfloor \bar{p}(d,\pi) \rfloor$ to the others. This yields the following karma transition probabilities, conditional on the outcome o:

$$\mathbb{P}[k^+ \mid k,a,o] = \begin{cases} f, & o = \text{fast},\ k^+ = k - b + \lceil \bar{p} \rceil, \\ 1-f, & o = \text{fast},\ k^+ = k - b + \lfloor \bar{p} \rfloor, \\ f, & o \neq \text{fast},\ k^+ = k + \lceil \bar{p} \rceil, \\ 1-f, & o \neq \text{fast},\ k^+ = k + \lfloor \bar{p} \rfloor, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

Finally, we can construct the karma transition function as

$$\kappa_\tau[k^+ \mid k,a](d,\pi) = \sum_{o \in \mathcal{O}} \psi^\epsilon[o \mid a]\, \mathbb{P}[k^+ \mid k,a,o]. \tag{11}$$

C. Existence of Stationary Nash Equilibrium (SNE)

We introduce some notation before defining the equilibrium concept of the proposed DPG. Given the constituents of the coupled MDPs $\zeta_\tau(d,\pi)$ and $\kappa_\tau(d,\pi)$, we define the expected immediate rewards $R_\tau(d,\pi)$, state transition matrix $P_\tau(d,\pi)$, infinite horizon value function $V_\tau(d,\pi)$, and single-stage deviation rewards $Q_\tau(d,\pi)$ as

$$
\begin{aligned}
&R_\tau[u,k] = \sum_a \pi_\tau[a \mid u,k]\,\zeta_\tau[u,a],\\
&P_\tau[u^+,k^+ \mid u,k] = \phi_\tau[u^+ \mid u] \sum_a \pi_\tau[a \mid u,k]\,\kappa_\tau[k^+ \mid k,a],\\
&V_\tau[x] = R_\tau[x] + \delta \sum_{x^+} P_\tau[x^+ \mid x]\,V_\tau[x^+],\\
&Q_\tau[u,k,a] = \zeta_\tau[u,a]\\
&\qquad + \delta \sum_{u^+,k^+} \phi_\tau[u^+ \mid u]\,\kappa_\tau[k^+ \mid k,a]\,V_\tau[u^+,k^+].
\end{aligned}
$$

Note that V_τ is the Bellman recursion for the fixed policy π_τ, which is well-known to have a unique and continuous solution for $\delta \in [0,1)$, see [25, Lemma 1].

Definition 1. A social state (d^*,π^*) is a Stationary Nash Equilibrium (SNE) if, for all $\tau \in \Gamma$, $[u,k] \in \mathcal{U} \times \mathbb{N}$,

$$
d^*_\tau = P_\tau(d^*,\pi^*)^\top\, d^*_\tau, \tag{13a}
$$

$$
\pi^*_\tau[\cdot \mid u,k] \in \operatorname*{arg\,max}_{\sigma \in \Delta(\mathcal{A}_\tau[k])} \sigma^\top Q_\tau[u,k,\cdot]. \tag{13b}
$$

At a SNE (d^*,π^*), d^* is stationary under the dynamics induced by π^* (13a), and π^*_τ is optimal for each type τ's MDP (13b). The existence of a SNE was shown for general karma economies for resource allocation in [25] and specialized to the bottleneck model in [26]. The main technical difficulty lies in that the karma state k belongs to the countably infinite space $\mathbb{N}$. In addition to the standard assumption of continuity of $\zeta_\tau(d,\pi)$ and $\kappa_\tau(d,\pi)$ in (d,π) (which can be guaranteed by the continuous approximation ψ^ϵ in (8)), this requires that $\kappa_\tau(d,\pi)$ preserves the average amount of karma in the system.

Assumption 1. Karma is preserved in expectation for all (d,π), i.e., $\mathbb{E}[k^+] = \mathbb{E}[k]$, which expands to

$$
\sum_{\tau,u,k} d_\tau[u,k] \sum_a \pi_\tau[a \mid u,k] \sum_{k^+} \kappa_\tau[k^+ \mid k,a]\,k^+ = \sum_{\tau,u,k} d_\tau[u,k]\,k.
$$

Note that under the presence of elastic commuters that do not always participate in the game, the karma held by the fixed commuters need not be preserved. Nonetheless, since our formulation incorporates the elastic commuters as a sub-population, we can guarantee that the average karma over the whole population (both fixed and elastic commuters) is preserved.

Proposition 1. *The karma transition function* (11) *satisfies Assumption 1, i.e., karma is preserved in expectation.*

The proof is analogous to [26, Proposition 1] and is omitted for brevity. The following is then immediate from [25, Theorem 1].

Theorem 1. *For every average karma level $\bar{k} \in \mathbb{N}$, a SNE (d^*,π^*) satisfying $\sum_{\tau,u,k} d^*_\tau[u,k]\,k = \bar{k}$ is guaranteed to exist in* `CARMA` *with elastic commuters.*

The proof relies on Assumption 1 to guarantee that the set of state distributions $\mathcal{D}$ is compact, which enables invoking an infinite-dimensional version of Kakutani's fixed point theorem, see [25]. An algorithm for computing SNE of dynamic population games is developed in [18], [25], which is based on showing an equivalency to a standard Nash equilibrium of a suitably defined (static) population game and employing standard evolutionary dynamics [24] to compute the equilibrium. In what follows, we utilize these tools to illustrate how the dynamic karma mechanism performs at the SNE in numerical case studies.

III. Numerical Analysis of Rebound Effect

In this section, we perform numerical computations of the SNE under varying number of $N_{\texttt{elast}}$ in order to shed light on the performance of the karma-based bottleneck congestion management scheme. We consider two cases: in the former, both fixed and elastic commuters belong to the same income class, and in the latter, the elastic commuters belong to a higher income class than the fixed commuters. We compare our results to those obtained under no policy intervention (denoted by "`NOM`") as well as the Vickrey-optimal monetary tolling [1] of the fast lane (denoted by "`TOLL`"). This serves as an example to illustrate the benefit of our dynamic karma scheme, and to showcase how our model can assist in making informed policy decisions.

A. Performance measures and benchmarks

As performance measures, we consider the average travel cost at the equilibrium traffic assignment, both at the commuter-type level (measure of *fairness*) and the overall system level (measure of *efficiency*). Let $\mathbb{P}_\tau[u]$ be the probability that type τ commuters have VOT level u, which is given by the stationary distribution of ϕ_τ, and let $\bar{u}_\tau = \sum_u \mathbb{P}_\tau[u]\,u$ be the average VOT of type τ. Empirical evidence has shown a strong correlation between the (monetary) VOT and the income of the commuters [31], [32]. Thus, we consider $\bar{u}_\tau$ to represent the *income class* of type τ. To define income-invariant performance metrics, we express costs on a scale normalized by $\bar{u}_\tau$ respectively for each τ. Accordingly, let $\bar{c}_\tau$ be the (normalized) average travel cost of type τ, and $\bar{c} = g_{\texttt{fix}}\,\bar{c}_{\texttt{fix}} + g_{\texttt{elast}}\,\bar{c}_{\texttt{elast}}$ be the system-level (normalized) average travel cost. Table I gives the expression of $\bar{c}_\tau$ for both benchmarks `NOM` and `TOLL` as well as the karma-based scheme `CARMA`. In the equilibrium of `NOM`, all elastic commuters prefer to not travel regardless of u since $c^{\text{other}} \leq c^{\text{NOM}}$. The equilibrium of `TOLL` is determined by the distribution of the immediate VOT in the population, namely, higher VOT commuters occupy the fast lane closer to the desired arrival time t^*

TABLE I: Average travel cost per commuter type.

	Fixed ($\bar{c}_{\mathtt{fix}}$)	Elastic ($\bar{c}_{\mathtt{elast}}$)
No intervention (“NOM”)	c^{NOM}	c^{other}
Optimal tolling (“TOLL”)	$\frac{1}{\bar{u}_\tau} \sum_u \mathbb{P}_\tau[u]\, c_\tau^{\mathtt{TOLL}}[u]$	
Karma scheme (“CARMA”)*	$-\frac{1}{\bar{u}_\tau\, g_\tau} \sum_{u,k} d^*_\tau[u,k]\, R_\tau[u,k]$	
*computed at the SNE (d^*, π^*).		

TABLE II: Default values of model parameters.

Name	Notation	Unit	Value
Number of fixed commuters	$N_{\mathtt{fix}}$		9000
Bottleneck capacity	s	veh/min	60 (100%)
- Fast lane	s_{fast}		24 (40%)
- Slow lane	s_{slow}		36 (60%)
Length of discrete time step	Δt	min	15
Normalized VOT		cost/hour*	
- queuing delay	α		6.4
- early arrival	β		4
- late arrival	γ		16
Nominal equilibrium cost	c^{NOM}		8
Cost of elastic alternative	c^{other}		7.2
Desired arrival time	t^*	min	120
Discount factor	δ		0.99
Parameter for model continuity	ϵ		10^{-4}
Average karma per commuter	$\bar{k}$		10
*in TOLL the unit is \$/hour.			

as they can afford to pay higher tolls. The resulting average travel cost of commuters of type τ and VOT u, denoted by $c_\tau^{\mathtt{TOLL}}[u]$, was derived in [26] following basic principles of the classical bottleneck model [30] under no elastic demand. In the presence of elastic commuters, those commuters will have an incentive to use the fast lane at all times t for which $u \max\{\beta\,(t^*-t), \gamma\,(t-t^*)\} + p(t) \leq u\, c^{\text{other}}$, where $p(t)$ denotes the toll price. This could affect the equilibrium by shifting more fixed commuters to the slow lane. The exact derivation of the mixed-population tolling equilibrium is omitted for brevity as it follows similar principles to [26].

B. Results and discussion

The default values of the model parameters used in our numerical investigation are reported in Table II. For the number of elastic commuters $N_{\mathtt{elast}}$, we perform a sweep with values lying in the grid $\{0, 0.1, \dots, 0.9, 1\}\, N_{\mathtt{fix}}$. Moreover, we consider the following two cases:

1) *Homogeneous income:* both fixed and elastic commuters have the same independent and identically distributed (i.i.d.) VOT process with $u \in \mathcal{U} = \{u^{\text{low}}, u^{\text{high}}\}$, $u^{\text{low}} = 1$, $u^{\text{high}} = 2$, $\mathbb{P}[u^{\text{low}}] = 0.8$, and $\mathbb{P}[u^{\text{high}}] = 0.2$;
2) *Heterogeneous income:* fixed commuters have the same VOT process as in Case 1, whereas the elastic commuters’ process is characterized by $u^{\text{low}} = 3$, $u^{\text{high}} = 6$, and the same parameters otherwise. This translates to elastic commuters having three times the income level of the fixed commuters.

Figure 2 shows the results attained in the above two cases. In the homogeneous income case (Figure 2a–2b), both CARMA and TOLL achieve qualitatively similar efficiency in terms of the system-level average travel costs, see Figure 2b. However, there are considerable differences when investigating the average costs per type, see Figure 2a. From the fixed commuters’ point of view, CARMA is less sensitive than TOLL to low penetration of elastic demand, as illustrated by the more gradual slope for $N_{\mathtt{elast}} \leq 0.4\, N_{\mathtt{fix}}$. The trend reverses, however, for $N_{\mathtt{elast}} \geq 0.6\, N_{\mathtt{fix}}$. In this regime, a sharp rise in the average travel cost of fixed commuters is observed in CARMA, while it remains constant in TOLL. The former occurs because in CARMA, elastic commuters bid to enter the fast lane less often than fixed commuters, meanwhile commuters of both types receive the same karma redistribution. This has a minor effect at low penetration of elastic demand but eventually leads to elastic commuters holding a higher share of the system karma and degrading the performance for fixed commuters. A possible countermeasure is to redistribute less karma to commuters that do not travel, which can be achieved through a suitable design of $\kappa_\tau(d,\pi)$, see Section II-B.2. The latter occurs because in TOLL, the slow lane eventually becomes costly to the extent that low VOT fixed commuters are willing to pay more toll to enter the fast lane than additional high VOT elastic commuters.

Despite the degradation of fixed commuter costs in CARMA at high penetration of elastic demand, a detrimental rebound effect does not manifest in the homogeneous income case, where both types of commuters experience strict benefits with respect to NOM in CARMA and TOLL. This is however not the case when considering the heterogeneous income case (Figure 2c–2d), where the higher income elastic commuters experience serious benefits in comparison to the fixed commuters in TOLL. Since the elastic commuters can afford higher toll prices, they quickly occupy the fast lane to the extent that fixed commuters experience *higher* average travel costs than if no policy was in place for $N_{\mathtt{elast}} \geq 0.2\, N_{\mathtt{fix}}$, see Figure 2c. In contrast, CARMA is completely invariant to income, since $\bar{u}_\tau$ only introduces a constant scaling of the costs in the dynamic optimization of the commuters. This also implies that CARMA is *more efficient* than TOLL with respect to the income-normalized average travel costs $\bar{c}$ for the wide range of $0.1 \leq N_{\mathtt{elast}}/N_{\mathtt{fix}} \leq 0.7$, see Figure 2d. The peak in $\bar{c}$ for TOLL coincides with when no additional elastic commuters find it benefitial to use the fast lane at the equilibrium toll prices.

IV. Conclusions

We demonstrated that dynamic models allow devising new classes of fair congestion mitigation schemes by formulating a non-monetary karma economy for bottleneck congestion management as a Dynamic Population Game (DPG). Our model incorporates the presence of an elastic demand of

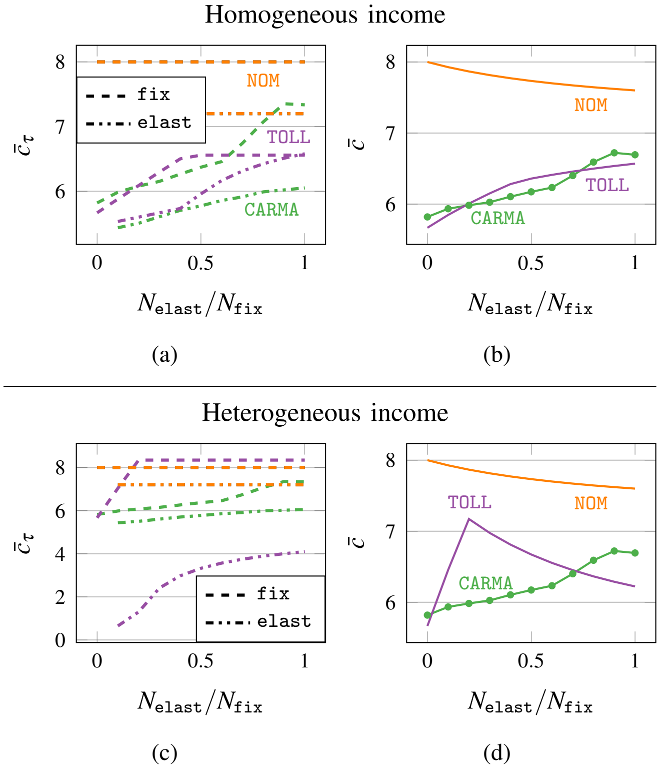

Fig. 2: Performance measures plotted against the ratio of elastic to fixed demand for the cases of homogeneous income and heterogeneous income.

commuters that only travel if congestion is reduced. We guarantee the existence of a Stationary Nash Equilibrium (SNE) through a suitable design of the karma exchange protocol which preserves the amount of karma in the system. Through numerical analysis we illustrate the benefit of our dynamic karma scheme in comparison to a static monetary tolling scheme. Namely, the karma scheme is invariant to income heterogeneity, and robust against a detrimental rebound effect manifesting when high income elastic commuters cause low income fixed commuters to experience worse congestion than if the bottleneck is left uncontrolled.

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