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Achieving robustness against uncertain time delays using non-parametric IQCs

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Abstract— **With increasing control applications in large-scale distributed and networked systems, the impact of uncertain time delay is ever-increasing. This paper proposes the use of a novel non-parametric Integral Quadratic Constraint (IQC) to achieve robustness against uncertain time delays. The proposed IQC is then integrated into a frequency-domain controller synthesis approach for robustness guarantees. Numerical simulation of an active suspension system within an intra-car network shows the effectiveness of the proposed method.**

I. INTRODUCTION

Time delays in a dynamical system can degrade performance and even cause system instability. With the increasing application of control in large-scale distributed and networked systems, where delays are random and time-varying, research into robust control design and analysis for such delays is increasingly important. This paper addresses time-delay uncertainty for robust controller synthesis.

Stability analysis of time-delay systems has been a fundamental area of study in the literature. Methods leveraging eigenvalue locations, spectrum assignment and parametric techniques serve as valuable tools in assessing the impact of delay in various domains, such as biology, networks and supply chains [1]. Time-domain methodologies, including the direct Lyapunov method and Lyapunov-Krasovskii functional, have been demonstrated to be powerful tools in the arsenal of the researchers [2]. Meanwhile, several frequency-domain approaches leveraging the Integral Quadratic Constraints (IQCs) framework have also been proposed.

IQCs offer a flexible formalism for representing and analysing various nonlinearities and uncertainties, including time-delay. A sufficient condition for robust stability with IQC-type uncertainties has been described in [3]. While most IQC literature focuses on robustness analysis, a small subset also addresses controller synthesis using the IQC framework. For example, [4] presents a model-based control synthesis approach, supported by the MATLAB package 'IQClab' [5], which uses an iterative procedure alternating between nominal controller synthesis and IQC analysis. Similarly, [6] addresses robust synthesis for uncertain LPV systems with a similarly alternating approach. A recent non-smooth optimisation framework for structured controller synthesis is presented in [7], which provides an optimality certificate but requires a specific multiplier structure. Finally, [8] presents a frequency-domain synthesis framework ensuring robustness for any non-parametric IQC multiplier with local convergence guarantees.

There have been several studies on the analysis of systems with time delays using IQCs, where the system is represented as an interconnection of a linear time-invariant (LTI) system and a "delay-difference" operator. These studies mainly aim to derive IQC multipliers associated with the delay-difference operator. For example, [9] presents classes for continuous-time systems, while [10], [11] focus on discrete-time systems. A more graphical interpretation of IQCs is provided in [12], along with the introduction of a tighter IQC multiplier class. Parametric IQCs require multiple classes of multipliers to reduce uncertainty, complicating controller synthesis. Non-parametric IQCs simplify this by providing a single class capable of approximating multiple parametric IQCs, thereby easing the controller synthesis problem. Recent advances in frequency-domain data-driven controller synthesis have enabled the use of non-parametric IQCs. For example, [13] explores the IQC framework for LPV controller synthesis, while [14] investigates fixed-structure controller synthesis for multivariable systems represented in linear fractional representation (LFR) form, integrating non-parametric IQCs to guarantee robustness as discussed in [8]. Non-parametric IQCs define a set at each frequency that encompasses the true uncertainty, allowing frequency-dependent scaling on both axes and frequency-dependent rotations. This enables the consideration of elliptical sets which has been explored in [15], [16], leading to a substantial reduction in conservatism.

This paper derives a novel non-parametric IQC multiplier associated with the delay-difference operator over a range of time delays. The derived multiplier has different scaling for the real and imaginary axes and frequency-dependent rotation, enabling the use of an elliptical uncertainty set. It is then used in a frequency-domain controller synthesis to develop a robust performance controller.

The paper follows the following progression: First, some basic notations are established, followed by the problem description under consideration in Section II. The problem is reformulated as an interconnection of a linear time-invariant system, a controller and a so-called delay-difference operator. Section III briefly introduces the stability analysis using IQCs and presents one of the paper's main contributions: the non-parametric IQC associated with the delay-difference operator. Building upon the insights from [14], [8], Section IV formulates an optimisation problem for robust controller synthesis. Numerical simulations of an active suspension system within an intra-car network are presented in Section V, followed by

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concluding remarks in Section VI.

Notations: $\mathbb{R}$ represents the set of real numbers, while $\mathbb{C}$ denotes the set of complex numbers. The $2/\infty$-norm of a vector is given by $\|\cdot\|_{2/\infty}$. The set of real rational stable transfer functions with bounded infinity norm is denoted by $\mathcal{RH}_\infty$. The notation $M \succ (\succeq), N$ signifies that the matrix $M - N$ is positive (semi-) definite, and $M \prec (\preceq), N$ signifies that $M - N$ is negative (semi-) definite. The identity matrix of the appropriate size is denoted by I. The conjugate transpose of a complex matrix M is indicated as M^*. The conjugate transpose of the diagonally opposed element in a square matrix is denoted by $\star$. diag$(\cdot)$ denotes a diagonal matrix. If $M \in \mathbb{C}^{n\times m}$ is full row rank, its right inverse is defined as $M^{\mathsf{R}} = M^*(MM^*)^{-1}$, such that $MM^{\mathsf{R}} = I$ and $M^{\mathsf{R}}M$ are Hermitian. In the case of full column rank, the left inverse is denoted as $M^{\mathsf{L}} = (M^*M)^{-1}M^*$, such that $M^{\mathsf{L}}M = I$ and MM^{L} is Hermitian. For a square matrix with full rank, $M^{\mathsf{R}} = M^{\mathsf{L}} = M^{-1}$. The frequency spectrum is denoted by Ω. For continuous-time systems $\Omega := \mathbb{R}$ and for discrete-time systems $\Omega := [-\pi/T_s, \pi/T_s)$, where T_s is the sampling time. $G(j\omega)$ will be used to denote the system's frequency response in both cases. Define $\mathcal{D}_\tau$ as the time-delay operator and $\mathcal{S}_\tau$ as the time-difference operator.

$$\mathcal{D}_\tau(u) \triangleq u(t-\tau)\ ;\ \mathcal{S}_\tau(u) \triangleq (I-\mathcal{D}_\tau)(u) = u(t)-u(t-\tau).$$

II. Problem Description

Consider a controller synthesis problem where communication between the plant and the controller occurs over a network (see Fig. 1a), which can introduce delays characterised by a nominal value and known bounds. This networked feedback system can be represented as (see Fig. 1b):

$$y = \hat{G}u_d = \hat{G}(\mathcal{D}_{\tau_u}u) \tag{1a}$$

$$u = Ky_d = K\big(\mathcal{D}_{\tau_y}y\big) \tag{1b}$$

where τ_y and τ_u denote the delay in the measurements and the control signals respectively. These delays are assumed to be within the interval $[\underline{\tau}, \overline{\tau}]$ where $\underline{\tau} \le 0 \le \overline{\tau}$. $\hat{G}$ represents the estimated FRF model of the plant with the nominal delay.

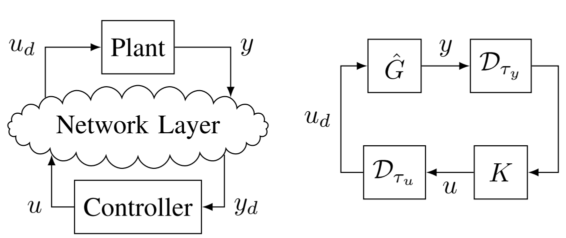


Fig. 1: Control system with a network layer

The time-delay operator $\mathcal{D}_\tau$ can be replaced by a feed-through connection and a feed-forward delay-difference operator $\mathcal{S}_\tau$ (see Fig. 2). This feedback connection can then be represented as a linear fractional representation (LFR) plant shown in Fig. 3.

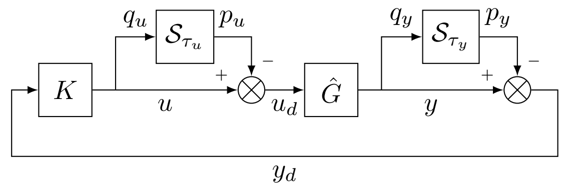


Fig. 2: System with delay-difference operator

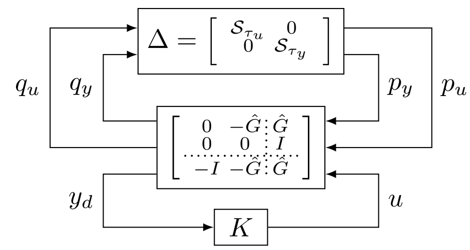


Fig. 3: Linear fractional representation of feedback system

The objective of the paper is to identify a non-parametric IQC multiplier for the time-delay operator, which is then utilised for robust controller synthesis to achieve the desired nominal performance.

III. Non-parametric IQC multiplier

First, a quick introduction to Integral Quadratic Constraints (IQCs) and their use in robust system analysis is provided. Then, a non-parametric IQC multiplier for the delay-difference operator is derived.

A. Integral Quadratic Constraint

Two signals p and q are said to satisfy the IQC defined by a multiplier Π, if

$$\int_\Omega \begin{bmatrix} \mathcal{F}_q(j\omega) \\ \mathcal{F}_p(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \mathcal{F}_q(j\omega) \\ \mathcal{F}_p(j\omega) \end{bmatrix} d\omega \ge 0 \tag{2}$$

where $\mathcal{F}_p(j\omega)$ and $\mathcal{F}_q(j\omega)$ are the Fourier transform of the signals p and q respectively.

From [3, Theorem 1], the feedback connection between H, a stable LTI system with bounded infinity norm, and a bounded causal operator Δ (see Fig. 4) is stable if,

1) Interconnection of H and $\tau\Delta$ is well-posed, $\forall \tau \in [0, 1]$;
2) $\tau\Delta$ satisfies the IQC defined by Π, $\forall\tau \in [0, 1]$;
3) $\exists\, \epsilon > 0$ such that the following frequency domain inequality (FDI) is satisfied,

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \quad \forall\omega \in \Omega \tag{3}$$

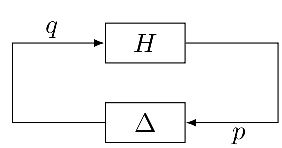


Fig. 4: Basic feedback configuration

Remark 1: If Π is partitioned as $\begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12}^* & \Pi_{22}\end{bmatrix}$ with $\Pi_{11} \succeq 0$ and $\Pi_{22} \preceq 0$, then using [3, Remark 2], $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0, 1]$ if and only if Δ satisfies the IQC. Most relevant IQCs can be represented in this form.

B. IQC multiplier for delay-difference operator

The uncertainty associated with the delay-difference operator can expressed as the set

$$\mathbb{S} = \{\mathcal{S}_\tau \mid \tau \in [\underline{\tau}, \overline{\tau}]\}.$$

In the frequency domain, this set can be depicted as an arc at each frequency point:

$$\mathbb{S}(j\omega) = \left\{1 - e^{-j\omega\tau} \mid \tau \in [\underline{\tau}, \overline{\tau}]\right\} \tag{4}$$

To formulate an IQC multiplier for the delay-difference operator involves identification of an enclosing uncertainty set. A classical approach has been to take the disk of radius 1 around the point $(1, 0)$ defined by the IQC multiplier:

$$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \tag{5}$$

Since this IQC formulation is independent of the range of time-delay uncertainty, it can result in significant conservatism in the solution. A more effective IQC multiplier can be computed by taking the maximal time delay into account [12]:

$$\Pi_2(j\omega) = \begin{bmatrix} |\phi(j\omega)|^2 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

where $|\mathcal{S}_\tau(j\omega)| \leq |\phi(j\omega)|$. An example of ϕ satisfying this constraint can be given as [11]:

$$\phi(s) = \frac{2\tilde{\tau}^2 s^2 + 2c\tilde{\tau}s}{\tilde{\tau}^2 s^2 + a\tilde{\tau}s + 2c} \tag{7}$$

where $\tilde{\tau} = \max(|\overline{\tau}|, |\underline{\tau}|)$, $a = 2\sqrt{c}$ and $c \in \mathbb{R}^+$. This IQC multiplier yields smaller conservatism but is heavily dependent on the selection of ϕ. The conservatism can be further reduced by combining the multipliers into a single IQC multiplier. This usually requires iterative optimisation over parameters a_1 and a_2 for $a_1\Pi_1 + a_2\Pi_2$.

In this paper, an 'optimal' elliptical uncertainty set $\mathcal{E}(j\omega)$ which encloses the uncertainty in the delay-difference operator is identified such that the area of uncertainty is minimum at each frequency point (see Fig. 5)

$$\mathbb{S}(j\omega) \subseteq \mathcal{E}(j\omega) \triangleq \left\{ x \;\middle|\; \left\| A(\omega) \begin{bmatrix} \mathrm{Re}\{x\} \\ \mathrm{Im}\{x\} \end{bmatrix} + b(\omega) \right\|_2 \leq 1 \right\}. \tag{8}$$

Theorem 1: Let the uncertainty in the time-difference operator be symmetric about zero, i.e., $\overline{\tau} = -\underline{\tau}$. Then, the 'optimal' elliptical uncertainty set $\mathcal{E}(j\omega)$ that encompasses the uncertainty in the time-difference operator, which minimises the area of uncertainty at each frequency point, as in (8) is characterised by,

$$A(\omega) = \begin{cases} \mathrm{diag}\left(\frac{3}{2(1-\cos\theta)}, \frac{\sqrt{3}}{2\sin\theta}\right) & \text{, if } \theta \leq \frac{2\pi}{3} \\ I & \text{, otherwise} \end{cases} ; \; b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

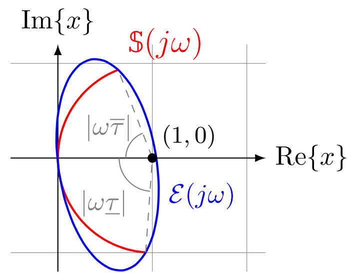


Fig. 5: Uncertainty in time delay at a given frequency ω and the corresponding elliptical uncertainty set

where, $\theta = \omega\overline{\tau}$.

Proof: First, observe that the arc of $\mathbb{S}$ is symmetric around the x-axis. Consequently, the covering ellipse should also demonstrate the symmetry about the x-axis with the semi-major axis aligned to the y-axis. Furthermore, the centre of the arc $(0, 0)$ and the end points of the arc $(1 - \cos(\theta), \pm\sin(\theta))$ should lie at the boundary of the ellipse. Then, it can be easily verified that a class of ellipses covering the arc is given as:

$$\mathcal{E} = \left\{ x \;\middle|\; \begin{array}{l} \left(\frac{\mathrm{Re}\{x\}-a}{a}\right)^2 + \left(\frac{\mathrm{Im}\{x\}}{b}\right)^2 \leq 1, \; a \in \left(\frac{1}{2}(1-\cos\theta), 1\right], \\ b = \frac{a\sin\theta}{\sqrt{2a(1-\cos\theta)-(1-\cos\theta)^2}} \quad \text{and} \quad \theta \in [0, \pi]. \end{array} \right\}$$
$$\bigcup \left\{ x \;\middle|\; \begin{array}{l} \left(\frac{\mathrm{Re}\{x\}-a}{a}\right)^2 + \left(\frac{\mathrm{Im}\{x\}}{b}\right)^2 \leq 1, \\ b \geq 1, \quad a = 1 \quad \text{and} \quad \theta > \pi. \end{array} \right\}.$$

Now, a minimisation problem over this class of ellipses can be solved to minimise the area of the ellipse πab. This minimisation problem can be solved analytically to give:

$$(a,\, b) = \begin{cases} \left(\frac{2(1-\cos\theta)}{3}, \frac{2\sin\theta}{\sqrt{3}}\right) & \text{, if } \theta \leq \frac{2\pi}{3} \\ (1,\, 1) & \text{, otherwise} \end{cases} \tag{9}$$

Expressing the equation of the ellipse in norm representation gives the desired values for $A(\omega)$ and b. ■

In contrast to the parametric IQC (6), the proposed non-parametric can address the non-symmetric delay by rotating the ellipse around the centre of the arc, i.e., $(1,\, 0)$.

Corollary 2: Let the uncertainty in the time-difference operator be $[\underline{\tau},\, \overline{\tau}]$. Then, the 'optimal' elliptical uncertainty set $\mathcal{E}(j\omega)$ that encompasses the uncertainty in the time-difference operator at each frequency point as in (8) can be characterised by,

$$A(\omega) = \begin{cases} \mathrm{diag}\left(\frac{3}{2(1-\cos\theta)}, \frac{\sqrt{3}}{2\sin\theta}\right) R & \text{, if } \theta \leq \frac{2\pi}{3} \\ R & \text{, otherwise} \end{cases}$$

$$b(\omega) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - A(\omega) \begin{bmatrix} 1 - \cos\tilde{\theta} \\ -\sin\tilde{\theta} \end{bmatrix}$$

where,

$$\theta = \omega\left(\frac{\overline{\tau} - \underline{\tau}}{2}\right); \; \tilde{\theta} = -\omega\left(\frac{\overline{\tau} + \underline{\tau}}{2}\right); \; R = \begin{bmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{bmatrix}.$$

Since the parametric IQC multipliers cannot represent the elliptical uncertainty set, a non-parametric IQC multiplier is identified for this elliptical uncertainty set. To determine the multiplier Π for the time-difference uncertainty, first consider a transformation matrix $J = \mathrm{diag}(1, j)$ such that $J^*J = I$.

The delay-difference uncertainty lies within an elliptical uncertainty set defined by (8) which can be written as,

$$\left\| A \begin{bmatrix} \mathrm{Re}\{x\} \\ \mathrm{Im}\{x\} \end{bmatrix} + b \right\|_2 \leq 1 \Leftrightarrow \left\| AJ^* \begin{bmatrix} \mathrm{Re}\{x\} \\ j\,\mathrm{Im}\{x\} \end{bmatrix} + b \right\|_2 \leq 1$$

$$\Leftrightarrow \begin{bmatrix} 1 \\ \mathrm{Re}\{x\} \\ j\,\mathrm{Im}\{x\} \end{bmatrix}^* \begin{bmatrix} 1 - b^*b & -b^*\bar{A} \\ -\bar{A}^*b & -\bar{A}^*\bar{A} \end{bmatrix} \begin{bmatrix} 1 \\ \mathrm{Re}\{x\} \\ j\,\mathrm{Im}\{x\} \end{bmatrix}^* \geq 0$$

where $\bar{A}(j\omega) = A(\omega)J^*$. So, the IQC multiplier Π is:

$$\Pi(j\omega) = \begin{bmatrix} 1 - b^*b & -b^*\bar{A} \\ -\bar{A}^*b & -\bar{A}^*\bar{A} \end{bmatrix} (j\omega) \tag{10}$$

Since Π satisfies the condition given in Remark 1, $\tau\Delta$ also satisfies the IQC defined by Π for all $\tau \in [0, 1]$.

It's worth noting that this non-parametric multiplier necessitates the splitting of the uncertainty into its real and imaginary components, which is not feasible with parametric IQC multipliers. Additionally, the switching condition at $\frac{2\pi}{3}$ offers another natural rationale for non-parametric IQCs.

IV. Frequency-domain robust controller synthesis

In this section, first, a summary of the frequency-domain controller synthesis method from [14] is provided. Then, the conditions for robustness using IQC framework are provided using [8]. Finally, both methods are combined for robust controller synthesis with nominal performance.

A. Robustness conditions from IQC

Given a non-parametric IQC multiplier, the approach presented in [8] can be used for robust controller synthesis in the frequency domain. The robust control synthesis problem for the feedback connection between the perturbation Δ and the generalised plant G is considered,

$$q = G_{11}p + G_{12}u \tag{11a}$$

$$y = G_{21}p + G_{22}u \tag{11b}$$

$$p = \Delta(q) \tag{11c}$$

$$u = Ky \tag{11d}$$

The objective is to determine a fixed-structure controller $K \in \mathcal{K}$ that ensures the robustness of the closed-loop system against the perturbation Δ, which satisfies the IQC defined by the non-parametric multiplier Π. Subsequently, employing [8], all controllers in the form of $K = XY^{-1}$, which produce closed-loop systems satisfying the frequency domain inequality (3), are encompassed within the set defined by the following linear matrix inequalities (LMIs):

$$\begin{bmatrix} (\Pi^+)^{-1} & L \\ L^* & -\mathfrak{L} \end{bmatrix} (j\omega) \succeq 0, \quad \forall \omega \in \Omega \tag{12}$$

$$(\Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c)(j\omega) \succeq 0, \quad \forall \omega \in \Omega \tag{13}$$

where, $K_c = X_cY_c^{-1}$ is an initial stabilising controller,

$$\begin{aligned}
\mathfrak{L} &= L^*\Pi^- L_c + L_c^*\Pi^- L - L_c^*\Pi^- L_c \\
L &= \begin{bmatrix} G_{11}\Phi + G_{12}X & G_{11}\Psi \\ \Phi & \Psi \end{bmatrix} \\
L_c &= \begin{bmatrix} G_{11}\Phi_c + G_{12}X_c & G_{11}\Psi \\ \Phi_c & \Psi \end{bmatrix} \\
\Phi &= G_{21}^{\mathsf{R}}(Y - G_{22}X) \\
\Phi_c &= G_{21}^{\mathsf{R}}(Y_c - G_{22}X_c) \\
\Psi &= I - \Phi\Phi^{\mathsf{L}} = I - G_{21}^{\mathsf{R}}G_{21}
\end{aligned}$$

where $\Pi^+ \succ 0$ and $\Pi^- \preceq 0$ are chosen such that for some $\epsilon > 0$, $\Pi + \left[\begin{smallmatrix} 0 & 0 \\ 0 & \epsilon I \end{smallmatrix}\right] = \Pi^+ + \Pi^-$.

B. Controller synthesis with performance criterion

Furthermore, a desired control performance criterion can be added to the optimisation. The frequency-domain controller synthesis approach presented in [14] can be utilised allowing for the combination of both performance metrics and robustness condition into a single optimisation.

Consider a generalised LTI system, mapping exogenous disturbances $w \in \mathbb{R}^{n_w}$ and control inputs $u \in \mathbb{R}^{n_u}$ to performance channels $z \in \mathbb{R}^{n_z}$ and measurements $y \in \mathbb{R}^{n_y}$ given as:

$$\begin{aligned}
z &= G_{p,11}w + G_{p,12}u \\
y &= G_{p,21}w + G_{p,22}u
\end{aligned}$$

The synthesis objective is to design a fixed-structure feedback controller K that regulates the effect of the exogenous disturbances w onto the performance channels z given by:

$$T_{zw} = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

Under the assumption that the closed loop is stable, the norms of T_{zw} can be expressed as:

$$\begin{aligned}
\|T_{zw}\|_2^2 &= \frac{1}{2\pi}\int_\Omega \mathrm{trace}\left(T_{zw}(j\omega)T_{zw}^*(j\omega)\right) d\omega \\
\|T_{zw}\|_2^2 &= \sup_{\omega\in\Omega} \overline{\sigma}\left(T_{zw}(j\omega)T_{zw}^*(j\omega)\right)
\end{aligned}$$

where, $\overline{\sigma}(\cdot)$ is the maximum singular value. Then, the controller design problem can be formulated as the minimisation of an upper bound on the system norms

$$\begin{aligned}
&\min_{K\in\mathcal{K},\,\Gamma} \gamma \\
\text{s.t.,}\quad & K \text{ stabilises the closed-loop} \\
& T_{zw}(j\omega)T_{zw}^*(j\omega) \preceq \Gamma(j\omega), \qquad \forall \omega \in \Omega
\end{aligned} \tag{14}$$

where, $\mathcal{K}$ is the set of controllers with desired structure and $\Gamma(j\omega)$ is a Hermitian matrix. For the $\mathcal{H}_\infty$ norm, $\Gamma(j\omega) = \gamma I$, where $\gamma \in \mathbb{R}$ and for the $\mathcal{H}_2$ norm, we have:

$$\gamma = \frac{1}{2\pi}\int_\Omega \mathrm{trace}\left(\Gamma(j\omega)\right) d\omega. \tag{15}$$

Using [14], the desired optimisation can be written as a series of convex optimisations with $K_c = X_cY_c^{-1}$ as the stabilising controller from the previous iteration.

$$\min_{X,Y,\Gamma} \gamma \tag{16}$$

$$\begin{bmatrix} \Gamma - \Lambda_p & (G_{p,11}\Phi_p + G_{p,12}X) \\ \star & \Phi_p^*\Phi_{p,c} + \Phi_{p,c}^*\Phi_p - \Phi_{p,c}^*\Phi_{p,c} \end{bmatrix}(j\omega) \succ 0, \forall\omega \in \Omega$$

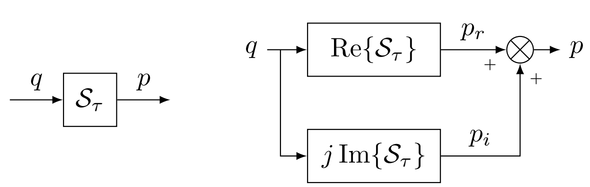
where,

$$\Phi_p = G_{p,21}^{\mathsf{R}}(Y - G_{p,22}X) \qquad \Psi_p = I - G_{p,21}^{\mathsf{R}}G_{p,21}$$
$$\Phi_{p,c} = G_{p,21}^{\mathsf{R}}(Y_c - G_{p,22}X_c) \quad \Lambda_p = (G_{p,11}\Psi_p)(G_{p,11}\Psi_p)^*$$

When the initial controller K_c is known to be stabilising, it can be shown that the controller K is also stabilising [14]. For a stable plant G_{22}, a controller with a sufficiently small gain can in general stabilise the closed-loop system. In the case of an unstable plant, a stabilising controller must be available already for system identification. To solve the optimisation problem, a grid-based approach can be employed, where the controller K is used as the initial stabilising control for the next optimisation. This sequence of convex optimisation problems will converge towards a local optimal solution of the original problem since the initial controller already satisfies the constraint and any optimisation can only improve the objective.

C. Robust controller synthesis

To utilise the proposed IQC multiplier, the real and imaginary component of the uncertainty needs to be split. The split representation of the uncertainty and the corresponding auxiliary signals p_r and p_i can be seen in Fig. 6. This split representation is feasible since the proposed controller synthesis is done in the frequency domain.



(a) True uncertainty (b) Split representation

Fig. 6: Real and imaginary splitting of the uncertainty

Then the time-delay problem under consideration can be represented in an LFR formulation as:

$$\begin{bmatrix} q_y \\ q_u \\ \hdashline y_d \end{bmatrix} = \left[\begin{array}{cccc:c} 0 & 0 & -\hat{G} & -\hat{G} & \hat{G} \\ 0 & 0 & 0 & 0 & I \\ \hdashline -I & -I & -\hat{G} & -\hat{G} & \hat{G} \end{array}\right] \begin{bmatrix} p_{yr} \\ p_{yi} \\ p_{ur} \\ p_{ui} \\ \hdashline u \end{bmatrix} \tag{17}$$

where, p_{yr} and p_{yi} are the outputs of the split $\mathcal{S}_{\tau_y}$, and similarly p_{ur} and p_{ui} are the outputs of the split $\mathcal{S}_{\tau_u}$. The corresponding Π is given as:

$$\Pi(j\omega) = \left[\begin{array}{cc:cc} \Pi_{y,11} & 0 & \Pi_{y,12} & 0 \\ 0 & \Pi_{u,11} & 0 & \Pi_{u,12} \\ \hdashline \Pi_{y,21} & 0 & \Pi_{y,22} & 0 \\ 0 & \Pi_{u,21} & 0 & \Pi_{u,22} \end{array}\right](j\omega) \tag{18}$$

where $\Pi_y(j\omega)$ and $\Pi_u(j\omega)$ are the non-parametric multipliers for the delay uncertainty for the measurements and the control signals respectively. In scenarios involving multiple delayed measurements and control signals, the uncertainty of each signal can be considered independently and structured diagonally. Π can be derived using similar logic.

Now, with an appropriate choice of Π^+ and Π^-, a single optimisation for a controller that is robust to time-delay variations and has good performance under nominal conditions can be written as:

$$\min_{X,Y} \gamma \tag{19}$$

$$\text{s.t.} \quad \begin{bmatrix} \Gamma - \Lambda_p & (G_{p,11}\Phi_p + G_{p,12}X) \\ \star & \Phi_p^*\Phi_{p,c} + \Phi_{p,c}^*\Phi_p - \Phi_{p,c}^*\Phi_{p,c} \end{bmatrix}(j\omega) \succ 0$$

$$\begin{bmatrix} (\Pi^+)^{-1} & L \\ L^* & -\mathfrak{L} \end{bmatrix}(j\omega) \succeq 0 \qquad \forall\omega \in \Omega$$

$$(\Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c)(j\omega) \succeq 0 \qquad \forall\omega \in \Omega$$

The final set of constraints is necessary to ensure the stability of the nominal closed loop. However, since the performance objective already guarantees this stability, the final set of constraints can be safely dropped.

V. Numerical Simulations

Conventional suspension systems consist of a spring and damper between the main body of the car and the wheel assembly. The spring-damper characteristics are usually selected based on the desired trade-off between passenger comfort, road handling, and suspension deflection. Active suspensions achieve a better balance of these objectives using a feedback-controlled hydraulic actuator between the chassis and wheel assembly. In this section, the active suspension system example from the MATLAB Robust Control Toolbox is taken and converted to discrete-time at a sampling frequency of $100\,\text{Hz}$. It is assumed that the controller is collocated with the actuator, i.e. there is no delay, while the sensors are connected via the intra-car network such as a CAN bus. Under nominal conditions, the measurements should arrive within a sampling time step, i.e., there is no delay. However, under peak congestion, priority is assigned to critical signals such as throttle control. Consequently, the sensor readings may experience delays of up to 20 samples or $0.2\,\text{s}$. Furthermore, each sensor can directly transmit sensor values onto the bus, so a combination of delays may occur.

Fig. 7 shows an example of an active suspension system. The system under consideration has two sensors: body acceleration (a_b) and suspension travel (s_d). The sensor values are sent to the controller via the CAN network which introduces delays τ_1 and τ_2 to the measurements, respectively.

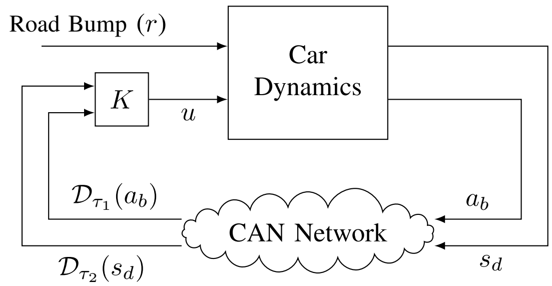


Fig. 7: Active suspension system in an intra-car network

The objective of the controller synthesis is to design a controller with a trade-off between passenger comfort and road handling. This paper considers the scenario where road handling is prioritised ($\beta = 0.99$). More details on the used filters and model parameters can be found in the MATLAB documentation [17].

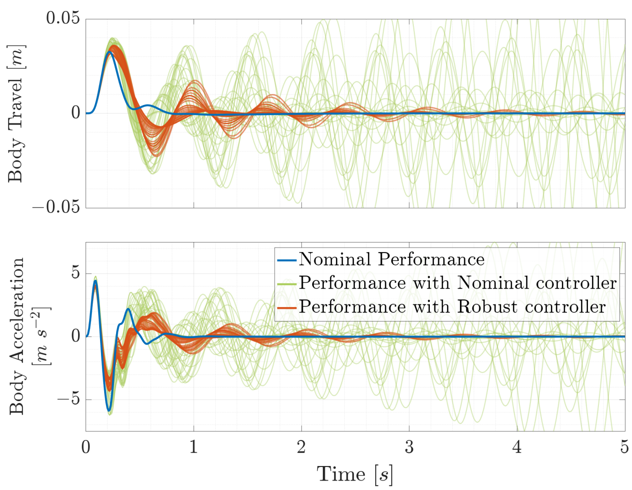


Fig. 8: Simulation of the active suspension system with the nominal and the robust controllers for a road bump of height 5 cm. System with nominal delay and 30 different realisation of the delays are shown.

The non-parametric IQC (10) is defined for both delays to account for robustness to time delays from 0 to 20 samples. Now a robust second-order controller can be synthesised. Fig. 8 shows the time-domain simulation results for a road bump of height 5 cm. The controller with nominal performance may lead to instability for certain delay combinations, whereas the robust controller maintains system stability regardless of delay variations.

Furthermore, the parametric IQC (6) is used for controller synthesis with $c = 1$. Although controllers synthesised using either the parametric or the non-parametric IQC achieve robustness to time delay, the controller obtained using the non-parametric IQC formulation demonstrates a better nominal performance of 1.5048 compared to the controller obtained using the parametric IQC formulation, which has a nominal performance of 1.6581. This highlights the reduced conservatism of the proposed non-parametric IQC compared to the parametric IQC.

VI. Conclusion

This paper presents a novel non-parametric IQC for the delay-difference operator, which can be used to guarantee robustness against time-delay uncertainty. This non-parametric IQC represents an elliptical set covering the uncertainty in the delay-difference operator. It can then be utilised for synthesis of a robust controller with desired frequency-domain performance metrics. Finally, numerical simulations of an active suspension system within an intra-car network are presented. The simulations show that the synthesised controller is robust to time-delay uncertainty in the prescribed range and exhibits reduced conservatism compared to the controller synthesised using the parametric IQC.

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