

Asymptotic Stability of Non-Demolition Quantum Trajectories with Measurement Imperfections

Maël Bompais¹ and Nina Amini¹

Abstract—We consider the question of asymptotic stability of quantum trajectories undergoing quantum non-demolition imperfect measurement, that is to say the convergence of the estimated trajectory towards the true trajectory whose parameters and initial state are not necessarily known. We give conditions on the estimated initial state and regions of validity for the estimated parameters so that this convergence is ensured. We illustrate these results through numerical simulations on the physical example [1] and discuss the asymptotic stability for a more realistic general case where decoherence acts on the system. In this case, the evolution is described by new Kraus operators which do not satisfy the quantum non-demolition property.

I. INTRODUCTION

Open quantum systems are systems which are in interaction with an environment. Such interaction causes the loss of information from the system to the environment or vice versa from the environment to the system, a phenomenon called decoherence [2]. Open quantum systems can be observed through measurement processes, see e.g., [3], [4].

Direct measurement of a quantum system will freeze its state, a phenomenon called Zeno effect [3]. Instead, the quantum system can be measured indirectly. This means that the system interacts with a meter, usually light beams, and then an observable of the scattered light is measured. This measure contains information about the system and also has random back action on it. Quantum measurements have completely probabilistic nature. In discrete time, the evolution is described by a Markov chain. In continuous time, it is described by stochastic master equations driven by Wiener and Poisson processes depending on the type of detection; Wiener processes for homodyne or heterodyne and Poisson processes for photon counting detection, see e.g., [5]. Such stochastic processes are referred to as quantum trajectories (see [6]) or quantum filters (see [7], [8], [9]).

The perturbation induced by the measurement on the system can be better overcome thanks to the quantum non-demolition (QND) property which is introduced in [10]. In essence, this condition means that it is possible to identify a basis for the system Hilbert space where every element of the basis remains unchanged by the measurement. The QND measurement has been implemented by Serge Haroche's group to realize the first experiment of a feedback stabilizing photon number states inside a quantum electrodynamics cavity [11], [12], [1].

In real experiments, there are many sources of imperfection related for instance to the detector efficiency, the error in the results, etc. Quantum trajectories in presence of imperfections take more complicated forms. The robustness of quantum trajectories with respect to the misspecification of parameters and initial states is fundamental. For instance, error correction codes and quantum feedback control play an important role in keeping the behavior of trajectories robust to errors, see e.g., [13], [14].

The asymptotic stability of quantum trajectories, i.e., the convergence of the estimated trajectories towards the true ones, in the case of lack of knowledge about the initial state, has been investigated in different papers with different approaches, whereas developing necessary conditions for generic quantum trajectories including measurement imperfections remains open. In [15], the author establishes a sufficient condition for asymptotic stability. In [16], [17], for the case of perfect QND measurement, the authors obtain asymptotic stability for discrete-time and continuous-time trajectories. In [18], we characterize necessary conditions for asymptotic stability of quantum trajectories for perfect measurements. In [19], the authors design an optimal discrete-time filter containing different sources of imperfections and show that such the filter in average becomes closer to the ideal filter with correct initial state by proving the submartingale property of the fidelity (see [20, Chapter 9] for the definition of the fidelity). Later in [21], this result is extended to the continuous-time filter.

Here we consider discrete-time QND quantum trajectories when measurements are imperfect, more precisely we consider the optimal filter developed in [19]. Firstly, we study a quantum state reduction property for these quantum trajectories where we obtain the rate of convergence. To this end, we adopt the approaches developed in [16] for the case of perfect QND measurements (see Section II). Secondly, we assume that the initial state and the physical parameters appearing in the measurement operators are unknown, and we develop asymptotic stability analyses in these cases where we provide conditions on the estimated parameters and estimated initial state which ensure the asymptotic stability (see Section III). This represents the first result of such kind of asymptotic stability including misspecification of parameters in addition to initial state. Finally, we consider the well-known example of the photon box [1], where we observe numerically such asymptotic stability. Furthermore, we consider the case of decoherence induced by the environment in this example, and we illustrate numerically that even though the new Kraus operators do not satisfy the QND condition, the asymptotic

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stability property with respect to the initial state is still ensured (see Section IV). This is encouraging for further investigations in this direction.

II. SELECTION OF THE POINTER STATE FOR QND IMPERFECT MEASUREMENT

A. Quantum trajectories with measurement imperfections

We consider a quantum system undergoing discrete-time indirect measurement. The Hilbert space of the system is denoted by $\mathcal{H} \cong \mathbb{C}^d$ and the state space is the set of density matrices

$$\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho = \rho^\dagger, \rho \geq 0, \text{tr}(\rho) = 1\}.$$

With an ideal detector, the state of the system evolving over time satisfies the following Markov chain

$$\rho_{n+1} = \frac{V_{i_n} \rho_n V_{i_n}^\dagger}{\text{tr}(V_{i_n} \rho_n V_{i_n}^\dagger)},$$

depending on the random measurement result i_n taking values in a finite set \mathcal{Y} . The probability to detect the measurement result $i_n = i$ at time n is given by $\text{tr}(V_i \rho_n V_i^\dagger)$. The operators V_i are referred to as Kraus operators and satisfy $\sum_{i \in \mathcal{Y}} V_i^\dagger V_i = I$, with I denoting the identity operator on \mathcal{H} .

Now suppose that the detector is biased and give some corrupted measurement results. The probability of error is supposed to be known and described by the correlation matrix η such that $\eta_{i,j}$ is the probability to detect i while an ideal detector would have given j . In this case, the best estimation of the state of the system knowing the sequence of measurement results is given by the optimal filter derived in [19]

$$\rho_{n+1} = \frac{\Phi_{i_n}(\rho_n)}{\text{tr}(\Phi_{i_n}(\rho_n))} \quad (1)$$

with $\Phi_i(\rho) = \sum_{j \in \mathcal{Y}} \eta_{i,j} V_j \rho V_j^\dagger$ and the measurement results are governed by the probability measure \mathbb{P} such that $\mathbb{P}(i_n = i) = \text{Tr}(\Phi_{i_n}(\rho_n))$.

B. Asymptotic behaviour under QND imperfect measurement

In this section, we aim to study the asymptotic behavior of the Markov chain (1) when the measurement satisfies a QND condition defined as follows:

Definition 2.1 (QND measurement): A measurement satisfies a non-demolition condition for a basis \mathcal{P} if any element $|\alpha\rangle$ of \mathcal{P} is not changed by the measurement. The states $|\alpha\rangle\langle\alpha|$ are called pointer states.

Remark 1: In particular, this property is equivalent to say that the measurement operators V_i are diagonal in this basis, see more details in e.g. [17].

We consider from now that this condition is verified. For $|\alpha\rangle$ an element in \mathcal{P} , we define

$$p(i|\alpha) = \text{tr}(\Phi_i(|\alpha\rangle\langle\alpha|))$$

which corresponds to the probability to observe i when we perform a measurement on the state $|\alpha\rangle\langle\alpha|$. Let us also define

$$p_n(i) = \text{tr}(\Phi_i(\rho_n))$$

which corresponds to the probability to observe i at time n .

Throughout this paper, we suppose that the following non-degeneracy assumption holds.

Assumption 2.1: For two different elements $|\alpha\rangle, |\beta\rangle$ in \mathcal{P} , there exists $i \in \mathcal{Y}$ such that

$$p(i|\alpha) \neq p(i|\beta).$$

In order to study the asymptotic behaviour of ρ_n , we will be interested in the quantities

$$q_\alpha(n) = \text{tr}(|\alpha\rangle\langle\alpha| \rho_n)$$

which determine the population in the pointer state $|\alpha\rangle\langle\alpha|$ at time n . Note that $q_\alpha(n) \in [0, 1]$ and $\sum_{\beta \in \mathcal{P}} q_\beta(n) = 1$.

The following theorem establishes the asymptotic selection of a pointer state.

Theorem 2.1: There exists a random variable $|\Upsilon\rangle$ taking values in the set \mathcal{P} such that

$$\lim_{n \rightarrow \infty} q_\Upsilon(n) = 1 \quad \text{a.s.},$$

equivalently

$$\lim_{n \rightarrow \infty} \rho_n = |\Upsilon\rangle\langle\Upsilon| \quad \text{a.s.}$$

Moreover, $\mathbb{P}(\Upsilon = \alpha) = q_\alpha(0)$.

The proof is a direct adaptation of the arguments applied in [22] for perfect measurement. This uses the martingale property of $q_\alpha(n)$, still valid for imperfect measurements, and Assumption 2.1, to state that $q_\alpha(\infty)$ and $q_\beta(\infty)$ cannot be non-zero simultaneously for $\alpha \neq \beta$.

Now let us set the following notation $\mathbb{P}_\alpha(\cdot) = p(\cdot|\alpha)$. The next theorem precises the speed of selection of the pointer state $|\Upsilon\rangle\langle\Upsilon|$.

Theorem 2.2: Let α be such that $q_\alpha(0) \neq 0$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{q_\alpha(n)}{q_\Upsilon(n)} \right) = -S(\mathbb{P}_\Upsilon || \mathbb{P}_\alpha) \quad \text{a.s.},$$

where $S(\mathbb{P}_\Upsilon || \mathbb{P}_\alpha) = \sum_i p(i|\Upsilon) \ln \left(\frac{p(i|\Upsilon)}{p(i|\alpha)} \right)$ is the relative entropy between the probability distributions \mathbb{P}_Υ and \mathbb{P}_α .

Proof: First we start by showing that the following recurrence

$$q_\alpha(n+1) = q_\alpha(n) \frac{p(i_n|\alpha)}{p_n(i_n)} \quad (2)$$

holds for $q_\alpha(n)$. We note that

$$\begin{aligned} q_\alpha(n+1) &= \text{tr}(|\alpha\rangle\langle\alpha|\rho_{n+1}) \\ &= \text{tr}\left(|\alpha\rangle\langle\alpha|\frac{\Phi_{i_n}(\rho_n)}{\text{tr}(\Phi_{i_n}(\rho_n))}\right) \\ &= \frac{\text{tr}\left(\Phi_{i_n}^\dagger(|\alpha\rangle\langle\alpha|)\rho_n\right)}{\text{tr}(\Phi_{i_n}(\rho_n))} \\ &= \frac{\text{tr}\left(\text{tr}(\Phi_{i_n}^\dagger(|\alpha\rangle\langle\alpha|))|\alpha\rangle\langle\alpha|\rho_n\right)}{\text{tr}(\Phi_{i_n}(\rho_n))}, \end{aligned}$$

where for the last equality, we use the QND property of measurement and Remark 1. Then we get the following

$$q_\alpha(n+1) = \frac{\text{tr}(\Phi_{i_n}^\dagger(|\alpha\rangle\langle\alpha|))\text{tr}(|\alpha\rangle\langle\alpha|\rho_n)}{\text{tr}(\Phi_{i_n}(\rho_n))},$$

which gives the desired recurrence relation (2). The rest of the proof uses similar arguments as in [16, Section 4.3]. We give the main parts of these arguments for the sake of readability.

First, note that by using recurrence, the relation (2) can be rewritten as follows

$$q_\alpha(n) = q_\alpha(0) \frac{\prod_{k=0}^{n-1} p(i_k|\alpha)}{\prod_{k=0}^{n-1} p_k(i_k)}.$$

Now we can evaluate the following ratio for α and ζ in \mathcal{P}

$$\begin{aligned} &\frac{1}{n} \ln\left(\frac{q_\alpha(n)}{q_\zeta(n)}\right) \\ &= \frac{1}{n} \ln\left(\frac{q_\alpha(0)}{q_\zeta(0)}\right) + \frac{1}{n} \sum_{k=0}^{n-1} \ln(p(i_k|\alpha)) - \ln(p(i_k|\zeta)). \end{aligned}$$

Under the new probability \mathbb{Q}_γ defined by $\mathbb{Q}_\gamma(\cdot) = \mathbb{P}(\cdot|\Upsilon = \gamma)$, it can be shown that the measurement results are identically independently distributed. Then, using the law of large numbers, we can state that

$$\mathbb{Q}_\zeta\left(\lim_{n \rightarrow \infty} \frac{1}{n} \ln\left(\frac{q_\alpha(n)}{q_\zeta(n)}\right) = -S(\mathbb{P}_\zeta||\mathbb{P}_\alpha)\right) = 1.$$

However we aim to show the following

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} \frac{1}{n} \ln\left(\frac{q_\alpha(n)}{q_\Upsilon(n)}\right) = -S(\mathbb{P}_\Upsilon||\mathbb{P}_\alpha)\right) = 1.$$

Now we remark that

$$\mathbb{P}(\cdot) = \sum_\gamma \mathbb{P}(\Upsilon = \gamma) \mathbb{P}(\cdot|\Upsilon = \gamma) = \sum_\gamma q_\gamma(0) \mathbb{Q}_\gamma(\cdot)$$

Then we have

$$\begin{aligned} &\mathbb{P}\left(\lim_{n \rightarrow \infty} \frac{1}{n} \ln\left(\frac{q_\alpha(n)}{q_\Upsilon(n)}\right) = -S(\mathbb{P}_\Upsilon||\mathbb{P}_\alpha)\right) \\ &= \sum_\gamma q_\gamma(0) \mathbb{Q}_\gamma\left(\lim_{n \rightarrow \infty} \frac{1}{n} \ln\left(\frac{q_\alpha(n)}{q_\gamma(n)}\right) = -S(\mathbb{P}_\gamma||\mathbb{P}_\alpha)\right) \\ &= \sum_\gamma q_\gamma(0) = 1. \end{aligned}$$

III. ASYMPTOTIC STABILITY

In this section, we show asymptotic stability of the Markov chain (1) for two cases, first in absence of knowledge about initial state and second for ignorance of both initial state and physical parameters.

A. Unknown initial state

When the initial state ρ_0 is unknown, a natural way to construct an estimation $\hat{\rho}_n$ of the true trajectory ρ_n is to fix an arbitrary estimated initial state $\hat{\rho}_0$ and make it evolve according to the measurement results i_n at our disposal (emitted by the true system ρ_n), with the recursive relation

$$\hat{\rho}_{n+1} = \frac{\Phi_{i_n}(\hat{\rho}_n)}{\text{Tr}(\Phi_{i_n}(\hat{\rho}_n))}. \quad (3)$$

Similarly to the previous study, we define $\hat{q}_\alpha(n) = \text{tr}(|\alpha\rangle\langle\alpha|\hat{\rho}_n)$. The following theorem shows the asymptotic stability with respect to the initial state.

Theorem 3.1: Let α be such that $\hat{q}_\alpha(0) \neq 0$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln\left(\frac{\hat{q}_\alpha(n)}{\hat{q}_\Upsilon(n)}\right) = -S(\mathbb{P}_\Upsilon||\mathbb{P}_\alpha)$$

almost surely, where Υ is the random variable designating the pointer state selected by the trajectory ρ_n . Consequently, if for all α , $q_\alpha(0) \neq 0$, then

$$\lim_{n \rightarrow \infty} \hat{q}_\Upsilon(n) = 1,$$

equivalently

$$\lim_{n \rightarrow \infty} \hat{\rho}_n = |\Upsilon\rangle\langle\Upsilon| \text{ a.s.}$$

In other words, the estimated trajectory $\hat{\rho}_n$ selects the same pointer state as the true trajectory ρ_n . A simple way to fulfill the condition $q_\alpha(0) \neq 0$ for all α is to choose $\hat{\rho}_0$ as a full rank state.

Proof: The proof is principally based on showing the same recurrence property announced in (2) for estimated trajectory (3), i.e., for $\hat{q}_\alpha(n)$, which can be obtained in the same manner as before. The rest of the proof is similar to [16, Section 4.3]. ■

B. Unknown initial state & unknown parameters

Suppose now that the Kraus maps Φ_i depend on an unknown parameter θ , we write them Φ_i^θ , we then have a new estimation of the trajectory, which is based on an estimation $\hat{\theta}$ of θ ; and again evolving according to the measurement results detected as

$$\hat{\rho}_{n+1}^{\hat{\theta}} = \frac{\Phi_{i_n}^{\hat{\theta}}(\hat{\rho}_n^{\hat{\theta}})}{\text{Tr}(\Phi_{i_n}^{\hat{\theta}}(\hat{\rho}_n^{\hat{\theta}}))}.$$

We define the quantities $\hat{q}_\alpha^{\hat{\theta}}(n) = \text{tr}(|\alpha\rangle\langle\alpha|\hat{\rho}_n^{\hat{\theta}})$. We also set the notation $\mathbb{P}_\alpha^{\hat{\theta}}(\cdot) = p^{\hat{\theta}}(\cdot|\alpha)$ where $p^{\hat{\theta}}(i|\alpha) = \text{tr}(\Phi_i^{\hat{\theta}}(|\alpha\rangle\langle\alpha|))$. The following theorem establishes the asymptotic stability of quantum trajectories in this case under an appropriate assumption. ■

Theorem 3.2: Let α be such that $\hat{q}_\alpha^\theta(0) \neq 0$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{\hat{q}_\alpha^\theta(n)}{\hat{q}_\Upsilon^\theta(n)} \right) = S(\mathbb{P}_\Upsilon | \mathbb{P}_\Upsilon^\theta) - S(\mathbb{P}_\Upsilon | \mathbb{P}_\alpha^\theta) \quad a.s.,$$

where Υ is the random variable designating the pointer state selected by the trajectory ρ_n . Therefore, if for all α , $\hat{q}_\alpha^\theta(0) \neq 0$, and

$$\operatorname{argmin}_{\beta \in \mathcal{P}} S(\mathbb{P}_\alpha | \mathbb{P}_\beta^\theta) = \alpha, \quad (4)$$

then

$$\lim_{n \rightarrow \infty} \hat{q}_\Upsilon^\theta(n) = 1 \quad a.s.,$$

equivalently

$$\lim_{n \rightarrow \infty} \rho_n^\theta = |\Upsilon\rangle\langle\Upsilon| \quad a.s.$$

Proof: One can obtain a similar recurrence relation for $\hat{q}_\alpha^\theta(n)$ as before. This allows us to show that the following limit holds

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{\hat{q}_\alpha^\theta(n)}{\hat{q}_\Upsilon^\theta(n)} \right) = \sum_i p(i|\Upsilon) \ln \left(\frac{p^\theta(i|\alpha)}{p^\theta(i|\Upsilon)} \right).$$

The above expression can be written as follows

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{\hat{q}_\alpha^\theta(n)}{\hat{q}_\Upsilon^\theta(n)} \right) &= \sum_i p(i|\Upsilon) \ln \left(\frac{p(i|\Upsilon)}{p^\theta(i|\Upsilon)} \right) \\ &\quad - \sum_i p(i|\Upsilon) \ln \left(\frac{p(i|\Upsilon)}{p^\theta(i|\alpha)} \right) \\ &= S(\mathbb{P}_\Upsilon | \mathbb{P}_\Upsilon^\theta) - S(\mathbb{P}_\Upsilon | \mathbb{P}_\alpha^\theta). \end{aligned}$$

Now if $\operatorname{argmin}_{\beta \in \mathcal{P}} S(\mathbb{P}_\alpha | \mathbb{P}_\beta^\theta) = \alpha$ for all α , then the above expression is negative for any value $\alpha \neq \Upsilon$, hence necessarily $\hat{q}_\alpha^\theta(n)$ tends to zero, and because these quantities sum to one, $\hat{q}_\Upsilon^\theta(n)$ converges to one. ■

Remark 2: In simpler terms, condition 4 means that for any pointer state, the probability distribution generated with the estimated parameter is closer to that generated by the same pointer state with the true parameter than those generated by the other pointer states, in the sense of relative entropy.

IV. NUMERICAL STUDY

In this section, we consider the experimental setup designed by Laboratoire Kastler-Brossel (LKB) at Ecole Normale Supérieure (ENS) de Paris. Two situations are considered, first QND measurements in presence of imperfections, second we also take into account the decoherence due to the interaction of the system with the environment.

A. QND measurement: photon box example

We are interested in the evolution of a quantum electrodynamics cavity state considered in [1]. The system corresponds to a quantized trapped mode inside the cavity whose state is described in the Fock basis $|n\rangle\langle n|$ representing n photons inside the cavity. The photon number states are the pointer states in this example. Here we suppose that the

Hilbert space is finite-dimensional and we can have no more than n^{\max} photons inside the cavity. The state of the cavity is estimated through QND measurements. This is done by considering Rydberg atoms as the meter which are sent one by one inside the cavity and measured just after. The atom can be in the ground state $|g\rangle\langle g|$ or excited state $|e\rangle\langle e|$. In this experiment, there are various sources of imperfection. The sample of atoms interacting with the cavity can be empty of atoms with probability p_0 , one atom with probability p_1 and two atoms with probability p_2 . Other sources of imperfection can be the efficiency of the detector denoted by ε_d which corresponds to the probability that the detector detects an atom. The final type of error corresponds to the possibility of a false detection result, we denote by η_g (resp. η_e) the probability that the atom is detected in g (resp. e) while the correct one is e (resp. g). Formally, here we have 7 possibilities for the detection result $i \in \{no, g, e, gg, ge, ee\}$ corresponding to have no atom, or one atom in the state g or e , or two atoms both in the state g , one atom in g and the other in e or finally the possibility to have two atoms both on the state e . The following expressions give the forms of the Kraus operators corresponding to each of such detection possibilities:

$$\begin{aligned} V_{no} &= \sqrt{p_0} I, & V_g &= \sqrt{p_1} \cos \phi_N, & V_e &= \sqrt{p_1} \sin \phi_N \\ V_{gg} &= \sqrt{p_2} \cos^2 \phi_N, & V_{ge} &= V_{eg} = \sqrt{p_2} \cos \phi_N \sin \phi_N, \\ V_{ee} &= \sqrt{p_2} \sin^2 \phi_N, \end{aligned}$$

where $\phi_N = \frac{\phi_0(N + \frac{1}{2}) + \phi_R}{2}$, with ϕ_0 and ϕ_R corresponding to physical parameters and N denoting the photon number operator.

The elements η_{ij} of the correlation matrix for $i, j \in \{no, g, e, gg, ge, ee\}$ is determined by the table shown in Figure 6. The system dynamics is then described by Equation (1).

The values we used to simulate the true trajectory are $p_0 = 0.05$, $p_1 = 0.9$, $p_2 = 0.05$, $\phi_0 = 0.78$, $\phi_R = -0.44$. For the correlation matrix, $\varepsilon_d = 0.9$, $\eta_g = 0.1$, and $\eta_e = 0.1$. The true initial state ρ_0 was chosen as a random pure state, and the estimated initial state $\hat{\rho}_0$ as the completely mixed state.

Unknown initial state and parameters

Here we seek to numerically verify the asymptotic stability of trajectories whose initial states and physical parameters are unknown. For the stability purpose, we have seen that there is no real constraint on the estimated initial state (it is enough to choose it full rank). Concerning the estimated parameters, according to Theorem 3.2, it is sufficient that Equation 4 holds true, that is to say that for any pointer state, the probability distribution generated with estimated parameter is closer to the distribution generated by the same pointer state with the true parameter. For the sake of simplicity, we suppose that only the parameters ϕ_0 and ϕ_R suffer from imprecision. More precisely, we compute numerically the relative entropy between \mathbb{P}_α and \mathbb{P}_β^θ where $\hat{\theta} = (\hat{\phi}_0, \hat{\phi}_R)$ is an estimation of $\theta = (\phi_0, \phi_R)$. In Figure 1, we plot the parameters that verify the mentioned condition.

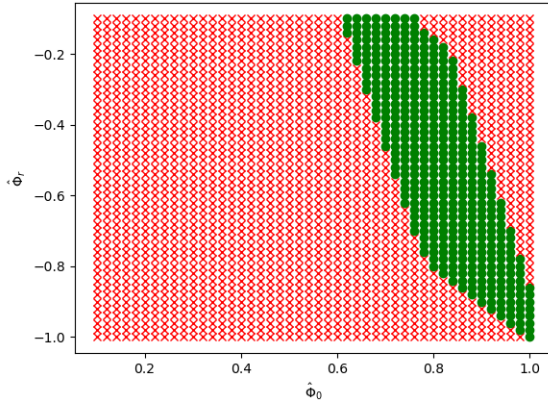


Fig. 1. Validity of condition (4). In green when the condition is verified; in red when it is not. Maximum photon number is $n^{\max} = 4$. True parameters are $\phi_0 = 0.78$ and $\phi_R = -0.44$.

Here is an example of the evolution of the populations in the pointer states, i.e. the quantities $q_\alpha(n)$ and $\hat{q}_\alpha^\theta(n)$, from $\alpha = 0$ to $\alpha = 4$.

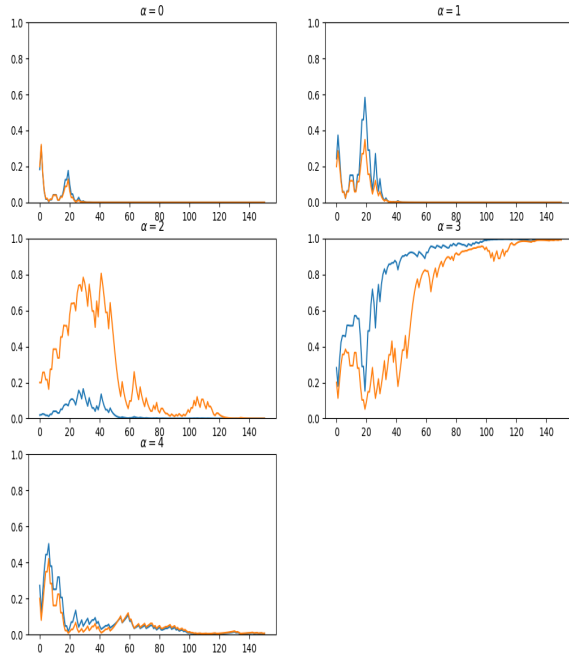


Fig. 2. Populations in the different pointer states over time. Blue for the true trajectory, yellow for the estimated trajectory. Estimated parameters are $\hat{\phi}_0 = 0.83$ and $\hat{\phi}_R = -0.40$.

In the following, similar to Figure 1, we plot the parameters which verify the condition (4) for a greater n^{\max} , which shows that the region is now much narrower (the figure is enlarged for a better visibility).

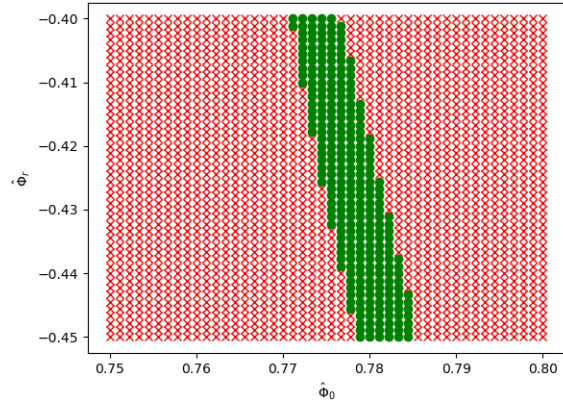


Fig. 3. Validity of condition (4). In green when the condition is verified; in red when it is not. Maximum photon number is $n^{\max} = 9$. True parameters are $\phi_0 = 0.78$ and $\phi_R = -0.44$.

B. Beyond the QND case: Photon box example taking into account the effect of decoherence

In this section, we still consider the photon box example, however we now take into account the interaction between the cavity field and the environment, i.e., decoherence. The decoherence can be described by the action of the superoperator T on the cavity state as follows

$$T(\rho) = L_0 \rho L_0^\dagger + L_+ \rho L_+^\dagger + L_- \rho L_-^\dagger,$$

where $L_0 = I - \frac{\varepsilon(1+2n_{th})}{2}N - \frac{\varepsilon n_{th}}{2}I$ corresponds to the no-photon jump operator, $L_+ = \sqrt{\varepsilon(1+n_{th})}a$ means the capture of a photon from the environment and $L_- = \sqrt{\varepsilon n_{th}}a^\dagger$ represents the loss of a photon to the environment. The experimental parameters are $0 \leq \varepsilon, n_{th} \ll 1$. The notations a and a^\dagger correspond to the annihilation and creation operators respectively.

The whole evolution can be described by 21 Kraus operators which are in the form $L_d V_i$ with $i \in \{no, g, e, gg, ge, ee\}$ and $d \in \{0, +, -\}$. However, clearly these new Kraus operators do not satisfy the QND property.

Here, we aim to study the asymptotic stability in presence of decoherence and imperfections through simulations.

This time, we cannot look at the populations in the pointer states because the Kraus operators are no longer QND. In this case, the true trajectory does not converge to a state in general. Nevertheless, we can consider the fidelity¹ between the true trajectory and the estimated one. We assume first that the parameters are known but we do not have access to the initial state. Figure 4 represents the convergence of the fidelity between the true state ρ_n and the estimated one $\hat{\rho}_n$ towards one.

¹The fidelity between two states ρ and $\hat{\rho}$ is defined by $\mathcal{F}(\rho, \hat{\rho}) = \text{tr}^2 \sqrt{\sqrt{\rho} \hat{\rho} \sqrt{\rho}} \in [0, 1]$.

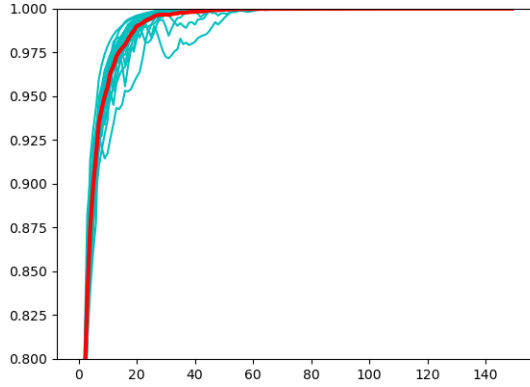


Fig. 4. Fidelity between the true trajectories and the estimated trajectories in presence of decoherence with unknown initial state. The red curve represents the mean value of 20 samples. Here $n^{\max} = 4$.

In Figure 5, we assume that in addition the parameters are unknown, so the estimated trajectory evolves with estimated parameters $\hat{\phi}_0, \hat{\phi}_R$. We observe that in this case the fidelity does not converge to one.

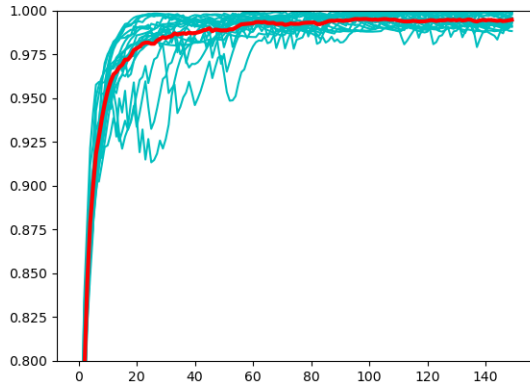


Fig. 5. Fidelity between the true trajectories and the estimated trajectories in presence of decoherence with unknown initial state and parameters. The red curve represents the mean value of 20 samples. Here $n^{\max} = 4$, $\hat{\phi}_0 = 0.83$, and $\hat{\phi}_R = -0.40$.

V. CONCLUSION

In this paper, we show that discrete-time quantum trajectories undergoing QND imperfect measurement are asymptotically stable with respect to initial state. This means the convergence of the estimated trajectories towards the true ones. Moreover, we consider the situation where the physical parameters are unknown and we provide a condition which ensures the asymptotic stability in this new case, not considered before in the literature as far as we know. Numerically, for the famous example of the photon box [1], we observe that in presence of decoherence and measurement imperfections, the estimated trajectories with arbitrary initial state converge towards the true trajectory with correct initial state even though the Kraus operators representing this

situation do not satisfy QND property. It seems however that stability with respect to parameters is not ensured in general. These would be interesting questions to be further investigated.

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$i \setminus j$	no	g	e	gg	ee	ge or eg
no	1	$1 - \varepsilon_d$	$1 - \varepsilon_d$	$(1 - \varepsilon_d)^2$	$(1 - \varepsilon_d)^2$	$(1 - \varepsilon_d)^2$
g	0	$\varepsilon_d(1 - \eta_g)$	$\varepsilon_d\eta_e$	$2\varepsilon_d(1 - \varepsilon_d)(1 - \eta_g)$	$2\varepsilon_d(1 - \varepsilon_d)\eta_e$	$\varepsilon_d(1 - \varepsilon_d)(1 - \eta_g + \eta_e)$
e	0	$\varepsilon_d\eta_g$	$\varepsilon_d(1 - \eta_e)$	$2\varepsilon_d(1 - \varepsilon_d)\eta_g$	$2\varepsilon_d(1 - \varepsilon_d)(1 - \eta_e)$	$\varepsilon_d(1 - \varepsilon_d)(1 - \eta_e + \eta_g)$
gg	0	0	0	$\varepsilon_d^2(1 - \eta_g)^2$	$\varepsilon_d^2\eta_e^2$	$\varepsilon_d^2\eta_e(1 - \eta_g)$
ge	0	0	0	$2\varepsilon_d^2\eta_g(1 - \eta_g)$	$2\varepsilon_d^2\eta_e(1 - \eta_e)$	$\varepsilon_d^2((1 - \eta_g)(1 - \eta_e) + \eta_g\eta_e)$
ee	0	0	0	$\varepsilon_d^2\eta_g^2$	$\varepsilon_d^2(1 - \eta_e)^2$	$\varepsilon_d^2\eta_g(1 - \eta_e)$

Fig. 6. The table showing the elements of η_{ij} , borrowed from [19].