2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

# A Blending Based Multiple Model Reference Adaptive Approach to Lateral Vehicle Motion Control

Alex Lovi[1] Barış Fidan[2] Christopher Nielsen[1]

***Abstract*— This paper studies reference tracking control of uncertain lateral vehicle dynamics, using a blending based multiple-model reference adaptive control (MMRAC) approach to overcome the parametric uncertainties and time-variations, including those in the tire force capacities and cornering stiffness. The lateral vehicle dynamics model under consideration is multiple-input, multiple-output, linear, and parameter varying. The design will assume a time-invariant system, such that all uncertain parameter variations lie inside of a known, compact, and convex set. The proposed MMRAC law guarantees perfect tracking of the desired state values generated by a linear reference model representing ideal driving conditions, and the system parameter estimates asymptotically converge to the unknown true values. We present simulations to show the stability and effectiveness of the proposed MMRAC scheme, even in the presence of slow time variations, as well as a performance comparison with existing lateral vehicle motion controllers.**

***Index Terms*— Multiple model, Adaptive control, Polytopic uncertainty, Model reference adaptive control, Lateral vehicle dynamics.**

## I. Introduction

On average 94% of car crashes are due to human error [1]. Moreover, 39% of crash-related fatalities in the US are due to lane departures [2]. Such statistics exhibits the importance of improving the safety standards of the systems that intervene in the steering of vehicles. When it comes to lateral vehicle dynamics, research mainly splits into lane departure warning systems [3], [4], lane keeping systems [5], [6], and yaw stability control systems [7], [8]. In this paper, we consider the problem of road accidents due to excessive side-slip of vehicles due to road conditions by applying a novel controller that improves the performance of the closed-loop.

Extensive research has been proposed, and implemented in the literature to mitigate excessive side-slip of vehicles on roads, and to restore the yaw velocity of the vehicle to the nominal motion expected by the driver. In [9], a model reference adaptive controller (MRAC) is developed to improve the vehicle's manoeuvrability, and a nonlinear MRAC is designed to compensate the disturbances. In [10], a fast identification technique using Kalman filters was developed based on available signals to identify the road friction coefficients, and evaluated on single and double-lane maneuvers. Studies on disturbance rejection when applying direct yaw moment control to autonomous vehicles can be found in [11]. In [12] a model predictive controller is applied to improve the yaw stability of a rear-wheel-drive vehicle with an electronic limited slip differential. In [13], a multiple model adaptive parameter identification scheme is combined with optimal controllers to achieve fast adaptation of the lateral motion control to changing road conditions. In [14] an MRAC scheme is applied to the problem of active front steering and direct yaw moment control that can overcome variations of the vehicle mass and tire-road friction coefficients.

Using multiple models in the control design of uncertain systems has been proven to achieve faster convergence of parameters and better closed-loop performance than single model approaches [15]–[18]. Considering a system that works in different dynamic regimes, multiple-model techniques allows the selection of the *best* model (the one that minimizes an error signal) [19], [20] or using the information of all the models to obtain a unique description of the system [13], [17], [21], [22].

In this paper we use multiple model reference adaptive control (MMRAC) with blending developed in [23] to the problem of lateral vehicle motion control. The controller generates a continuous input signal calculated using the identification errors from all the models, similar to [15], [24]. First, we perform a numeric study of the convergence speed for the control scheme. Second, we verify the performance of the MMRAC compared with MRAC, and an optimal controller LQR when we consider an unknown *constant* plant. Finally, we compare the MMRAC with MRAC and LQR when there is an explicit slow time variation of the parameters, without reinitialization. The lateral vehicle dynamics problem of interest is presented in Section II. In Section III, we present our blending based MMRAC approach to this problem, along with its generic design procedure and stability and convergence properties. In Section IV we design the controller for the lateral vehicle dynamics application. A set of numeric simulations are presented comparing lateral vehicle lane maneuver control with the studied blending based MMRAC scheme to MRAC and LQR in Section V. Concluding remarks are given in Section VI.

*Notation:* For a set $S \subset \mathbb{R}^n$, $\operatorname{conv}(S)$ denotes the closed convex hull of $S$, the interior of $S$ is written $\operatorname{int}(S)$. For two signals $f_1$ and $f_2$ we write $f_1 \overset{\lambda}{\doteq} f_2$ if there exists $\lambda > 0$ such that $e_f = f_1 - f_2$ satisfies $\dot{e}_f = -\lambda e_f$. For a vector $x \in \mathbb{R}^N$, $x_i$ denotes the $i$-th component, and $\bar{x} = [x_1, \cdots, x_{N-1}]^\top \in \mathbb{R}^{N-1}$. A signal $\psi : [0, \infty) \to \mathbb{R}^k$

This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

[1] Department of Electrical and Computer Engineering, University of Waterloo, Waterloo ON, N2L 3G1 Canada. {alovi;cnielsen}@uwaterloo.ca

[2] Department of Mechanical and Mechatronics Engineering, University of Waterloo, Waterloo ON, N2L 3G1 Canada. {fidan}@uwaterloo.ca

is persistently exciting (PE) if there exist $\alpha_{\Phi 1}, \alpha_{\Phi 2}, T > 0$ such that for every $t \geq 0$

$$\alpha_{\Phi 1}\mathbb{I} \preceq \int_{t}^{t+T} \Phi(\tau)\Phi^{\top}(\tau)\mathrm{d}\tau \preceq \alpha_{\Phi 2}\mathbb{I}. \tag{1}$$

## II. Lateral Vehicle Dynamics Problem

In this section, we provide the mathematical model of lateral vehicle dynamics of interest. We make the following widely used modeling simplifying assumptions [2]:

- All forces act on a plane flat road. The effect of the altitude of the center of gravity is neglected.
- The equations of motion are linearized. The tire force is assumed proportional to the slip angle.

Under these assumptions, the free-body diagram of the lateral vehicle dynamics using a bicycle model is presented in Figure 1. The system is represented by the state-space model

$$\dot{x}_p(t) = A_p(\eta)x_p(t) + B_p(\eta)u(t),\ x_p(0) = x_{p0}, \tag{2}$$

with state $x_p(t) \coloneqq \begin{bmatrix}\beta(t) & \dot{\psi}(t)\end{bmatrix}^{\top} \in \mathbb{R}^2$ and input $u(t) \coloneqq \begin{bmatrix}\delta(t) & M_z(t)\end{bmatrix}^{\top} \in \mathbb{R}^2$, where $\beta$ is the side-slip angle, $r$ is the yaw rate, $M_z$ is the yaw moment about the $z$-axis, $\delta = \delta_d + \delta_c$ is the total steering angle of the front wheels resulting from addition of the driver command $\delta_d$ and the corrective steering control input $\delta_c$. The system matrices $A_p(\eta) \in \mathbb{R}^{2\times 2}$ and $B_p(\eta) \in \mathbb{R}^{2\times 2}$ are unknown due to uncertain variations in cornering stiffness and longitudinal force capacities of the tires and are parameterized in the form

$$A_p(\eta) = A_{p1} + \eta_f A_{p2} + \eta_r A_{p3}, \tag{3}$$

$$B_p(\eta) = \eta_f B_{p1} + \eta_x B_{p2}, \tag{4}$$

$$A_{p1} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, A_{p2} = \begin{bmatrix} \dfrac{-C_f}{mv_x} & \dfrac{l_f C_f}{mv_x^2} \\ \dfrac{-l_f C_f}{I_z} & \dfrac{-l_f^2 C_f}{I_z v_x} \end{bmatrix},$$

$$A_{p3} = \begin{bmatrix} \dfrac{-C_r}{mv_x} & \dfrac{l_r C_r}{mv_x^2} \\ \dfrac{l_r C_r}{I_z} & \dfrac{-l_r^2 C_r}{I_z v_x} \end{bmatrix}, \tag{5}$$

$$B_{p1} = \begin{bmatrix} \dfrac{C_f}{mv_x} & 0 \\ \dfrac{l_f C_f}{I_z} & 0 \end{bmatrix}, B_{p2} = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{I_z} \end{bmatrix},$$

where $m$ is the mass of the vehicle, $I_z$ is the moment of inertia about the $z$-axis, $v_x$ is the longitudinal velocity, $l_f$ and $l_r$ are the distance from the center of gravity to the front and rear wheels, respectively, $C_f$ and $C_r$ are the cornering stiffness coefficients for the front and rear tires, respectively. The entries of $\eta = \begin{bmatrix}\eta_f & \eta_r & \eta_x\end{bmatrix}^{\top}$, in order, parameterize the uncertainties in front tire cornering stiffness, rear tire cornering stiffness, and longitudinal tire force capacity, where $\eta = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\top}$ represents the nominal model case. The upper and lower bounds of the uncertainty parameters $\eta_i$ for $i \in \{f, r, x\}$ are assumed to be known in the form

$$\eta_{i\,\min} \leq \eta_i \leq \eta_{i\,\max}, \tag{6}$$

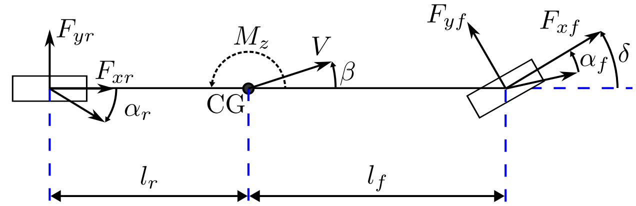


Fig. 1. Free-body diagram for lateral vehicle dynamics using a bicycle model.

where $0 < \eta_{i\,\min} < 1$ and $\eta_{i\,\max} > 1$. These parameter ranges correspond to the polytope generated by taking the minimum and maximum values of every entry of the matrices $A_p$ and $B_p$.

Under these assumptions, the model is fit to accurately describe the dynamics under normal driving conditions [25]. The control objective is to design a state-feedback controller such that the closed-loop plant state $x_p$ asymptotically tracks the desired state $x_r$ generated by the reference model

$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t),\ x_r(0) = x_{r0}, \tag{7}$$

which represents ideal driving behavior on optimal dry road conditions. The constant matrices $A_r \in \mathbb{R}^{2\times 2}$ and $B_r \in \mathbb{R}^{2\times 2}$ are known, $A_r$ is Hurwitz, and $r : [0, \infty) \to \mathbb{R}^2$ is a known, bounded, piece-wise continuous input to the reference model. For the case of lateral vehicle dynamics, we expect the car to achieve the yaw rate, with the minimum possible side-slip, i.e., we expect the car to behave as if the road had a very high friction coefficient.

## III. Multiple Model Reference Adaptive Controller

In this section, we provide the details to design a blending based MMRAC scheme [23] to track the states generated by a linear model for the lateral vehicle dynamics problem.

### A. The General Multiple-Model Reference Control Problem

Consider the MIMO LTI system (2) and the linear reference model (7). We assume the exact matching conditions [26], as stated in the following assumptions.

**Assumption 1.** *There exist matrices $K^* \in \mathbb{R}^{2\times 2}$ and $L^* \in \mathbb{R}^{2\times 2}$ such that*

$$A_p + B_p K^* = A_r, \tag{8a}$$

$$B_p L^* = B_r. \tag{8b}$$

Moreover, it is assumed that the uncertain system is contained inside of a polytope generated by $N$ fixed plant models with system matrices $A_i$, $B_i$, satisfying the following assumption.

**Assumption 2.** *There exist $N \in \mathbb{N}$ known matrices $A_i \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times 2}$, $i \in \{1, \cdots, N\}$, such that $\begin{bmatrix}A_p & B_p\end{bmatrix} \in \operatorname{int}\left(\operatorname{conv}\left\{\begin{bmatrix}A_i & B_i\end{bmatrix} : i \in \{1, \cdots, N\}\right\}\right)$, and every convex combination of the $B_i$ matrices is invertible.*

Assumption 2 implies the existence of the non-empty set

$$\mathcal{W} := \left\{ w \in [0,1]^N : [A_p \ B_p] = \sum_{i=1}^{N} w_i \left[A_i \ B_i\right], \sum_{i=1}^{N} w_i = 1 \right\}, \tag{9}$$

and means that the problem of identifying the matrices $A_p$ and $B_p$ is equivalent to identifying a weight vector $w \in \mathcal{W}$. Another consequence of Assumption 2 is that there exist matrices $K_i \in \mathbb{R}^{2\times 2}$ and $L_i \in \mathbb{R}^{2\times 2}$ such that, for all $i \in \{1, \cdots, N\}$,

$$A_i + B_i K_i = A_r, \tag{10a}$$

$$B_i L_i = B_r. \tag{10b}$$

Given this structure, the MMRAC consists of two parts. First, a parameter identification scheme that generates estimates $\hat{A}_p(t)$ and $\hat{B}_p(t)$ of $A_p$ and $B_p$, respectively, as a weighted sum of the estimates generated by $N$ estimators based on the $N$ fixed models. Second, generation of a control input based on the estimated parameters developed in the identification part. In the following subsections we present details of our multiple model parameter identification and MMRAC scheme designs for the specific problem setting in Section II.

### B. Multiple-Model Parameter Identifier Design

Filtering both sides of (2) by the linear filter $\dfrac{1}{s+\lambda}$, where $\lambda > 0$ is a design parameter, we obtain the parametric model

$$z(t) \overset{\lambda}{=} \Theta_p \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} =: \Theta_p \Phi(t), \tag{11}$$

where $z(t), \phi_1(t), \phi_2(t) \in \mathbb{R}^2$ are generated by the filters

$$\begin{bmatrix} \dot{\phi}_1(t) \\ \dot{\phi}_2(t) \end{bmatrix} = -\lambda \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} + \begin{bmatrix} x_p(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$z(t) = \dot{\phi}_1(t) = -\lambda \phi_1(t) + x_p(t). \tag{12}$$

For each of the fixed models, define

$$z_i(t) := \Theta_i \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \Theta_i \Phi(t), \ \forall i \in \{1, \cdots, N\}. \tag{13}$$

Then, the filtered state estimation error for each of the $N$ fixed models is defined as

$$\varepsilon_i(t) := z(t) - z_i(t), \forall i \in \{1, \cdots, N\}. \tag{14}$$

For any $w \in \mathcal{W}$, (11) is related to (13) via

$$z(t) = \sum_{i=1}^{N} w_i z_i(t) + e_z(t).$$

Hence, using $\sum_{i=1}^{N} w_i = 1$ we have

$$\sum_{i=1}^{N} w_i z(t) - \sum_{i=1}^{N} w_i z_i(t) = \sum_{i=1}^{N} w_i \varepsilon_i(t) = e_z(t). \tag{15}$$

Subtracting $\varepsilon_N(t)$ from both sides of (15), we obtain

$$\sum_{i=1}^{N-1} w_i \left(\varepsilon_i(t) - \varepsilon_N(t)\right) = e_z(t) - \varepsilon_N(t). \tag{16}$$

Defining the $2 \times (N-1)$ time-varying matrix

$$E(t) := \left[\varepsilon_1(t) - \varepsilon_N(t) \quad \cdots \quad \varepsilon_{N-1}(t) - \varepsilon_N(t)\right], \tag{17}$$

we can rewrite (16) in matrix form as

$$E(t)\bar{w} = e_z(t) - \varepsilon_N(t). \tag{18}$$

Defining the compact set

$$\Pi := \left\{ \hat{\bar{w}} \in [0,1]^{N-1} : \sum_{i=1}^{N-1} \hat{\bar{w}}_i \leq 1 \right\}, \tag{19}$$

let $\mathrm{Pr}_{\Pi,\hat{\bar{w}}} : \mathbb{R}^{N-1} \to \Pi \subset \mathbb{R}^{N-1}$ denote the parameter projection operator used to keep $\hat{\bar{w}}$ within $\Pi$ [27]: If $\hat{\bar{w}}$ is on the boundary $\partial(\Pi)$ of $\Pi$ and $v \in \mathbb{R}^{N-1}$ points to outside of $\Pi$ then $\mathrm{Pr}_{\Pi,\hat{\bar{w}}}(v)$ is equal to the projection of $v$ on the tangent plane of $\partial(\Pi)$ of $\Pi$ at $\hat{\bar{w}}$; otherwise $\mathrm{Pr}_{\Pi,\hat{\bar{w}}}(v) = v$. With $\hat{\bar{w}}(0) \in \mathrm{int}(\Pi)$, the following adaptive law based on Equation (18) is used to generate the weight vector estimate $\hat{w}(t)$:

$$\dot{\hat{\bar{w}}}(t) = \mathrm{Pr}_{\Pi,\hat{\bar{w}}}\left(-\Gamma\left(E^\top(t) E(t) \hat{\bar{w}}(t) + E^\top(t)\varepsilon_N(t)\right)\right),$$

$$\hat{w}_N(t) = 1 - \sum_{i=1}^{N-1} \hat{w}_i(t), \tag{20}$$

where $\Gamma \in \mathbb{R}^{(N-1)\times(N-1)}$ is a preset symmetric positive definite matrix, which tunes the convergence speed. We establish the following proposition along Theorem 1 and Corollary 1 of [28]:

**Proposition 1.** *Consider the system* (2)*. Under Assumption 2, and assuming that $\Phi(t)$ is bounded and PE, the estimation scheme* (20) *guarantees that for any initial estimate $\hat{\bar{w}}(0)$, $\sum_{i=1}^{N} \hat{w}_i(t)\Theta_i$ asymptotically converges to $\Theta_p$.*

The complete analysis can be found in the proof of Theorem 2 of [23].

### C. Multiple Model Reference Adaptive Control Design

Let $w \in \mathcal{W}$, then multiplying both sides of (10b) by $w_i$ and summing over $i$ yields

$$\sum_{i=1}^{N} w_i B_i L_i = \sum_{i=1}^{N} w_i B_r = B_r,$$

which implies, together with (8b) from Assumption 1, that

$$L^* = B_p^{-1} \sum_{i=1}^{N} w_i B_i L_i. \tag{21}$$

Applying the same steps on (10a) we get

$$A_r = \sum_{i=1}^{N} w_i A_r = A_p + \sum_{i=1}^{N} w_i B_i K_i. \tag{22}$$

Comparing (22) to (8a) we get that

$$K^* = B_p^{-1} \sum_{i=1}^{N} w_i B_i K_i. \tag{23}$$

Equations (21) and (23) motive us to generate estimates of the gains $K^*$ and $L^*$ using the estimates $\hat{w}(t)$, keeping in mind the invertibility supposition in Assumption 2, as

$$\hat{K}(t) = \hat{B}_p^{-1}(t)\sum_{i=1}^{N}\hat{w}_i(t)\,B_iK_i, \tag{24}$$

$$\hat{L}(t) = \hat{B}_p^{-1}(t)\sum_{i=1}^{N}\hat{w}_i(t)\,B_iL_i. \tag{25}$$

The control law we will consider to achieve asymptotic tracking of the reference model is

$$u(t) := \hat{K}(t)\,x_p(t) + \hat{L}(t)\,r(t). \tag{26}$$

We have the following proposition for the closed loop system:

**Proposition 2.** *Consider the plant (2) and the reference model (7). Under Assumptions 1 and 2, and assuming that $\Phi(t)$ is PE, the control law (26), with the adaptation scheme (20) guarantees that for any initial state $x_{p0}$ of the plant (2), and any initial state $x_{r0}$ and any piecewise continuous and bounded reference signal $r : [0,\infty) \to \mathbb{R}^2$ in the reference model (7), all closed-loop signals are bounded and $x_p(t)$ asymptotically converges to $x_r(t)$.*

The complete analysis can be found in the proof of Theorem 3 of [23].

## IV. Application of MMRAC to Lateral Vehicle Motion Control

In this section we verify that the model (2) satisfies Assumptions 1 and 2 from Section III for any physically meaningful values of the plant parameters, and finally we give an expression for the complete controller.

In order to verify Assumption 1 we need to be able to invert the matrix $B_p$. Re-writing (4) with the definitions from (5) we obtain

$$B_p(\eta) = \eta_f \begin{bmatrix} \dfrac{C_f}{mv_x} & 0 \\ \dfrac{l_fC_f}{I_z} & 0 \end{bmatrix} + \eta_x \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{I_z} \end{bmatrix}. \tag{27}$$

Note that the parameters $C_f$, $m$, $v_x$, $l_f$, and $I_z$ need to be strictly greater than zero to be physically meaningful during forward motion. This implies that, since the values of $\eta_{f\,\min}$ and $\eta_{x\,\min}$ are strictly positive, the matrix $B_p$ will be lower triangular with positive values on the main diagonal, hence it will always have a positive determinant, which implies that it will always be invertible. With these considerations we can calculate $K^*$ and $L^*$ from (8a) and (8b), respectively.

To satisfy Assumption 2 we need to select the fixed models pairs $\begin{bmatrix} A_i & B_i \end{bmatrix}$, for $i \in \{1, \cdots, N\}$. We consider all possible combination of minimums and maximums of $\eta_f$, $\eta_r$, and $\eta_x$ to obtain eight corner models. This selections guarantees that $B_p$ is in the interior of some convex combination of the corner model, and Assumption 2 is satisfied. Note that this

TABLE I

Values used to run simulations.

| Symbol | Value | Meaning |
|---|---|---|
| $m$ | 1140 kg | Mass of the car |
| $I_z$ | 1020 kgm$^2$ | Moment of inertia about the yaw-axis |
| $l_f$ | 1.165 m | Distance from CG to the front wheel |
| $l_r$ | 1.165 m | Distance from CG to the rear wheel |
| $v_x$ | 100 m/s | Longitudinal velocity |
| $C_f$ | 86849 N/rad | Front tire cornering stiffness |
| $C_r$ | 90950 N/rad | Rear tire cornering stiffness |
| $\lambda$ | 20 | Linear filter design parameter |
| $\Gamma$ | $50\mathbb{I}_{7\times 7}$ | Symmetric matrix to tune convergence speed |

selection process does not minimize the number of corner models. The corner model pairs are as follows:

$$A_i = A_{p1} + \eta_{fi}A_{p2} + \eta_{ri}A_{p3}, \tag{28}$$

$$B_i = \eta_{fi}B_{p1} + \eta_{xi}B_{p2}, \tag{29}$$

where $i \in \{1, \cdots, 8\}$, and

$$\eta_{fi} = \begin{cases} \eta_{f\,\min}, & \text{if } i = \{1,3,5,7\} \\ \eta_{f\,\max}, & \text{if } i = \{2,4,6,8\}, \end{cases} \tag{30}$$

$$\eta_{ri} = \begin{cases} \eta_{r\,\min}, & \text{if } i = \{1,2,5,6\} \\ \eta_{r\,\max}, & \text{if } i = \{3,4,7,8\}, \end{cases} \tag{31}$$

$$\eta_{xi} = \begin{cases} \eta_{x\,\min}, & \text{if } i = \{1,2,3,4\} \\ \eta_{x\,\max}, & \text{if } i = \{5,6,7,8\}. \end{cases} \tag{32}$$

From (4), (5) and strictly positive $\eta_f$ and $\eta_x$ are, the determinant of $B_i$ is strictly greater than zero, for every $i \in \{1, \cdots, 8\}$. This implies that any convex combination of the $B_i$ matrices will have a strictly positive determinant, which further implies that any convex combination of the $B_i$ matrices is invertible, hence the proposed controller can be applied to the uncertain model of lateral vehicle dynamics.

Finally, we can calculate the gains $K_i$ and $L_i$, for $i \in \{1, 2 \cdots, 8\}$ as

$$K_i = B_i^{-1}(A_r - A_i), \tag{33}$$

$$L_i = B_i^{-1}B_r. \tag{34}$$

Selecting a value of $\lambda > 0$, and a positive definite matrix $\Gamma \in \mathbb{R}^{7\times 7}$, the full controller is described by Equations (12), (20) and (24)–(26), with $N = 8$.

## V. Simulations

In this section we present simulation results, and compare the performance of the MMRAC with other controllers. The parameters used to run the simulations are in Table I. Moreover, all the uncertainty of the system and the time-variations satisfies (6) with

$$\eta_{i\,\min} = 0.1,\ \eta_{i\,\max} = 1.3,\ i \in \{f, r, x\}. \tag{35}$$

With these values, we fixed the corner systems, and we track the states generated by the reference model defined by

$$A_r = \begin{bmatrix} -13.6 & 1.96 \\ 17 & -18.85 \end{bmatrix}, B_r = \begin{bmatrix} 6.8 & 0 \\ 124.67 & 0.001 \end{bmatrix}. \tag{36}$$

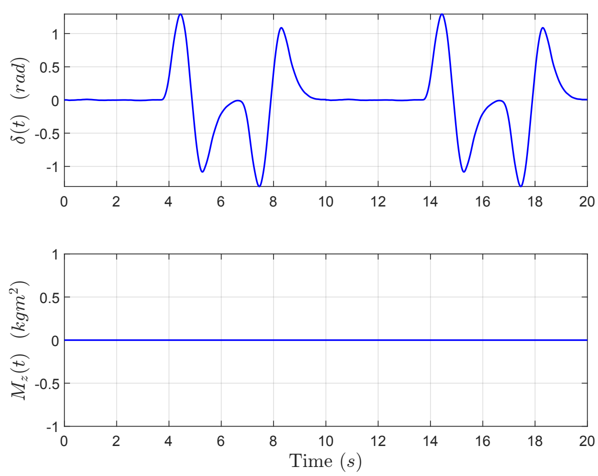


Fig. 2. $r(t)$ for the reference model.

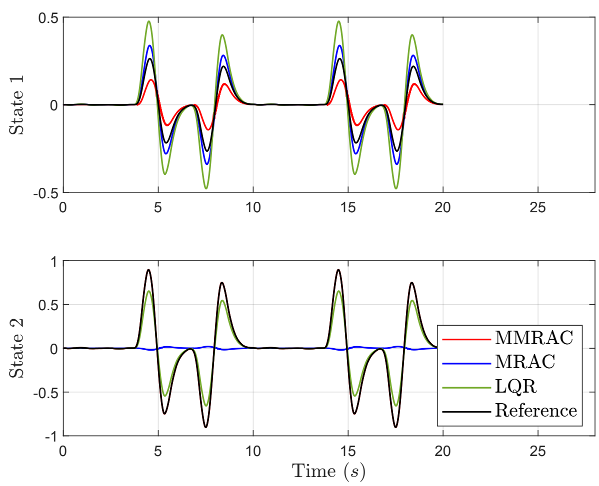


Fig. 3. States of the system for different control techniques assuming $\eta = [1 \quad 1 \quad 1]^\top$.

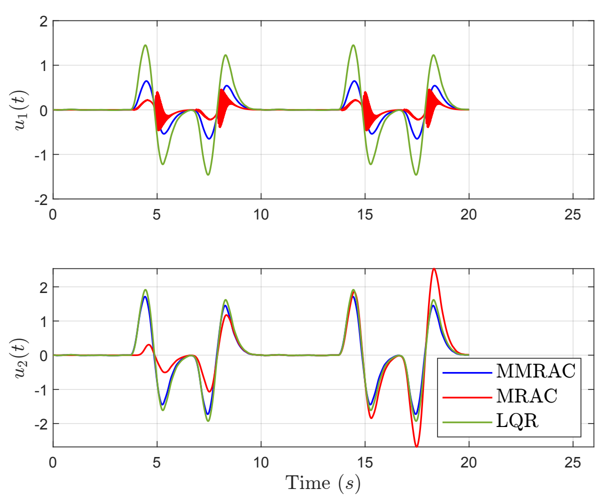


Fig. 4. Control effort of the different control techniques assuming $\eta = [1 \quad 1 \quad 1]^\top$.

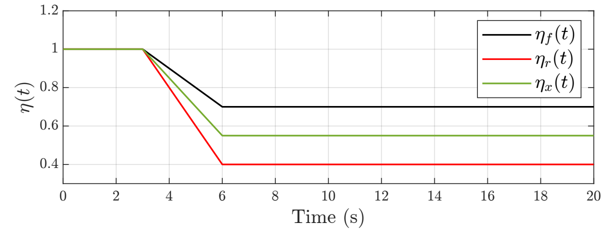


Fig. 5. Time-variation of $\eta(t)$.

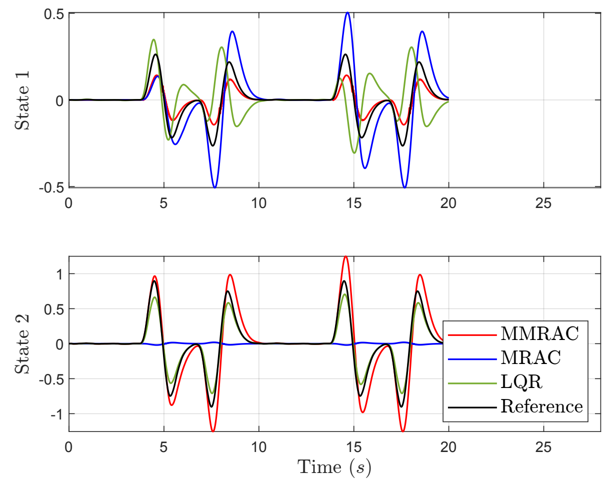


Fig. 6. States of the system for different control techniques assuming $\eta = \eta(t)$.

First, we present a set of simulations that assumes a constant unknown value of $\eta_f = \eta_r = \eta_x = 1$. We compare the proposed multiple-model technique with a single model reference adaptive controller, and a single model optimal LQR controller. The reference signal $r(t)$ shown in Figure 2 represents a change of lanes, and was obtained on CarSim. In Figure 3 we observe that MMRAC and MRAC are able to track the desired side-slip and LQR has the largest error. When we look at the yaw rate instead, we see that LQR and MMRAC are able to track the desired state, but not the MRAC control. As observed in Figure 4 the control efforts applied are comparable, which suggests that there is an advantage in using the proposed MMRAC scheme.

Finally, we present a set of simulations that assumes that the unknown parameter $\eta$ is slowly time-varying, and takes values contained in the set (35) as shown in Figure 5. Once again, we compare the results with an MRAC technique and an LQR controller. In Figure 6 we observe that the MMRAC scheme alone tracks the desired side-slip, and MMRAC and LQR are able to track the desired yaw rate, with similar control efforts, as shown in Figure 7. Simulations with time-varying parameters show clear advantages of using the proposed MMRAC scheme over single model controllers.

In the presence of process and measurement noise, all closed loop signals remain bounded. A more comprehensive example of the behavior under process and measurement noise can be found in [28].

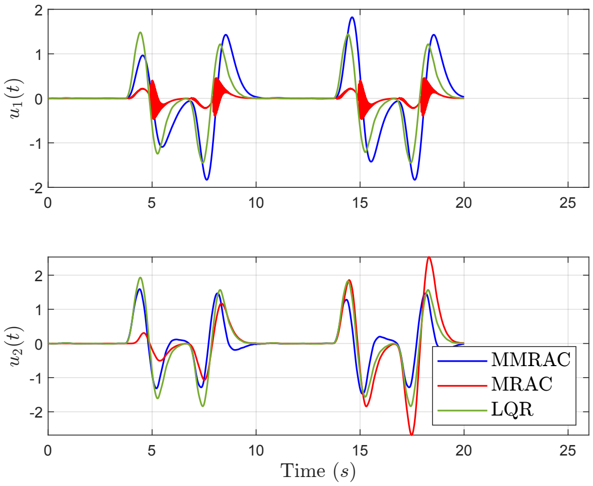


Fig. 7. Control effort of the different control techniques assuming $\eta = \eta(t)$.

## VI. Conclusions

In this article, we have applied an MMRAC scheme to reference tracking control of uncertain lateral vehicle dynamics. We have established that the conditions to apply MMRAC scheme are satisfied by the assumed lateral vehicle dynamics model. Simulations show that the tracking error goes to zero, even in the case of unknown slowly time-varying parameters. The presented simulations compare the MMRAC scheme with a single model MRAC controller, and an optimal LQR controller. The simulation results indicate a superior performance by the proposed method, most notably in the case of slowly time-varying parameters.

## References

[1] S. Singh, "Critical reasons for crashes investigated in the national motor vehicle crash causation survey," U.S. Department of Transportation, Tech. Rep., 2015.

[2] R. Rajamani, *Vehicle dynamics and control*, 2nd ed., ser. Mechanical engineering series. New York: Springer, 2012.

[3] W. Wang, D. Zhao, W. Han, and J. Xi, "A learning-based approach for lane departure warning systems with a personalized driver model," *IEEE Tr. on Vehicular Technology*, vol. 67, no. 10, pp. 9145–9157, 2018.

[4] P. Angkititrakul, R. Terashima, and T. Wakita, "On the use of stochastic driver behavior model in lane departure warning," *IEEE Tr. on Intelligent Transportation Systems*, vol. 12, no. 1, pp. 174–183, 2011.

[5] Y. Bian, J. Ding, M. Hu, Q. Xu, J. W., and K. Li, "An advanced lane-keeping assistance system with switchable assistance modes," *IEEE Tr. on Intelligent Transportation Systems*, vol. 21, no. 1, pp. 385–396, 2020.

[6] Y. S. Son, W. Kim, S. Lee, and C. Chung, "Robust multirate control scheme with predictive virtual lanes for lane-keeping system of autonomous highway driving," *IEEE Tr. on Vehicular Technology*, vol. 64, no. 8, pp. 3378–3391, 2015.

[9] N. Ahmadian, A. Khosravi, and P. Sarhadi, "Managing driving disturbances in lateral vehicle dynamics via adaptive integrated chassis control," *Proceedings of the Institution of Mechanical Engineers. Part K, Journal of multi-body dynamics*, vol. 235, no. 1, pp. 122–133, 2021.

[7] E. Esmailzadeh, A. Goodarzi, and G. R. Vossoughi, "Optimal yaw moment control law for improved vehicle handling," *Mechatronics*, vol. 13, no. 7, pp. 659–675, 2003.

[8] B. Lenzo, M. Zanchetta, A. Sorniotti, P. Gruber, and W. De Nijs, "Yaw rate and sideslip angle control through single input single output direct yaw moment control," *IEEE Tr. on Control Systems Technology*, vol. 29, no. 1, pp. 124–139, 2021.

[10] J. Hu, S. Rakheja, and Y. Zhang, "Real-time estimation of tire–road friction coefficient based on lateral vehicle dynamics," *Proceedings of the Institution of Mechanical Engineers. Part D, Journal of automobile engineering*, vol. 234, no. 10-11, pp. 2444–2457, 2020.

[11] L. Jiang, S. Wang, Y. Xie, J. Meng, S. Zheng, X. Zhang, and H. Wu, "Anti-disturbance direct yaw moment control of a four-wheeled autonomous mobile robot," *IEEE Access*, vol. 8, pp. 174 654–174 666, 2020.

[12] R. Hajiloo, A. Khajepour, A. Kasaiezadeh, S. Chen, and B. Litkouhi, "A model predictive control of electronic limited slip differential and differential braking for improving vehicle yaw stability," *IEEE Tr. on Control Systems Technology*, vol. 31, no. 2, pp. 797–808, 2023.

[13] H. Zengin, N. Zengin, B. Fidan, and A. Khajepour, "Blending based multiple-model adaptive control of multivariable systems with application to lateral vehicle motion control," *European Journal of Control*, vol. 58, pp. 1–10, Mar. 2021.

[14] N. Ahmadian, A. Khosravi, and P. Sarhadi, "Integrated model reference adaptive control to coordinate active front steering and direct yaw moment control," *ISA Transactions*, vol. 106, pp. 85–96, 2020.

[15] K. S. Narendra and Z. Han, "A new approach to adaptive control using multiple models," *International Journal of Adaptive Control and Signal Processing*, vol. 26, no. 8, pp. 778–799, Aug. 2012.

[16] A. S. Morse, "A simple example of an adaptive control system," *Com. in Information and Systems*, vol. 11, no. 2, pp. 105–118, 2011.

[17] M. Kuipers and P. Ioannou, "Multiple model adaptive control with mixing," *IEEE Tr. Automatic Control*, vol. 55, no. 8, pp. 1822–1836, Aug. 2010.

[18] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Overcoming the limitations of adaptive control by means of logic-based switching," *Systems & Control Letters*, vol. 49, no. 1, pp. 49–65, May 2003.

[19] A. K. Al-Jiboory and G. G. Zhu, "Robust Gain-Scheduling Control with Guaranteed Performance," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 528–533, 2016.

[20] J. P. Hespanha and A. S. Morse, "Switching between stabilizing controllers," *Automatica*, vol. 38, no. 11, pp. 1905–1917, Nov. 2002.

[21] K. Büyükkabasakal, B. Fidan, and A. Savran, "Mixing adaptive fault tolerant control of quadrotor UAV," *Asian Journal of Control*, vol. 19, no. 4, pp. 1441–1454, Jul. 2017.

[22] J. L. Mancilla-Aguilar and R. A. García, "An algorithm for the robust exponential stabilization of a class of switched systems," *International Journal on Robust and Nonlinear Control*, vol. 25, no. 13, pp. 2062–2082, Sep. 2015.

[23] A. Lovi, B. Fidan, and C. Nielsen, "Multiple model reference adaptive control with blending for non-square multivariable systems," *arXiv*, 2024.

[24] Z. Han and K. S. Narendra, "New concepts in adaptive control using multiple models," *IEEE Tr. Automatic Control*, vol. 57, no. 1, pp. 78–89, Jan. 2012.

[25] M. Harrer and P. E. Pfeffer, *Steering handbook*. Cham, Switzerland: Springer, 2017.

[26] G. Tao, *Adaptive Control Design and Analysis*. Wiley, 2003.

[27] P. Ioannou and B. Fidan, *Adaptive Control Tutorial*. SIAM, 2006.

[28] A. Lovi, B. Fidan, and C. Nielsen, "Multiple model reference adaptive tracking control of multivariable systems with blending," in *61st Conf. on Decision and Control*. IEEE, 2022, pp. 1362–1367.