2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Filtered High Gain Interval Observer for LPV Systems with Bounded Uncertainties

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Abstract— **In this paper, a new High-Gain Interval Observer (HGIO) structure and its filtered version, named Filtered High-Gain Interval Observer (FHGIO), are proposed for a class of Linear Parameter Varying (LPV) systems subject to additive disturbances and measurement noise. Those uncertainties are assumed to be unknown but bounded with known values. The HGIO is based on a high-gain observer structure from which an interval formulation is deduced taking into account the uncertainties bounds. Then, the proposed HGIO is extended to incorporate a filter for the output estimation error, leading to the FHGIO design whose goal is to reduce the measurement noise amplification. Usually, the design of such interval observers is based on monotone systems theory which is hard to satisfy in many cases. In this paper, suitable changes of coordinates are used to overcome this limitation. Moreover, a sufficient condition for the non-divergence of the radius dynamics and a procedure to design the observers gains ensuring the stability are given for each observer. The efficiency of the proposed observers is illustrated through a simulation on a numerical example.**

I. INTRODUCTION

Model-based state estimation of uncertain systems is a challenging problem when it comes to controlling complex applications. Uncertainties are often due to insufficient knowledge about the system itself or its environment. It can include unmodelled or neglected dynamics, parameter variations, disturbances, noise, and more. To tackle this challenge, different techniques have been developed. In the stochastic framework, Extended, Unscented Kalman Filters and Particle Filters are the most frequently used approaches as presented in [10]. However, they prove complex to setup because of the tuning process that requires to quantify the accuracy of the model and some insight on the probability density function. Moreover, the stability is not guaranteed. In the deterministic framework, Sliding Modes Observers, High Gain Observers (HGO), Interval Observers or Adaptive Observers offer robust tools to cope with uncertainties for state estimation. Neural Networks methods [17] have also been recently introduced for state estimation to tackle the problem of accounting for unmodelled uncertainties.

Among all these various possibilities, HGOs had become one of the most popular solution because of their simplicity of implementation. They were first developed in the eighties as a robust observer design for linear feedback systems and they have been progressively extended to specific classes of nonlinear systems. Nowadays, the most common class of systems for designing HGOs is the observability canonical form [11], however it is possible to carry out the design for unstructured systems by taking additional considerations [16]. Besides their simplicity coming from a Luenberger type of structure, HGOs can be tuned by a unique high gain parameter, usually denoted θ. This parameter should be taken large enough to ensure robustness and fast convergence. However, the high gain effect induces three major drawbacks. The first one concerns the numerical implementation for high dimension systems and large value of θ. The second drawback is the well-known peaking phenomenon of the state estimations in the transient periods. Finally, the last drawback is the effect of measurement noise that are amplified in the corrective term.

To overcome those issues, different techniques have been developed in the literature. In [22], a new HG/LMI design is presented where the standard high-gain methodology and the LMI-based observer design technique are combined in order to reduce the maximum power of θ. In [21] and [1], a filter is used inside a high-gain structure in order to reduce the impact of the measurement noise. Thus, the output estimation error is filtered before being amplified in the corrective term. Finally, a solution to reduce disturbances impact is to compensate by estimation such as presented by the survey in [3].

Another popular way to deal with the uncertainties in the deterministic context is to consider a set-membership framework. In this case, all the considered uncertainties are assumed unknown but bounded with known bounds. There are two main types of set-membership observers. On the one hand, those which are based on geometrical sets, such as polytopes, ellipsoids ([14]) or zonotopes ([4]). On the other hand, those based on the description of the sets under the interval forms ([7], [20]). They provide intervals containing the actual state values, usually defined by minimal and maximal values. They were first introduced in [9] and they still are an active topic of research since then. Set-membership observers have been frequently used for LPV systems. They are indeed able to represent a wide variety of non-linear systems, notably thanks to guaranteed transformations [12], and allow the use of well established tools for linear systems

This work was supported by the French National Research Agency (ANR) projects ArtISMo and DESIHR. ArtISMo is funded by ANR and DESIHR is co-funded by ANR, Normandie Region and Hauts De France Region. Their financial support is gratefully aknowledged.

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[2], [13], [8].
To the best of the authors knowledge, observers that combine both the high-gain and intervals advantages have been little studied so far [18], [19], [15]. In [18] and [19], two similar HGIO structures based on center-radius interval definition have been proposed and rely on three design steps. The first one is the observer gain selection, the second one deals with the time-varying change of coordinate settings and the third one establishes the interval observer structure. In [19], the observer gain is designed using a LMI pole placement technique without considering actual high-gain poles. Moreover, in both [18] and [19], the norm of the change of coordinates is exponentially decreasing over time which may induce numerical precision issues. All those limitations have been overcome in this work.
This paper presents a new High Gain Interval Observer (HGIO) and its filtered version named Filtered-HGIO (FHGIO) for a class of LPV systems subject to uncertainties. Based on the high-gain observers theory, a high-gain parameter θ will be used to reduce the width of the estimation bounds by increasing its value. To overcome the drawback of measurement noise amplification induced by large θ, the proposed FHGIO design incorporates an output error filter in the observer structure. The uncertainties are considered unknown but bounded with known values and the intervals are formulated in the centered form. A procedure to compute the observer gains is given and a suitable change of coordinates ensuring the cooperativity of the system in the new base is introduced. A sufficient condition ensuring the non-divergence of the radius dynamics is also detailed for each interval observer. To illustrate their efficiency, both observers are compared in simulation on a numerical example.
The paper is organized as follows: Section 2 presents some notations and preliminaries material before introducing the problem statement. Then, in Section 3, the HGIO and FHGIO designs are presented with their stability analysis. Section 4 deals with the numerical simulation of the proposed observers. Finally, a conclusion is drawn in Section 5.

II. Preliminaries and Problem statement

A. Notations and interval preliminaries

Throughout this paper, the following notations will be used. For any positive integers n and m, I_n and O_n are respectively the identity and zero matrices of dimension n. $O_{n\times m}$ is the $n\times m$ zero matrix. **1** and **0** are column vectors of respectively ones and zeros, with appropriate dimensions. For a vector $v\in\mathbb{C}^n$, diag(v) is the operator that creates a diagonal matrix with the components of v. Note that $\mathbb{C}$ is the set of complex numbers.
Complex intervals will be used in this work which will require to state some definitions. First, a partial order over $\mathbb{C}$ is introduced : $\forall(a,b)\in\mathbb{C}\times\mathbb{C}, a\star b\Leftrightarrow(a^R\star b^R)\wedge(a^I\star b^I)$ where $\star\in\{=,<,>,\leq,\geq\}$. The superscripts $.^R$ and $.^I$ stand respectively for the real and imaginary parts. Then if $a<b$, a complex interval can be defined as $[a,b]=[a^R,b^R]+i[a^I,b^I]$. It can also be expressed in the centered form using the operator $\pm$ defined as :

$$\begin{array}{rccl}\pm: & \mathbb{C}\times\mathbb{C}^+ & \to & \mathbb{IC}\\ & (c,r) & \mapsto & c\pm r=[c-r,c+r]\end{array}\tag{1}$$

The two forms are related by $c=\frac{a+b}{2}$ and $r=\frac{b-a}{2}$.
The superscripts $.^c$ and $.^r$ will be used to denote respectively the center and the radius of an interval.
Complex intervals arithmetic requires to introduce the operators *ctimes* and *cabs*, defined respectively as $a\diamond b=|a||b|+2|a^I||b^I|$ and $|a|=|a^R|+i|a^I|$ where $|.|$ applied to real scalars is the classic absolute value. The modulus of $a\in\mathbb{C}$ is denoted $\|a\|$. Note that the previous definitions can be directly extended to vectors and matrices considering element-by-element operations.
The following theorem establishes the expression of the linear image of a complex interval matrix.

Theorem 1. $\forall(A,B,C)\in\mathbb{C}^{n\times p}\times\mathbb{C}^{p\times q}\times\mathbb{C}^{p\times q}$,

$$A(B\pm C)=(AB)\pm(A\diamond C)\in\mathbb{IC}^{n\times q}\tag{2}$$

where $A\diamond C=|A||C|+2|A^I||C^I|$

The proof is given in [6]. The next propositions and corollaries have all been presented and proven in [5].

Proposition 1. $\forall(A,B,v)\in\mathbb{C}^{n\times p}\times\mathbb{C}^{n\times q}\times(\mathbb{R}^+)^n$,

$$\begin{aligned}|[A,B]|\boldsymbol{1}&=|A|\boldsymbol{1}+|B|\boldsymbol{1}\\ |A|\boldsymbol{1}&\leq\|A\|\boldsymbol{1}(1+i)\\ \|Adiag(v)\|\boldsymbol{1}&=\|A\|v\end{aligned}\tag{3}$$

Proposition 2. *Let* $z:\mathbb{R}^+\to\mathbb{C}^n$, $z^c:\mathbb{R}^+\to\mathbb{C}^n$ *and* $z^r:\mathbb{R}^+\to(\mathbb{C}^+)^n$ *be three continuous functions. If* $\forall t\in\mathbb{R}^+$, $z(t)\in z^c(t)\pm z^r(t)$, *then* $\exists\sigma:\mathbb{R}^+\to[-1,1]^{2n}$ *a continuous function returning bounded real vector values such that:*

$$z(t)=z^c(t)+\Delta(z^r(t))\sigma(t)\tag{4}$$

where $\forall v\in\mathbb{C}^n,\Delta(v)=\begin{bmatrix}diag(v^R) & i.diag(v^I)\end{bmatrix}\in\mathbb{C}^{n\times 2n}$

Corollary 1. *Consider the state* $z(t)\in\mathbb{C}^n$ *of the system*

$$\dot z(t)=diag(\xi)z(t)+\Phi(t,z(t))\tag{5}$$

where $\xi\in\mathbb{R}^n$ *and* $\Phi:\mathbb{R}^+\times\mathbb{C}^n\to(\mathbb{C}^+)^n$ *is a positive and locally Lipschitz function w.r.t* $z(t)$, *i.e:*

$$\forall t\in\mathbb{R}^+,\Phi(t,z(t))\geq 0\tag{6}$$

Then if $z(0)\geq 0,\forall t\in\mathbb{R}^+,z(t)\geq 0$

Proposition 3. *Consider the notations of Corollary 1 and* $\bar z(t)\in\mathbb{C}^n$ *as the state of the system*

$$\dot{\bar z}(t)=diag(\xi)\bar z(t)+\bar\Phi(t,\bar z(t))\tag{7}$$

where $\xi\in\mathbb{R}^n$, $\bar z(0)\geq z(0)$ *and* $\bar\Phi:\mathbb{R}^+\times\mathbb{C}^n\to(\mathbb{C}^+)^n$ *is a positive and locally Lipschitz function satisfying*

$$\forall t\in\mathbb{R}^+,\Phi(t,z(t))\leq\bar\Phi(t,\bar z(t))\tag{8}$$

Hence, $\bar\Phi(t,\bar z(t))$ *is an upper bound of* $\Phi(t,z(t))$ *and so is* $\bar z(t)$ *for* $z(t)$.
The upper bound $\bar\Phi(t,\bar z(t))$ *is written:*

$$\bar\Phi(t,\bar z(t))=g(t)+H(1+i)z^R(t)+H(1+i)z^I(t)\tag{9}$$

where $g : \mathbb{R}^+ \to (\mathbb{C}^+)^n$ is a positive continuous function and $H \in {\mathbb{R}^+}^{n\times n}$.

Using the previous proposition and based on the Corollary 14 in [5] where $M^R = M^I = N^R = N^I = H$, the following corollary can be deduced.

Corollary 2. *Let $\hat{M} = [diag(\xi) + H, H; H, diag(\xi) + H] \in \mathbb{R}^{2n\times 2n}$. If the Metzler matrix $\hat{M}$ is Hurwitz and $g(t)$ is bounded, then $\bar{z}(t)$ is bounded and follows a stable dynamics. $z(t)$ is also bounded by the upper bound $\bar{z}(t)$.*

B. Problem statement

Consider a class of Linear Parameter Varying (LPV) system described by :

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) + Ed(t) \\ y(t) = Cx(t) + w(t) \end{cases} \tag{10}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the known input and $y(t) \in \mathbb{R}^{n_y}$ is the output vector, $d(t) \in \mathbb{R}^{n_d}$ and and $w(t) \in \mathbb{R}^{n_y}$ denote respectively the unknown disturbances and the measurement noise, $\rho(t) \in \mathbb{R}^{n_r}$ is the unmeasurable scheduling vector.
The matrices $A(\rho(t))$ and $B(\rho(t))$ can be decomposed as:

$$\begin{aligned} A(\rho(t)) &= A_0 + \sum_{i=1}^{n_r} A_i\rho_i(t) \\ B(\rho(t)) &= B_0 + \sum_{i=1}^{n_r} B_i\rho_i(t) \end{aligned} \tag{11}$$

where A_0 and B_0 are known constant matrices that represent the nominal part of the system. A_i and B_i $(i = 1, ..., n_r)$ are known constant matrices representing the uncertainties related to the scheduling vector $\rho(t)$ whose i-th component is denoted by $\rho_i(t)$.
In order to make the link with classic HGOs designs, the system (10) is written as a partially linear system:

$$\begin{cases} \dot{x}(t) = A_0x(t) + \varphi(u(t), x(t)) + Ed(t) \\ y(t) = Cx(t) + w(t) \end{cases} \tag{12}$$

where $\varphi(u(t), x(t)) = \sum_{i=1}^{n_r} A_i\rho_i(t)x(t) + B(\rho(t))u(t)$ is locally Lipschitz w.r.t to $x(t)$ uniformly in $u(t)$, provided that $\rho(t)$ is differentiable and bounded.
Three assumptions are considered for the design of the proposed observers.

Assumption 1. *The pair (A_0, C) is observable.*

Assumption 2. *The input $u(t)$ is bounded $\forall t \in \mathbb{R}^+$.*

Assumption 3. *The initial state is unknown but bounded with known bounds, i.e : $x(0) \in x^c(0) \pm x^r(0)$ where $x^c(0) \in \mathbb{R}^{n_x}$ and $x^r(0) \in (\mathbb{R}^+)^{n_x}$.
The other considered uncertainties are unknown but bounded with known bounds such that:*

$$\forall t \in \mathbb{R}^+, \begin{cases} \rho(t) \in \boldsymbol{0} \pm \boldsymbol{1} \\ d(t) \in d^c \pm d^r \\ w(t) \in w^c \pm w^r \end{cases}$$

where $(d^c, d^r) \in \mathbb{R}^{n_d} \times (\mathbb{R}^+)^{n_d}$, $(w^c, w^r) \in \mathbb{R}^{n_y} \times (\mathbb{R}^+)^{n_y}$.

Remark 1. The assumption on the scheduling vector bounds is not restrictive since the matrices A_0, A_i, B_0 and B_i can be easily adapted so that it is satisfied.

III. MAIN RESULTS

A. High-Gain Interval Observer (HGIO)

*1) **HGIO design**:* In this section, the new HGIO design is presented for system (12). Three steps are required to build its structure. The first step is to establish a HGO for the considered system and to design the observer gain. Then, a time-varying change of coordinates is introduced to guarantee the Metzler character of the state matrix in the new base. Finally, the last step is the interval formulation of the observer which is derived based on the uncertainties bounds.

STEP 1: HGO structure and gain design.
The proposed HGO structure for system (12) is given by:

$$\dot{\hat{x}}(t) = (A_0 - LC)\hat{x}(t) + \varphi(u(t), \hat{x}(t)) + E\hat{d}(t) + Ly(t) \tag{13}$$

where $\hat{x}(t)$ is the state estimate and $\hat{d}(t)$ represents an estimation of the disturbances. It is assumed that $\hat{d}(t)$ has the same bounds as $d(t)$, i.e $\hat{d}(t) \in d^c \pm d^r$. L is the observer gain designed following the three steps below:

1) Compute a gain L' by a pole placement technique such that $(A_0 - L'C)$ is Hurwitz with distinct desired poles $\xi + i\zeta \in \mathbb{C}^{n_x}$ lying in the left half plane and close to the origin (the proximity depends on the studied system). ξ and ζ represent respectively the real and imaginary parts of the eigenvalues of $(A_0 - L'C)$.
2) Verify that L' satisfies the following stability sufficient condition (see Therorem 2 in [16]):
$$\min_{\omega\geq 0} \sigma_{min}(A_0 - L'C - i\omega I_{n_x}) > \gamma_\varphi \tag{14}$$
where σ_{min} is the minimum singular value and γ_φ is the Lipschitz constant of the function $\varphi(u(t), x(t))$.
If the condition is not satisfied then go back to step 1) and change the eigenvalues, else go to step 3).
3) Now, for a given $\theta \geq 1$, compute the observer gain L such that the poles of $(A_0 - LC)$ are $\theta(\xi + i\zeta)$. This allows the poles of $(A_0 - LC)$ to be placed further into the left half plane so that the high-gain effect is guaranteed while satisfying the condition (14) for L.

Remark 2. Notice that the following relation holds:

$$\begin{aligned} &\min_{\omega\geq 0} \sigma_{min}(A_0 - L'C - i\omega I_{n_x}) \\ &\leq \min_{\omega\geq 0} \sigma_{min}(A_0 - LC - i\omega I_{n_x}) \end{aligned} \tag{15}$$

Then, it is straightforward that equation (14) is satisfied for the computed observer gain L. On the other hand, it is worth noticing that the condition (14), with the observer gain L, allows to relax the assumption that the pair (A_0, C) is under the canonical form of the classic high-gain observer design framework as in [11].

Thus, this method allows to find a high-gain type observation gain L, indirectly parameterized by θ, that makes the HGO asymptotically stable as well as $(A_0 - LC)$ Hurwitz and $\mathbb{C}$-diagonalizable (diagonalizable in the field of complex numbers). The latter condition is required in order to define the time-varying change of coordinates in the next step which will ensure the cooperativity of the system in the new base.

STEP 2: Change of coordinates.
Assuming L has been designed as previously mentioned, $(A_0 - LC)$ is thus $\mathbb{C}$-diagonalizable:

$$A_0 - LC = v\text{diag}(\theta\xi + i\theta\zeta)v^{-1} \tag{16}$$

where $v \in \mathbb{C}^{n_x \times n_x}$ is formed from the eigenvectors. $\theta\xi < \mathbf{0}$ represents the real parts of the eigenvalues and $\theta\zeta$ the imaginary parts.
Consider the following time-varying change of coordinates:

$$z(t) = \Omega(t)\hat{x}(t) \tag{17}$$

where $\Omega(t) = \text{diag}(e^{-i\theta\zeta t})v^{-1}$.
Then, the observer dynamics in the new base (z) is:

$$\dot{z}(t) = \text{diag}(\theta\xi)z(t) + \Psi(t) \tag{18}$$

where $\Psi(t) = \Omega(t)(\varphi(u(t), \hat{x}(t)) + E\hat{d}(t) + Ly(t))$.

Proof. Using the usual derivative rules, equations (13), (16) and (17) as well as the commutativity of diagonal matrices, one could derive the following computations:

$$\begin{aligned}\dot{z}(t) &= \dot{\Omega}(t)\hat{x}(t) + \Omega(t)\dot{\hat{x}}(t)\\ &= \text{diag}(-i\theta\zeta)\Omega(t)\hat{x}(t) + \Omega(t)((A_0 - LC)\hat{x}(t) +\\ &\quad \varphi(u(t), \hat{x}(t)) + E\hat{d}(t) + Ly(t))\\ &= -\text{diag}(i\theta\zeta)z(t) + \Omega(t)v\text{diag}(\theta\xi + i\theta\zeta)v^{-1}\Omega^{-1}(t)z(t)\\ &\quad + \Omega(t)(\varphi(u(t), \hat{x}(t)) + E\hat{d}(t) + Ly(t))\\ &= \text{diag}(\theta\xi)z(t) + \Psi(t)\end{aligned}$$

□

Remark 3. The observer dynamics in the new base (z) is driven by $\text{diag}(\theta\xi)$ which is Hurwitz and Metzler by construction. Therefore, the change of coordinates overcomes the difficulty of designing L such that the system (13) is stable and monotone. Indeed, such properties are very hard to satisfy at the same time. Moreover, note that if the desired poles are real, the change of coordinate becomes time-invariant.

STEP 3: Interval observer structure.
Based on the proposed high-gain observer dynamics (18) in the new base and using the bounds of the uncertainties, the HGIO is given by the following theorem:

Theorem 2. *Consider the system described by (13) and the Assumptions 1-3. The system:*

$$\begin{cases}\dot{z}^c(t) = diag(\theta\xi)z^c(t) + \Psi^c(t)\\ \dot{z}^r(t) = diag(\theta\xi)z^r(t) + \Psi^r(t)\end{cases} \tag{19}$$

is an interval observer for system (13), named HGIO, described by its center $z^c(t)$ and radius $z^r(t)$ state dynamics in the new base (z) with:

$$\begin{cases}\Psi^c(t) = \Omega(t)(B_0u(t) + Ed^c + Ly(t))\\ \Psi^r(t) = |\Omega(t)[\ldots A_i\Omega^{-1}(t)z^c(t) + B_iu(t)\ldots,\\ \qquad\qquad \ldots A_i\Omega^{-1}(t)\Delta(z^r(t))\ldots, Ed^r]|\mathbf{1}\end{cases} \tag{20}$$

Moreover, the observer satisfies the inclusion property (21) in the new base which can also be expressed by (22) in the original base.

$$\forall t \in \mathbb{R}^+, z^r(t) \geq 0 \wedge z(t) \in z^c(t) \pm z^r(t) \subset \mathbb{C}^{n_x} \tag{21}$$

$$\forall t \in \mathbb{R}^+, \hat{x}^r(t) \geq 0 \wedge \hat{x}(t) \in \hat{x}^c(t) \pm \hat{x}^r(t) \subset \mathbb{R}^{n_x} \tag{22}$$

where:

$$\forall t \in \mathbb{R}^+, \hat{x}^c(t) = \Omega^{-1}(t)z^c(t), \hat{x}^r(t) = \Omega^{-1}(t)\diamond z^r(t) \tag{23}$$

Proof. Consider the bounds of the uncertainties given in Assumption 3 and $z^c(t)$ and $z^r(t)$ as being any two continuous functions such that $z(t) \in z^c(t) \pm z^r(t)$. Then, using Proposition 2 for $z(t)$, the following property for $\Psi(t)$ is deduced:

$$\begin{aligned}\Psi(t) \in \Omega(t)(B_0u(t) + Ed^c + Ly(t)) + \Omega(t)[\ldots A_i\Omega^{-1}(t)z^c(t)\\ + B_iu(t)\ldots, \ldots A_i\Omega^{-1}(t)\Delta(z^r(t))\ldots, Ed^r](\mathbf{0} \pm \mathbf{1})\end{aligned} \tag{24}$$

Then applying Theorem 1 to $\Psi(t)$, the expressions in equation (20) can be deduced. Hence, using Theorem 3 in [6], Theorem 2 is proven. □

*2) **HGIO stability analysis**:* The proof of stability of the HGIO presented in Therorem 2 is divided in two parts. The first one deals with the stability of the center dynamics whereas the second one is about the radius dynamics.

Proof. Stability of the center dynamics:
The first line of equation (19) gives the center dynamics of the observer. It is composed of the endogenous term $\text{diag}(\theta\xi)z^c(t)$ and the term $\Psi^c(t)$. The latter depends only on the bounded exogenous variables $u(t)$ and $y(t)$ according to Assumption 2. Thus $\Psi^c(t)$ is bounded. Hence, the center stability condition comes down to having $\text{diag}(\theta\xi)$ Hurwitz, which is achieved by the suitable design of L. □

Proof. Stability of the radius dynamics:
The stability of the radius dynamics, given by the second line of equation (19), can not be deduced directly. Indeed the term $\Psi^r(t)$ depends on the bounded exogenous variables $u(t)$, $y(t)$ and $z^c(t)$ but also depends on the endogenous term $z^r(t)$. Thus, a non-divergence sufficient condition to ensure the stability is proposed. It is formulated in the following proposition settled using Proposition 3 and Corollary 2 with $H = \sum_{i=1}^{n_r} \|v^{-1}A_iv\|$.

Proposition 4. *Let $H = \sum_{i=1}^{n_r} \|vA_iv^{-1}\| \in (\mathbb{R}^+)^n$. If $diag(\theta\xi)$ is Hurwitz and if the Metzler matrix $[diag(\theta\xi) + H, H; H, diag(\theta\xi) + H]$ is Hurwitz, then $\forall t \in \mathbb{R}^+, 0 \leq z^r(t) \leq \bar{z}^r(t)$ where $\bar{z}^r(t)$ follows a stable dynamics and so is $z^r(t)$.*

□

B. Filtered High-Gain Interval Observer (FHGIO)

1) ***FHGIO design****:* This section presents the design of a FHGIO for system (10). It is an extension of the work done with HGIO where an estimation error filter is incorporated within the observer structure. As a first step, the observer and filter structures are established. Then, a suitable time-varying change of coordinates is proposed to guarantee the Metzler character of the state matrix in the new base. Finally, the assumptions on the bounded uncertainties are used to derived the interval observer structure.

STEP 1: FHGO structure and gains design.
The proposed structure for the FHGO is composed of two subsystems. The first one, dedicated to state estimation, is similar to the HGO presented before where the corrective term depends now on the state of the second subsystem. The latter is a linear low-pass filter applied to the estimation error. Hence, the FHGO structure for the system (12) is:

$$\begin{cases} \dot{\hat{x}}(t) = A_0\hat{x}(t) + \varphi(u(t), \hat{x}(t)) + E\hat{d}(t) + \bar{L}\eta(t) \\ \dot{\eta}(t) = D\eta(t) + \theta(y(t) - C\hat{x}(t)) \end{cases} \tag{25}$$

where $\hat{x}(t)$ is the state estimate and $\eta(t) \in \mathbb{R}^{n_y}$ is the filtered estimation error. $\bar{L} \in \mathbb{R}^{n_x \times n_y}$ and $D \in \mathbb{R}^{n_y}$ are two matrices which will be designed later in order to guarantee the stability. Again, $\hat{d}(t)$ is a disturbance estimation which satisfy $\hat{d}(t) \in d^c \pm d^r$. $\theta \geq 1$ is the high-gain parameter.
A compact writing of (25) is deduced by considering the augmented state $\tilde{x}(t) = \begin{bmatrix}\hat{x}^T(t) & \eta^T(t)\end{bmatrix}^T$ and the augmented input $\tilde{u}(t) = \begin{bmatrix}\hat{d}^T(t) & y^T(t)\end{bmatrix}^T$:

$$\dot{\tilde{x}}(t) = M\tilde{x}(t) + \Phi(u(t), \tilde{x}(t)) + N\tilde{u}(t) \tag{26}$$

where $M = \begin{bmatrix} A_0 & \bar{L} \\ -\theta C & D \end{bmatrix}$, $N = \begin{bmatrix} E & O_{n_x \times n_y} \\ O_{n_y \times n_d} & \theta I_{n_y} \end{bmatrix}$ and
$\Phi(u(t), \tilde{x}(t)) = \begin{bmatrix} \sum_{i=1}^{n_r} A_i\rho_i(t) & O_{n_x \times n_y} \\ O_{n_y \times n_x} & O_{n_y} \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} B(\rho(t)) \\ O_{n_y \times n_u} \end{bmatrix} u(t).$
To design the gains $\bar{L}$ and D, the following procedure is proposed:

1) Consider the matrices $\bar{A} = \begin{bmatrix} A_0 & O_{n_x \times n_y} \\ \theta C & O_{n_y} \end{bmatrix}$ and $\bar{C} = \begin{bmatrix} O_{n_y \times n_x} & -I_{n_y} \end{bmatrix}$ that is an observable pair if (A_0, C) is observable (the proof is straightforward and thus omitted due to space limitation).
   Compute $\bar{K}' = \begin{bmatrix} \bar{L}'^T & D'^T \end{bmatrix}^T$ by a pole placement technique such that $M' = \bar{A} - \bar{K}'\bar{C}$ is Hurwitz with the distinct desired poles $\bar{\xi} + i\bar{\zeta} \in \mathbb{C}^{n_x+n_y}$ lying in the left half plane and close to the origin. Note that there are $n_x + n_y$ poles to place. The first n_x poles are for the state estimation dynamics and the last n_y poles tune the filter dynamics. If the filter poles are taken close to the state estimation poles, then the filter will be more efficient but it will lead to a greater conservatism in the state estimation intervals. Conversely, when the filter poles are selected further from the other poles, the effect of filtering is reduced, inducing less conservatism.
2) Verify that M' satisfies the following stability sufficient condition (see Therorem 2 in [16]):

$$\min_{\omega \geq 0} \sigma_{min}(M' - i\omega I_{n_x+n_y}) > \gamma_\Phi \tag{27}$$

   where γ_Φ is the Lipschitz constant of the function $\Phi(u(t), \tilde{x}(t))$. Note that $\gamma_\Phi = \gamma_\varphi$.
   If the condition is not satisfied then go back to step 1) and change the eigenvalues, else go to step 3).
3) Now, for a given $\theta \geq 1$, compute $\bar{K}$ such that the poles of M are $\theta(\bar{\xi} + i\bar{\zeta})$. This allows the poles of M to be placed further into the left half plane so that the high-gain effect is guaranteed while satisfying the condition (27) for M.

Hence, this method allows to compute the gains $\bar{L}$ and D, indirectly parameterized by θ, such that the FHGO is stable and the matrix M is Hurwitz as well as $\mathbb{C}$-diagonalizable.

STEP 2: Change of coordinates.
Assuming that the gains $\bar{K}$ and D have been computed following the previous method. M is thus Hurwitz and $\mathbb{C}$-diagonalizable such that:

$$M = \bar{v}\text{diag}(\theta\bar{\xi} + i\theta\bar{\zeta})\bar{v}^{-1} \tag{28}$$

where $\bar{v} \in \mathbb{C}^{(n_x+n_y)\times(n_x+n_y)}$ is formed from the eigenvectors and $\theta\bar{\xi} + i\theta\bar{\zeta} \in \mathbb{C}^{n_x+n_y}$ is the vector of the eigenvalues. The following change of coordinates is thus considered:

$$\tilde{z}(t) = \bar{\Omega}(t)\tilde{x}(t) \tag{29}$$

where $\bar{\Omega}(t) = \text{diag}(e^{-i\theta\bar{\zeta}t})\bar{v}^{-1}$.
Hence, the observer dynamics in the new base is:

$$\dot{\tilde{z}}(t) = \text{diag}(\theta\bar{\xi})\tilde{z}(t) + \bar{\Psi}(t) \tag{30}$$

where $\bar{\Psi}(t) = \bar{\Omega}(t)(\Phi(u(t), \bar{\Omega}^{-1}(t)\tilde{z}(t)) + N\tilde{u}(t))$.
The proof is similar to the one of equation (18) and Remark 3 applied to (30) still holds.

STEP 3: Interval observer structure.
Based on the FHGO dynamics (30) in the new base and using the bounds of the uncertainties, the FHGIO is given by the following theorem:

Theorem 3. *Consider the system described by (25) and the Assumptions 1-3. The system:*

$$\begin{cases} \dot{\tilde{z}}^c(t) = diag(\theta\bar{\xi})\tilde{z}^c(t) + \bar{\Psi}^c(t) \\ \dot{\tilde{z}}^r(t) = diag(\theta\bar{\xi})\tilde{z}^r(t) + \bar{\Psi}^r(t) \end{cases} \tag{31}$$

is an interval observer for system (25), named FHGIO, described by its center $\tilde{z}^c(t)$ and radius $\tilde{z}^r(t)$ state dynamics in the new base $(\tilde{z})$ with:

$$\begin{cases} \bar{\Psi}^c(t) = \bar{\Omega}(t)(\Phi^c(u(t)) + N\tilde{u}^c(t)) \\ \bar{\Psi}^r(t) = |\bar{\Omega}(t)[...\Phi_i^r(u(t), \bar{\Omega}^{-1}(t)\tilde{z}^c(t), \bar{\Omega}^{-1}(t)\Delta(\tilde{z}^r(t)))..., \\ \qquad N\tilde{u}^r]|\mathbf{1} \end{cases} \tag{32}$$

Moreover, the observer satisfies the inclusion property (33) in the new base which can also be expressed by (34) in the original base.

$$\forall t \in \mathbb{R}^+, \tilde{z}^r(t) \geq 0 \wedge \tilde{z}(t) \in \tilde{z}^c(t) \pm \tilde{z}^r(t) \subset \mathbb{C}^{n_x+n_y} \quad (33)$$

$$\forall t \in \mathbb{R}^+, \tilde{x}^r(t) \geq 0 \wedge \tilde{x}(t) \in \tilde{x}^c(t) \pm \tilde{x}^r(t) \subset \mathbb{R}^{n_x+n_y} \quad (34)$$

where:

$$\forall t \in \mathbb{R}^+, \tilde{x}^c(t) = \Omega^{-1}(t)\tilde{z}^c(t), \tilde{x}^r(t) = \Omega^{-1}(t) \diamond \tilde{z}^r(t) \quad (35)$$

Proof. Consider the bounds of the uncertainties given in Assumption 3 and $\tilde{z}^c(t)$ and $\tilde{z}^r(t)$ as being any two continuous functions such that $\tilde{z}(t) \in \tilde{z}^c(t) \pm \tilde{z}^r(t)$. Then, using Proposition 2 for $\tilde{z}(t)$, the following property for $\bar{\Psi}(t)$ is deduced:

$$\bar{\Psi}(t) \in \bar{\Omega}(t)(\Phi^c(u(t)) + N\tilde{u}^c(t)) + \bar{\Omega}(t)[...\Phi_i^r(u(t), \bar{\Omega}^{-1}(t)\tilde{z}^c(t), \bar{\Omega}^{-1}(t)\Delta(\tilde{z}^r(t)))..., N\tilde{u}^r(t)](\mathbf{0} \pm \mathbf{1}) \quad (36)$$

where $\Phi^c(u(t)) = \begin{bmatrix} B_0 \\ O_{n_y \times n_u} \end{bmatrix} u(t)$, and for $i = 1, ..., n_r$,

$$\Phi_i^r(u(t), \bar{\Omega}^{-1}(t)\tilde{z}^c(t), \bar{\Omega}^{-1}(t)\Delta(\tilde{z}^r(t))) = \begin{bmatrix} \bar{A}_i\bar{\Omega}^{-1}(t)\tilde{z}^c(t) + \bar{B}_i u(t), & \bar{A}_i\bar{\Omega}^{-1}\Delta(\tilde{z}^r(t)) \end{bmatrix}$$

with $\bar{A}_i = \begin{bmatrix} A_i & O_{n_x \times n_y} \\ O_{n_y \times n_x} & O_{n_y} \end{bmatrix}$, $\bar{B}_i = \begin{bmatrix} B_i \\ O_{n_y \times n_u} \end{bmatrix}$,
$\tilde{u}^c(t) = \begin{bmatrix} d^c \\ y(t) \end{bmatrix}$ and $\tilde{u}^r(t) = \tilde{u}^r = \begin{bmatrix} d^r \\ \mathbf{0} \end{bmatrix}$.
Then, applying Theorem 1 to $\bar{\Psi}(t)$ yields to the expressions in equation (32). Hence, using Theorem 3 in [6], Theorem 3 is proven. □

*2) **FHGIO stability analysis**:* The proof of stability of the FHGIO presented in Therorem 3 is divided in two parts as for the HGIO. The first one deals with the stability of the center dynamics whereas the second one is about the radius dynamics.

Proof. Stability of the center dynamics:
The first line of equation (31) gives the center dynamics of the observer. It is composed of the endogenous term $\text{diag}(\theta\bar{\xi})\tilde{z}^c(t)$ and the term $\bar{\Psi}^c(t)$. The latter depends only on the bounded exogenous variables $u(t)$ and $y(t)$, thus it is also bounded. Hence, the center stability condition comes down to having $\text{diag}(\theta\bar{\xi})$ Hurwitz, i.e M Hurwitz, which is achieved by the suitable design of $\bar{L}$ and D. □

Proof. Stability of the radius dynamics:
The stability of the radius dynamics given by the second line of equation (31) can not be deduced directly. Indeed, the term $\bar{\Psi}^r(t)$ depends on the bounded exogenous variables $u(t)$ and $z^c(t)$ but it also depends on the endogenous term $\tilde{z}^r(t)$. Thus, a non-divergence sufficient condition to ensure the stability is formulated in the following proposition based on Proposition 3 and Corollary 2 where $H = \bar{H} = \sum_{i=1}^{n_r} \|\bar{v}\bar{A}_i\bar{v}^{-1}\|$.

Proposition 5. *Let $\bar{H} = \sum_{i=1}^{n_r} \|\bar{v}\bar{A}_i\bar{v}^{-1}\| \in (\mathbb{R}^+)^{n_x+n_y}$. If $\text{diag}(\theta\bar{\xi})$ is Hurwitz and if the Metzler matrix $[\text{diag}(\theta\bar{\xi}) + \bar{H}, \bar{H}; \bar{H}, \text{diag}(\theta\bar{\xi}) + \bar{H}]$ is Hurwitz, then $\forall t \in \mathbb{R}^+, 0 \leq \tilde{z}^r(t) \leq \bar{\tilde{z}}^r(t)$ where $\bar{\tilde{z}}^r(t)$ follows a stable dynamics and so is $\tilde{z}^r(t)$.*

□

IV. SIMULATION RESULTS

Consider the LPV system described by (10) with:
$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
$B_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $B_1 = \begin{bmatrix} 0 & 0.5 & -0.2 \end{bmatrix}^T$,
$E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
The input of the system is defined as $u(t) = \sin(2\pi t)$. The scheduling vector is taken as $\rho(t) = \cos(\pi t)$. The disturbance is considered as $d(t) = d^c + d^r \sin(0.4\pi t)$ with $d^c = 0$ and $d^r = 0.3$. For simulation purpose, the measurement noise is defined as a high frequency sine wave $w(t) = w^c + w^r \sin(400\pi t)$ with $w^c = 0$ and $w^r = 0.5$. The initial condition for the system is set to $x(0) = [0, 0, 0]^T$. However, it is assumed unknown but bounded by known values for the observers. Thus, their initial conditions are taken as $x^c(0) = [0, 0, 0]^T$ and $x^r(0) = [0.1, 0.1, 0.1]^T$. Hence, one could check that the Assumptions 1-3 are verified.
The gains for each observer have respectively been designed using the methods detailed in the STEPS 1. The poles without amplification by θ have been taken real for both HGIO and FHGIO, respectively equal to $\xi + i\zeta = [-0.9, -0.95, -1]$ and $\bar{\xi} + i\bar{\zeta} = [-0.9, -0.95, -1, -9, -10]$. The stability conditions (14) and (27) are thus satisfied. To illustrate the influence of the high-gain parameter θ, two different values were taken for the simulations, $\theta = 10$ and $\theta = 30$, leading to different observer and filter gains that are summarized in the Table I. For each θ, the non-divergence conditions of Proposition 4 and 5 are satisfied. Finally, the sampling time of simulation is set to $T_s = 10^{-4}$s.

θ	10	30
L	$\begin{bmatrix} 11.4884 & 16.3414 \\ 9.2112 & 3.8601 \\ 5.0660 & 13.2888 \end{bmatrix}$	$\begin{bmatrix} 120.0186 & 171.3237 \\ 34.2511 & 12.5020 \\ 14.1245 & 45.2489 \end{bmatrix}$
$\bar{L}$	$\begin{bmatrix} 95.2144 & 157.5594 \\ 87.0322 & 39.8196 \\ 47.7480 & 126.8238 \end{bmatrix}$	$\begin{bmatrix} 0.9827 & 1.6738 \\ 0.3303 & 0.1333 \\ 0.1306 & 0.4381 \end{bmatrix} .10^3$
D	$\begin{bmatrix} -108.7399 & -4.1485 \\ -5.2844 & -103.7601 \end{bmatrix}$	$\begin{bmatrix} -332.6057 & -13.4004 \\ -14.2297 & -316.8943 \end{bmatrix}$

TABLE I
COMPUTED GAINS FOR EACH HIGH-GAIN PARAMETER θ.

The Fig. 1 and 2 show together an increase of the high-gain parameter θ reduces the estimation bounds for both the HGIO and the FHGIO. Hence, both observers can compute precise state estimations provided that they are fine tuned using θ. However, it can be noticed in Fig. 2 that having large θ amplify the measurement noise, as expected and clearly visible in the HGIO bounds (in blue). This issue is addressed by the output observation error filter incorporated in the FHGIO structure. A much smaller noise amplification can be seen in the FHGIO bounds (in green) compared to the HGIO.

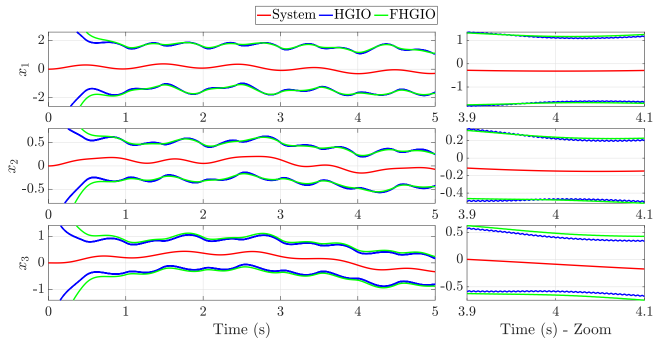

Fig. 1. HGIO and FHGIO state estimations for $\theta = 10$

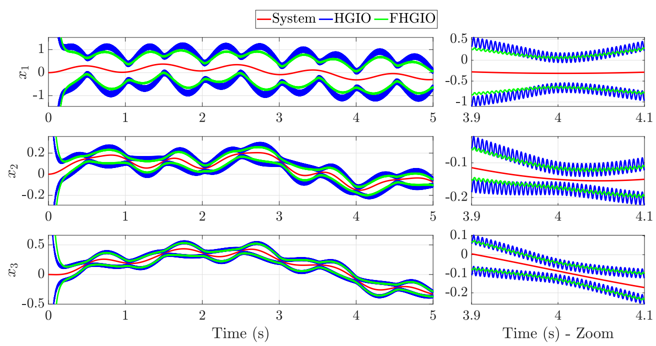

Fig. 2. HGIO and FHGIO state estimations for $\theta = 30$

V. CONCLUSIONS AND FUTURE WORKS

This paper has presented two new high-gain interval observer structures, namely the HGIO and the FHGIO, for a class of LPV systems subject to uncertainties. The proposed HGIO provides an effective way for state interval estimation when there is little to no measurement noise. For stronger measurement noise and fast convergence requirements (large θ), the FHGIO shows its interest by its ability to reduce the measurement noise amplification using an output estimation error filter. Moreover, constructive procedures for computing the observation and filter gains have been given. To ensure the interval observers stability, a sufficient condition ensuring the non-divergence of the radius dynamics is also proposed for each structure. The simulation illustrated the main advantages of such designs, i.e the ability to reduce the estimation bounds by increasing the high-gain parameter and filter the measurement noise when required. Future works will deal with the maximum authorized value of θ such that the inclusion property is also satisfied with respect to the studied system. Discrete and delayed measurements may also be considered.

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