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Pressure Stabilized POD Reduced Order Model for Control of Viscous Incompressible Flows

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Abstract— **In this paper, we propose a new pressure-stabilized proper orthogonal decomposition reduced order model (POD-ROM) for the control of viscous incompressible flows. It is a velocity-pressure ROM that uses pressure modes as well to compute the reduced order pressure needed for instance in the control drag and lift forces on bodies in the flow. We also propose and analyze a decoupled time-stepping scheme that uncouples the computation of velocity and pressure variables. It allows us at each time step to solve linear problems, uncoupled in pressure and velocity, which can greatly improve computational efficiency. Numerical studies are performed to discuss the accuracy and performance of the new pressure-stabilized ROM in the simulation of control of flow past a forward-facing step channel.**

I. Introduction

Control of viscous fluid flows is an important application area of tremendous benefits including drag reduction, lift enhancement, mixing augmentation and flow induced noise suppression. It can also improve agility and maneuverability for military aircraft and weapons. Successful implementation of flow control requires among other things efficient computational algorithms for real-time simulation and control. Modern nonlinear control system theoretic methods to flow control is hindered by the fact that fluid flow systems are nonlinear and high dimensional, see [5], [9]. Recognizing this complexity, a great deal of effort has been put into developing efficient computational methods for accurately solving those problems. One of the major developments is the nonlinear reduced-order controller approach [9], [11]. A popular method for constructing nonlinear reduced-order model is based on finding a suitable low dimensional basis by proper orthogonal decomposition (POD) and forming a reduced-order model (ROM) by Galerkin projection of the infinite dimensional model onto the basis, see[9].

For incompressible flows, the reduced order models generated via POD leads to velocity only reduced order models because in this framework, velocity POD modes are usually assumed to be at least weakly divergence free. This assumption holds true if, for instance, the POD modes are generated by snapshots computed using inf-sup stable finite elements for the velocity-pressure pair. However, the weakly divergence free property does not hold for many popular finite element discretizations of Navier-Stokes equations [7]. Despite the appealing computational efficiency of velocity only reduced-order models, the pressure is needed in many flow control problems such as minimizing drag or maximizing lift force on bodies in the flow.

In this paper, we propose a pressure stabilized POD reduced order model which is a coupled velocity-pressure reduced order model that uses pressure modes as well to compute the reduced order pressure. Also, unlike other existing approached such as pressure Poisson equation approach that provides velocity pressure approximations, in our approach the velocity modes do not have to be either strongly or weakly divergence free. The new method draws inspiration from successful pressure stabilization techniques used in the context of finite element methods for incompressible flows [2]. The main contribution of the present work is the study of a decoupled time stepping scheme (uncouples velocity and pressure) for the reduced-order optimality system for the optimal control of Navier-Stokes equations. The scheme we study allow us at each time step to solve linear problems, uncoupled in pressure and velocity which can greatly improve the computational efficiency. We prove error estimates for the reduced basis discretizations of the pressure stabilized reduced order optimality system from which control can be computed. We also investigate numerically the new pressure stabilized ROM in the simulation of control of flow past forward facing step channel.

II. Optimal Control of Navier-Stokes equations

In this paper, we are concerned with the numerical approximation of optimal control of incompressible flows using pressure stabilized reduced-order models. We consider the following optimal control problem constrained by the unsteady Navier-Stokes equations:

$$\text{Minimize}\ \frac{1}{2}\int_0^T \|\nabla\times\mathbf{u}\|^2+\gamma\|\mathbf{g}\|_{\mathscr{U}}^2\,dt\,,$$

subject to

$$\begin{aligned}\frac{\partial\mathbf{u}}{\partial t}-\nu\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{f},\ \text{in}\ \Omega\times(0,T]\,,\\ \nabla\cdot\mathbf{u}=0\,,\ \text{in}\ \Omega\times(0,T]\end{aligned}\tag{1}$$

with Navier boundary conditions [8] :

$$\mathbf{u}\cdot\mathbf{n}|_\Gamma=0\ \text{ and }\ (\alpha\mathbf{u}\cdot\tau+2\nu\mathbf{n}\cdot\mathbb{D}(\mathbf{u})\cdot\tau)|_\Gamma=\mathbf{g}\cdot\tau\,,$$

and initial conditions: $\mathbf{u}|_{t=0}=\mathbf{u}_0(\mathbf{x})$

where the spatial domain $\Omega\subset\mathbb{R}^d$ $(d\in\{2,3\})$ is bounded open convex polygon with Lipschitz boundary Γ, $\mathbf{n}$ and τ denote the outward unit normal and tangent vectors, respectively, $\alpha>0$ the coefficient of proportionality, $\gamma>0$

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the relative weight, $\mathscr{U} := \{\mathbf{g} \in L^2(\Gamma) : \mathbf{g}\cdot\mathbf{n}|_\Gamma = 0\}$ the set of admissible controls and $\mathbf{g}$ the tangential control input function [4], [12]. Moreover, in (1), $\mathbf{u}(\mathbf{x},t)$ is the velocity field, $p(\mathbf{x},t)$ the kinematic pressure, $\nu > 0$ the kinematic viscosity of the fluid, $\mathbf{f}$ the density of external forces and $\mathbb{D}(\mathbf{u}) := (1/2)(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$.

In the sequel, $L^2(\Omega)$ denotes the usual space of square-integrable functions equipped with the usual L^2-inner product $(\cdot,\cdot)$ and L^2-norm $\|u\| := \|u\|_{L^2(\Omega)}$. We denote the norms on Hilbertian Sobolev space $\mathbf{H}^k(\Omega)$ and the trace space $\mathbf{H}^k(\Gamma)$ by $\|\cdot\|_k$ and $\|\cdot\|_{k,\Gamma}$, respectively. Moreover, let $\{\mathscr{T}_h\}_{h>0}$ be a family of affine equivalent, conforming and regular triangulation of $\overline{\Omega}$, formed by triangles or quadrilaterals (d=2), or tetrahedra or hexahedra $(d = 3)$. For any mesh cell $K \in \mathscr{T}_h$, its diameter will be denoted by h_K and $h = \max_{K\in\mathscr{T}_h} h_K$. We denote conforming velocity and pressure finite element spaces based on a regular triangulation of spatial domain Ω having maximum triangle diameter h by $X_h \subset X$ and $Q_h \subset Q$. Let the time interval $[0,T]$ be partitioned as $0 = t_0 < t_1 < \cdots < t_N = T$, $\Delta t := t_n - t_{n-1}$.

The fully discrete approximation of the optimal control problem we will consider is to find $\mathbf{u}_h^n \in \mathbf{X}_h$, $\mathbf{g}_h^n \in \mathscr{U}_h$ and $p_h^n \in Q_h$, $n = 1,2,\ldots,N$, such that

$$\text{Minimize} \quad \tfrac{1}{2}\sum_{i=1}^{N} \|\nabla\times\mathbf{u}_h^n\|^2 + \gamma\|\mathbf{g}_h^n\|^2_{\mathscr{U}} \,,$$

subject to

$$\begin{aligned}(\tfrac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{\Delta t},\mathbf{v}_h) &+ 2\nu(\mathbb{D}(\mathbf{u}_h^n),\mathbb{D}(\mathbf{v}_h)) + c(\mathbf{u}_h^{n-1},\mathbf{u}_h^n,\mathbf{v}_h) \\ &+ \alpha(\mathbf{u}_h^n\cdot\tau,\mathbf{v}_h\cdot\tau)_\Gamma - (p_h^n,\nabla\cdot\mathbf{v}_h) \\ &= (\mathbf{g}_h^n\cdot\tau,\mathbf{v}_h\cdot\tau)_\Gamma\,,\ \forall\mathbf{v}_h\in\mathbf{X}_h\end{aligned} \tag{2}$$

$$\varepsilon(\frac{p_h^n - p_h^{n-1}}{\Delta t},\phi_h) + (\nabla\cdot\mathbf{u}_h^n,\phi_h) = 0\,,\ \forall\phi_h\in Q_h\,, \tag{3}$$

where the trilinear form $c(\cdot,\cdot,\cdot)$ is defined as $c(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}((\nabla\cdot\mathbf{u})\mathbf{v},\mathbf{w})$ and $0 < \varepsilon \ll 1$ is the penalty parameter.

In (2), the term ∇p_h^n can be eliminated by using

$$(p_h^n,\phi_h^n) = (p_h^{n-1},\phi_h) - \frac{\Delta t}{\varepsilon}(\nabla\cdot\mathbf{u}_h^n,\phi_h)\,,\ \forall\phi_h\in Q_h\,. \tag{4}$$

therefore, the calculation of $(\mathbf{u}_h^n, p_h^n)$ proceeds as follows: given $\mathbf{u}_h^n$ and p_h^n, solve for $\mathbf{u}_h^n$:

$$\begin{aligned}(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{\Delta t},\mathbf{v}_h) &+ 2\nu(\mathbb{D}(\mathbf{u}_h^n),\mathbb{D}(\mathbf{v}_h)) + c(\mathbf{u}_h^{n-1},\mathbf{u}_h^n,\mathbf{v}_h) \\ &+ \alpha(\mathbf{u}_h^n\cdot\tau,\mathbf{v}_h\cdot\tau)_\Gamma + \tfrac{\Delta t}{\varepsilon}(\nabla\cdot\mathbf{u}_h^n,\nabla\cdot\mathbf{v}_h) \\ &= (\mathbf{g}_h^n\cdot\tau,\mathbf{v}_h\cdot\tau)_\Gamma + (p_h^{n-1},\nabla\cdot\mathbf{v}_h)\,,\ \forall\mathbf{v}_h\in\mathbf{X}_h\end{aligned} \tag{5}$$

Perform an algebraic update for p_h^n using (4).

III. Pressure Stabilized POD Reduced Order Model

We briefly describe the POD method, following [9], and apply it to the pressure stabilized finite element model (2)-(3) to construct the reduced-order model. To compute the snapshots, we use the stabilized FEM model (2)-(3). Let us consider the velocity snapshot subspace $X_{d_u} := \text{span}\{\mathbf{u}_h^i\}_{i=1}^{N_u}$, pressure snapshot subspace $Q_{d_p} := \text{span}\{p_h^i\}_{i=1}^{N_p}$, adjoint velocity snapshot subspace $X_{d_\zeta} := \text{span}\{\zeta_h^i\}_{i=1}^{N_\zeta}$ and $Q_{d_\sigma} := \text{span}\{\sigma_h^i\}_{i=1}^{N_\sigma}$ with dimensions d_u, d_p, d_ζ, and d_σ, respectively. The POD method then seeks a low-dimesional subspaces $\mathbf{X}_R^u := \text{span}\{\phi_i^u\}_{i=1}^R \subset \mathbf{X}_h$, $Q_M^p := \text{span}\{\psi_i^p\}_{i=1}^M \subset Q_h$, $\mathbf{X}_R^\zeta := \text{span}\{\phi_i^\zeta\}_{i=1}^R \subset \mathbf{X}_h$ and $Q_M^\sigma := \text{span}\{\psi_i^\sigma\}_{i=1}^M \subset Q_h$, which optimally approximate the snapshot subspaces in discrete L^2-norms. Let $K_u := (k_{i,j}^u) \in \mathbb{R}^{N_u\times N_u}$, $K_\zeta := (k_{i,j}^\zeta) \in \mathbb{R}^{N_\zeta\times N_\zeta}$, $K_p := (k_{i,j}^p) \in \mathbb{R}^{N_p\times N_p}$ and $K_\sigma := (k_{i,j}^\sigma) \in \mathbb{R}^{N_\sigma\times N_\sigma}$ be the correlation matrices corresponding to snapshots, where $k_{i,j}^u := \frac{1}{N_u}(\mathbf{u}_h^i,\mathbf{u}_h^j)$, $k_{i,j}^\zeta := \frac{1}{N_\zeta}(\zeta_h^i,\zeta_h^j)$, $k_{i,j}^p := \frac{1}{N_p}(p_h^i,p_h^j)$ and $k_{i,j}^\sigma := \frac{1}{N_\sigma}(\sigma_h^i,\sigma_h^j)$ Let $\{\lambda_i^u\}_{i=1}^{d_u}$, $\{\lambda_i^p\}_{i=1}^{d_p}$, $\{\lambda_i^\zeta\}_{i=1}^{d_\zeta}$, $\{\lambda_i^\sigma\}_{i=1}^{d_\sigma}$ be the positive eigenvalues of K_u, K_p, K_ζ and K_σ, respectively. Then, the following error formula hold

$$\begin{aligned}\frac{1}{N_u}\sum_{j=1}^{N_u}\|\mathbf{u}_h^j - \sum_{k=1}^{R}(\mathbf{u}_h^j,\phi_k^u)\phi_k^u\|^2 &= \sum_{k=R+1}^{d_u}\lambda_k^u\,, \\ \tfrac{1}{N_p}\sum_{j=1}^{N_p}\|p_h^j - \sum_{k=1}^{M}(p_h^j,\psi_k^p)\psi_k^p\|^2 &= \sum_{k=M+1}^{d_p}\lambda_k^p\,, \\ \frac{1}{N_\zeta}\sum_{j=1}^{N_\zeta}\|\zeta_h^j - \sum_{k=1}^{R}(\zeta_h^j,\phi_k^\zeta)\phi_k^\zeta\|^2 &= \sum_{k=R+1}^{d_\zeta}\lambda_k^\zeta\,, \\ \tfrac{1}{N_\sigma}\sum_{j=1}^{N_\sigma}\|\sigma_h^j - \sum_{k=1}^{M}(\sigma_h^j,\psi_k^\sigma)\psi_k^\sigma\|^2 &= \sum_{k=M+1}^{d_\sigma}\lambda_k^\sigma\,.\end{aligned} \tag{6}$$

An analogous set of error formulae (albeit bounds) in $\|\nabla\cdot\|$ norm holds instead of $\|\cdot\|$ norm as in (6) due to inverse inequality [6], [10]. In the sequel, we will denote by P_r^u, P_r^p, P_r^ζ and P_r^σ the L^2 orthogonal projections onto X_R^u, Q_M^p, X_R^ζ and Q_M^σ, respectively.

We will also need the following a priori bounds for the orthogonal projections $P_r^u\mathbf{u}_h^n$ and $P_r^\zeta\zeta_h^n$ that can be obtained from the finite element approximations and the inverse inequality. Therefore in the sequel, we will assume $\|\nabla P_r^u\mathbf{u}_h^n\|_{L^{2d}}$, $\|P_r^u\mathbf{u}_h^n\|_\infty$, $\|\nabla P_r^u\mathbf{u}_h^n\|_\infty$, $\|\mathbf{u}_h^n\|_\infty$, $\|\nabla\mathbf{u}_h^n\|_\infty$, $\|\nabla\mathbf{u}_h^n\|_{L^{2d}}$, $\|P_r^\zeta\zeta_h^n\|_\infty$, $\|\nabla P_r^\zeta\zeta_h^n\|_\infty$, $n = 1,\ldots,N$, are bounded.

A. Pressure Stabilized Reduced Order Model Optimality System

The pressure stabilized reduced-order optimality system we consider is to find $(\mathbf{u}_r^n,\zeta_r^n) \in \mathbf{X}_R^u \times \mathbf{X}_R^\zeta$, $\mathbf{g}_r^n \in \mathscr{U}_r$ and $(p_r^n,\sigma_r^n) \in Q_M^p \times Q_M^\sigma$, $n = 1,2,\ldots,N$, such that

$$\begin{aligned}(\frac{\mathbf{u}_r^n-\mathbf{u}_r^{n-1}}{\Delta t},\mathbf{v}_r) &+ 2\nu(\mathbb{D}(\mathbf{u}_r^n),\mathbb{D}(\mathbf{v}_r)) + c(\mathbf{u}_r^{n-1},\mathbf{u}_r^n,\mathbf{v}_r) \\ &+ \alpha(\mathbf{u}_r^n\cdot\tau,\mathbf{v}_r\cdot\tau)_\Gamma - (p_r^n,\nabla\cdot\mathbf{v}_r) \\ &= (\mathbf{g}_r^n\cdot\tau,\mathbf{v}_r\cdot\tau)_\Gamma\,,\ \forall\mathbf{v}_r\in\mathbf{X}_R^u \\ \varepsilon(\tfrac{p_r^n-p_r^{n-1}}{\Delta t},\phi_r) &+ (\nabla\cdot\mathbf{u}_r^n,\phi_r) = 0\,,\ \forall\phi_r\in Q_M^p\end{aligned} \tag{7}$$

$$-(\frac{\zeta_r^n-\zeta_r^{n-1}}{\Delta t},\mathbf{v}_r)\;+\;2\nu(\mathbb{D}(\zeta_r^{n-1}),\mathbb{D}(\mathbf{v}_r))+c(\mathbf{u}_r^n,\mathbf{v}_r,\zeta_r^{n-1})$$
$$+\;c(\mathbf{v}_r,\mathbf{u}_r^n,\zeta_r^{n-1})$$
$$+\;\alpha(\zeta_r^{n-1}\cdot\tau,\mathbf{v}_r\cdot\tau)_\Gamma-(\sigma_r^{n-1},\nabla\cdot\mathbf{v}_r)$$
$$=\;(\nabla\times\mathbf{u}_r^{n-1},\nabla\times\mathbf{v}_r),\;\;\forall\mathbf{v}_r\in\mathbf{X}_R^\zeta$$
$$\varepsilon(\tfrac{\sigma_r^n-\sigma_r^{n-1}}{\Delta t},\phi_r)\;+\;(\nabla\cdot\zeta_r^{n-1},\phi_r)=0,\;\;\forall\phi_r\in Q_M^\sigma \tag{8}$$

$$(\gamma\mathbf{g}_r^n\quad+\quad\zeta_r^{n-1},\widehat{\mathbf{v}}_r)_\Gamma=0,\;\;\forall\widehat{\mathbf{v}}_r\in\mathscr{U}_R \tag{9}$$

final condition $\zeta_r^N(\mathbf{x})=0$ and the initial condition $\mathbf{u}_r^0(\mathbf{x})=\mathbf{u}_0^r(\mathbf{x})$.

IV. Error Analysis

This section is devoted to deriving error estimates for the reduced order optimality system (7)-(9).

Lemma 4.1. Let $(\mathbf{u}_h^n,p_h^n)\in\mathbf{X}_h\times Q_h$ be the solutions of (2) and $(\mathbf{u}_r^n,p_r^n)\in\mathbf{X}_R^u\times Q_M^p$ be the solutions of (7), for $n=1,2,\ldots,N$. Then, we have

$$\|\mathbf{u}_h^n-\mathbf{u}_r^n\|^2+\varepsilon\|p_h^n-p_r^n\|^2+\sum_{i=1}^N\Delta t\|\nabla(\mathbf{u}_h^i-\mathbf{u}_r^i)\|^2$$
$$\le\|\mathbf{u}_h^0-\mathbf{u}_r^0\|^2+\varepsilon\|p_h^0-p_r^0\|^2$$
$$+C\Delta t\left[\sum_{i=R+1}^{d_u}\lambda_i^u+\sum_{i=M+1}^{d_p}\lambda_i^p+\sum_{i=1}^N\|\mathbf{g}_h^i-\mathbf{g}_r^i\|_{0,\Gamma}^2\right], \tag{10}$$

where C is independent of h and Δt.

Proof. To get the error bounds of the method, we decompose the error as follows errors: $\mathbf{e}_u^n:=\mathbf{u}_r^n-P_r^u\mathbf{u}_h^n$, $e_p^n:=p_r^n-P_r^p p_h^n$, $\alpha_u^n:=P_r^u\mathbf{u}_h^n-\mathbf{u}_h^n$ and $\beta_p^n:=P_r^p p_h^n-p_h^n$. As the errors α_u^n and β_p^n can be easily estimated from (6), we will only estimate the errors $\mathbf{e}_u^n$ and e_p^n. It can be easily shown that the pair $(P_r^u\mathbf{u}_h^n,P_r^p p_h^n)$ satisfies

$$(\frac{P_r^u\mathbf{u}_h^n-P_r^u\mathbf{u}_h^{n-1}}{\Delta t},\mathbf{v}_r)+2\nu(\mathbb{D}(P_r^u\mathbf{u}_h^n),\mathbb{D}(\mathbf{v}_r))$$
$$+c(P_r^u\mathbf{u}_h^{n-1},P_r^u\mathbf{u}_h^n,\mathbf{v}_r)$$
$$-(P_r^p p_h^n,\nabla\cdot\mathbf{v}_r)+\alpha(P_r^u\mathbf{u}_h^n\cdot\tau,\mathbf{v}_r\cdot\tau)_\Gamma$$
$$=(\mathbf{g}_h^n\cdot\tau,\mathbf{v}_r\cdot\tau)_\Gamma+2\nu(\mathbb{D}(\alpha_u^n),\mathbb{D}(\mathbf{v}_r))$$
$$-(\beta_p^n,\nabla\cdot\mathbf{v}_r)+c(P_r^u\mathbf{u}_h^{n-1},P_r^u\mathbf{u}_h^n,\mathbf{v}_r)$$
$$+\;\alpha(\alpha_u^n\cdot\tau,\mathbf{v}_r\cdot\tau)_\Gamma$$
$$-\;c(\mathbf{u}_h^{n-1},\mathbf{u}_h^n,\mathbf{v}_r),\;\;\forall\mathbf{v}_r\in\mathbf{X}_R^u$$
$$\varepsilon(\tfrac{P_r^p p_h^n-P_r^p p_h^{n-1}}{\Delta t},\phi_r)\;+\;(\nabla\cdot P_r^u\mathbf{u}_h^n,\phi_r)=(\nabla\cdot\alpha_u^n,\phi_r)\,,\;\forall\phi_r\in Q_M^p\,. \tag{11}$$

For $n=1,2,\ldots,N$, subtracting (11) from (7) and setting $(\mathbf{v}_r,\phi_r)=\Delta t(\mathbf{e}_u^n,e_p^n)$ yields

$$\frac{1}{2}(\|\mathbf{e}_u^n\|^2+\varepsilon\|e_p^n\|^2)-\tfrac{1}{2}(\|\mathbf{e}_u^{n-1}\|^2+\varepsilon\|e_p^{n-1}\|^2)+\tfrac{1}{2}\|\mathbf{e}_u^n-\mathbf{e}_u^{n-1}\|^2$$
$$+\;\tfrac{\varepsilon}{2}\|e_p^n-e_p^{n-1}\|^2+2\Delta t\nu\|\nabla\mathbf{e}_u^n\|^2+\alpha\Delta t\|\mathbf{e}_u^n\cdot\tau\|_{0,\Gamma}^2$$
$$=\Delta t[(c(P_r^u\mathbf{u}_h^{n-1},P_r^u\mathbf{u}_h^n,\mathbf{e}_u^n)-c(\mathbf{u}_r^{n-1},\mathbf{u}_r^n,\mathbf{e}_u^n))$$
$$+\;2\nu(\mathbb{D}(\alpha_u^n),\mathbb{D}(\mathbf{e}_u^n))-(\beta_p^n,\nabla\cdot\mathbf{e}_u^n)$$
$$-\;(c(P_r^u\mathbf{u}_h^{n-1},P_r^u\mathbf{u}_h^n,\mathbf{e}_u^n)-c(\mathbf{u}_h^{n-1},\mathbf{u}_h^n,\mathbf{e}_u^n))$$
$$-\;(\nabla\cdot\alpha_u^n,e_p^n)+((\mathbf{g}_h^n-\mathbf{g}_r^n)\cdot\tau,\mathbf{e}_u^n\cdot\tau)_\Gamma$$
$$+\;\alpha(\alpha_u^n\cdot\tau,\mathbf{e}_u^n\cdot\tau)_\Gamma]=:\sum_{i=1}^7\Sigma_i\,. \tag{12}$$

We now proceed to estimate the terms on the right-hand side of (12). For the first term we use the skew symmetry to obtain

$$\begin{aligned}|\Sigma_1|\;&\le\;|\Delta t c(\mathbf{e}_u^{n-1},P_r^u\mathbf{u}_h^n,\mathbf{e}_u^n)|\\&\le\;C\Delta t[\|\nabla P_r^u\mathbf{u}_h^n\|_\infty\|\mathbf{e}_u^{n-1}\|\\&+\;\|\mathbf{e}_u^{n-1}\|\|P_r^u\mathbf{u}_h^n\|_\infty]\|\nabla\mathbf{e}_u^n\|\\&\le\;C\Delta t\|\mathbf{e}_u^{n-1}\|^2+\tfrac{\nu\Delta t}{4}\|\nabla\mathbf{e}_u^n\|^2\,.\end{aligned} \tag{13}$$

For the second term, we have

$$|\Sigma_2|\le\frac{\nu\Delta t}{4}\|\nabla\mathbf{e}_u^n\|^2+C\Delta t\|\nabla\alpha_u^n\|^2\,. \tag{14}$$

For the third term, we have

$$|\Sigma_3|\le\frac{\nu\Delta t}{4}\|\nabla\mathbf{e}_u^n\|^2+C\Delta t\|\nabla\beta_p^n\|^2\,. \tag{15}$$

For the fifth term, we first rewrite it as

$$\begin{aligned}\Sigma_5\;&:=\;\Delta t(\nabla\cdot\alpha_u^n,e_p^n)\\&=\;\Delta t(\nabla\cdot\alpha_u^n,e_p^n-e_p^{n-1})+\Delta t(\nabla\cdot\alpha_u^n,e_p^{n-1})\,.\end{aligned}$$

Employing the extended Cauchy-Buniakowskii-Schwarz inequality [3] yields

$$|\Sigma_5|\le\eta\Delta t\|\nabla\alpha_u^n\|\|e_p^n-e_p^{n-1}\|+\eta\Delta t\|\nabla\alpha_u^n\|\|e_p^{n-1}\|$$

for some constant $\eta\in[0,1)$. Therefore by the Young's inequality we have

$$\begin{aligned}|\Sigma_5|\;&\le\;\tfrac{\varepsilon\Delta t}{2}\|e_p^{n-1}\|^2+\tfrac{\eta^2\Delta t^2}{2\varepsilon}\|\nabla\alpha_u^n\|^2+\tfrac{\varepsilon}{2}\|e_p^n-e_p^{n-1}\|^2\\&+\;\tfrac{\eta^2\Delta t}{2\varepsilon}\|\nabla\alpha_u^n\|^2\,.\end{aligned} \tag{16}$$

To bound the fourth term, we use Sobolev embedding and Holder's inequality

$$\begin{aligned}|\Sigma_4|\;&\le\;\Delta t|c(P_r^u\mathbf{u}_h^{n-1},\alpha_u^n,\mathbf{e}_u^n)+c(\alpha_u^{n-1},\mathbf{u}_h^n,\mathbf{e}_u^n)|\\&\le\;\Delta t[\|P_r^u\mathbf{u}_h^{n-1}\|_{L^{\frac{2d}{d-1}}}\|\nabla\alpha_u^n\|\|\mathbf{e}_u^n\|_{L^{2d}}\\&+\;\|P_r^u\mathbf{u}_h^n\|_{L^{\frac{2d}{d-1}}}\|\nabla\mathbf{e}_u^n\|\|\alpha_u^n\|_{L^{2d}}]\\&+\;\Delta t[\|\alpha_u^{n-1}\|_{L^{\frac{2d}{d-1}}}\|\nabla\mathbf{u}_h^n\|\|\mathbf{e}_u^n\|_{L^{2d}}\\&+\;\|\alpha_u^n\|_{L^{\frac{2d}{d-1}}}\|\nabla\mathbf{e}_u^n\|\|\mathbf{u}_h^n\|_{L^{2d}}]\\&\le\;\tfrac{\nu\Delta t}{4}\|\nabla\mathbf{e}_u^n\|^2+C\Delta t[\|\nabla\alpha_u^n\|^2+\|\nabla\alpha_u^{n-1}\|^2]\,.\end{aligned} \tag{17}$$

Finally, we estimate the sixth and seventh terms using trace inequality as follows

$$|\Sigma_6| \leq \frac{\alpha\Delta t}{2}\|\mathbf{e}_u^n\cdot\tau\|_{0,\Gamma}^2 + C\Delta t\|\mathbf{g}_h^n - \mathbf{g}_r^n\|_{0,\Gamma}^2 \tag{18}$$

and

$$|\Sigma_7| \leq \frac{\alpha\Delta t}{2}\|\mathbf{e}_u^n\cdot\tau\|_{0,\Gamma}^2 + C\Delta t\|\nabla\alpha_u^n\|^2. \tag{19}$$

Inserting (13)-(18) into (12) yields

$$\begin{aligned}
&(\|\mathbf{e}_u^n\|^2 + \varepsilon\|e_p^n\|^2) - (\|\mathbf{e}_u^{n-1}\|^2 + \varepsilon\|e_p^{n-1}\|^2) + \|\mathbf{e}_u^n - \mathbf{e}_u^{n-1}\|^2 \\
&\quad + \tfrac{\varepsilon}{2}\|e_p^n - e_p^{n-1}\|^2 + 2\Delta t\nu\|\nabla\mathbf{e}_u^n\|^2 \\
&\quad \leq C\Delta t(\|\mathbf{e}_u^{n-1}\|^2 + \varepsilon\|e_p^{n-1}\|^2) + \tfrac{\eta^2\Delta t}{2\varepsilon}\|\nabla\alpha_u^n\|^2(1+\Delta t) \\
&\quad + C\Delta t\left[\|\nabla\alpha_u^n\|^2 + \|\nabla\beta_p^n\|^2 + \|\nabla\alpha_u^{n-1}\|^2 + \|\mathbf{g}_h^n - \mathbf{g}_r^n\|_{0,\Gamma}^2\right].
\end{aligned}$$

Finally summing from $n=1$ to $n=N$ and applying discrete Gronwall inequality we obtain the desired result. ■

Before we proceed to derive the error bounds for reduced order approximations of adjoint variables, we observe that by applying inverse inequality and the error bounds in Lemma 4.1, we obtain that $\|\mathbf{u}_r^n\|_\infty$ is bounded.

Let $\mathbf{e}_\zeta^n := \zeta_r^n - P_r^\zeta\zeta_h^n,\ e_\sigma^n := \sigma_r^n - P_r^\sigma\sigma_h^n,\ \alpha_\zeta^n := P_r^\zeta\zeta_h^n - \zeta_h^n$ and $\beta_\sigma^n := P_r^\sigma\sigma_h^n - \sigma_h^n$. It can be easily shown that the pair $(P_r^\zeta\zeta_h^n, P_r^\sigma\sigma_h^n)$ satisfies

$$\begin{aligned}
&-\left(\frac{P_r^\zeta\zeta_h^n - P_r^\zeta\zeta_h^{n-1}}{\Delta t}, \mathbf{v}_r\right) + 2\nu(\mathbb{D}(P_r^\zeta\zeta_h^{n-1}), \mathbb{D}(\mathbf{v}_r)) \\
&\quad + c(P_r^u\mathbf{u}_h^n, \mathbf{v}_r, P_r^\zeta\zeta_h^{n-1}) + c(\mathbf{v}_r, P_r^u\mathbf{u}_h^n, P_r^u\zeta_h^{n-1}) \\
&\quad + \alpha(P_r^\zeta\zeta_h^{n-1}\cdot\tau, \mathbf{v}_r\cdot\tau)_\Gamma - (P_r^p\sigma_h^{n-1}, \nabla\cdot\mathbf{v}_r) \\
&\quad = \alpha(\alpha_\zeta^{n-1}\cdot\tau, \mathbf{v}_r\cdot\tau)_\Gamma - (\beta_\sigma^{n-1}, \nabla\cdot\mathbf{v}_r) \\
&\quad + 2\nu(\mathbb{D}(\alpha_\zeta^n), \mathbb{D}(\nabla\mathbf{v}_r)) + c(P_r^u\mathbf{u}_h^n, \mathbf{v}_r, P_r^\zeta\zeta_h^{n-1}) \\
&\quad + c(\mathbf{v}_r, P_r^u\mathbf{u}_h^n, P_r^\zeta\zeta_h^{n-1}) \\
&\quad + (\nabla\times P_r^u\mathbf{u}_h^{n-1}, \nabla\times\mathbf{v}_r) - (\nabla\times\mathbf{e}_u^n, \nabla\times\mathbf{v}_r) \\
&\quad - c(\mathbf{u}_h^n, \mathbf{v}_r, \zeta_h^{n-1}) - c(\mathbf{v}_r, \mathbf{u}_h^n, \zeta_h^{n-1}),\ \ \forall\mathbf{v}_r\in\mathbf{X}_R \\
&\varepsilon\left(\frac{P_r^\sigma\sigma_h^n - P_r^\sigma\sigma_h^{n-1}}{\Delta t}, \phi_r\right) + (\nabla\cdot P_r^\zeta\zeta_h^n, \phi_r) \\
&\quad = (\nabla\cdot\alpha_\zeta^n, \phi_r),\ \forall\phi_r\in Q_M.
\end{aligned} \tag{20}$$

Lemma 4.2. Let $(\zeta_r^n, \sigma_r^n)\in\mathbf{X}_R^\zeta\times Q_M^\sigma$ be the solutions of (8) and $(\zeta_h^n, \sigma_h^n)\in\mathbf{X}_h\times Q_h$ be the corresponding finite element solutions, for $n=1,2,\ldots,N$. Then, we have

$$\begin{aligned}
&(\|\zeta_h^n - \zeta_r^n\|^2 + \varepsilon\|\sigma_h^n - \sigma_r^n\|^2) + \sum_{i=n+1}^{N}\Delta t\|\nabla(\zeta_h^i - \zeta_r^i)\|^2 \\
&\quad \leq C\Delta t\left[\sum_{i=R+1}^{d_u}\lambda_i^u + \sum_{i=M+1}^{d_p}\lambda_i^p + \left(1+\frac{\hat{\eta}^2}{\varepsilon}\right)\sum_{i=R+1}^{d_\zeta}\lambda_i^\zeta\right. \\
&\quad \left. + \sum_{i=M+1}^{d_\sigma}\lambda_i^\sigma + \sum_{i=1}^{N}\|\mathbf{g}_h^i - \mathbf{g}_r^i\|_{0,\Gamma}^2\right],
\end{aligned} \tag{21}$$

where C is independent of h and Δt.

Proof. For $n=1,2,\ldots,N$, subtracting (20) from (8) and setting $(\mathbf{v}_r, \phi_r) = \Delta t(\mathbf{e}_\zeta^{n-1}, e_\sigma^{n-1})$ yields

$$\begin{aligned}
&-\frac{1}{2}(\|\mathbf{e}_\zeta^n\|^2 + \varepsilon\|e_\sigma^n\|^2) + \tfrac{1}{2}(\|\mathbf{e}_\zeta^{n-1}\|^2 + \varepsilon\|e_\sigma^{n-1}\|^2) + \tfrac{1}{2}\|\mathbf{e}_\zeta^n - \mathbf{e}_\zeta^{n-1}\|^2 \\
&\quad + \tfrac{\varepsilon}{2}\|e_\sigma^n - e_\sigma^{n-1}\|^2 + 2\Delta t\nu\|\nabla\mathbf{e}_\zeta^{n-1}\|^2 + \alpha\Delta t\|\mathbf{e}_\zeta^{n-1}\cdot\tau\|_{0,\Gamma}^2 \\
&\quad = \Delta t[(c(P_r^u\mathbf{u}_h^n, \mathbf{e}_\zeta^{n-1}, P_r^\zeta\zeta_h^{n-1}) - c(\mathbf{u}_r^n, \mathbf{e}_\zeta^{n-1}, \zeta_r^{n-1})) \\
&\quad + (c(\mathbf{e}_\zeta^{n-1}, P_r^u\mathbf{u}_h^n, P_r^\zeta\zeta_h^{n-1}) - c(\mathbf{e}_\zeta^{n-1}, \mathbf{u}_r^n, \zeta_r^{n-1})) \\
&\quad - (c(P_r^u\mathbf{u}_h^n, \mathbf{e}_\zeta^{n-1}, P_r^\zeta\zeta_h^{n-1}) - c(\mathbf{u}_h^n, \mathbf{e}_\zeta^{n-1}, \zeta_h^{n-1})) \\
&\quad - (c(\mathbf{e}_\zeta^{n-1}, P_r^u\mathbf{u}_h^n, P_r^\zeta\zeta_h^{n-1}) - c(\mathbf{e}_\zeta^{n-1}, \mathbf{u}_h^n, \zeta_h^{n-1})) \\
&\quad + 2\nu(\mathbb{D}(\alpha_\zeta^{n-1}), \mathbb{D}(\mathbf{e}_u^n)) - (\beta_\sigma^{n-1}, \nabla\cdot\mathbf{e}_\zeta^{n-1}) \\
&\quad + (e_\sigma^{n-1}, \nabla\cdot\alpha_\zeta^n) - (\nabla\times\alpha_u^{n-1}, \nabla\times\mathbf{e}_\zeta^{n-1}) \\
&\quad + (\nabla\times\mathbf{e}_u^{n-1}, \nabla\times\mathbf{e}_\zeta^{n-1}) \\
&\quad + \alpha(\alpha_\zeta^{n-1}\cdot\tau, e_\zeta^{n-1}\cdot\tau)_\Gamma] = \sum_{i=1}^{10}\widehat{\Sigma}_i.
\end{aligned} \tag{22}$$

We will bound the terms on the right hand side of (22). We first note that using the skew-symmetry property, we get

$$\begin{aligned}
|\widehat{\Sigma}_1| &= \Delta t|c(\mathbf{e}_u^n, \mathbf{e}_\zeta^{n-1}, P_r^\zeta\zeta_h^{n-1})| \\
&\leq C\Delta t(\|\nabla P_r^\zeta\zeta_h^{n-1}\|_\infty + \|P_r^\zeta\zeta_h^{n-1}\|_\infty)\|\mathbf{e}_u^n\|\|\nabla\mathbf{e}_\zeta^{n-1}\| \\
&\leq \tfrac{\nu\Delta t}{7}\|\nabla\mathbf{e}_\zeta^{n-1}\|^2 + K_1\Delta t\|\mathbf{e}_u^n\|^2.
\end{aligned} \tag{23}$$

For the second term, we obtain

$$\begin{aligned}
|\widehat{\Sigma}_2| &= \Delta t|c(\mathbf{e}_\zeta^{n-1}, \mathbf{e}_u^n, P_r^\zeta\zeta_h^{n-1}) + c(\mathbf{e}_\zeta^{n-1}, \mathbf{u}_r^n, \mathbf{e}_\zeta^{n-1})| \\
&\leq C\Delta t(\|\nabla P_r^\zeta\zeta_h^{n-1}\|_\infty + \|P_r^\zeta\zeta_h^{n-1}\|_\infty)\|\mathbf{e}_\zeta^n\|\|\nabla\mathbf{e}_u^{n-1}\| \\
&+ C\Delta t\|\mathbf{u}_r^n\|_\infty\|\mathbf{e}_\zeta^n\|\|\nabla\mathbf{e}_\zeta^{n-1}\| \\
&\leq \tfrac{\nu\Delta t}{7}\|\nabla\mathbf{e}_\zeta^{n-1}\|^2 + K_2\Delta t\|\nabla\mathbf{e}_u^n\|^2 + K_3\Delta t\|\mathbf{e}_\zeta^{n-1}\|^2.
\end{aligned} \tag{24}$$

For the third term, we first note that using skew symmetry of the tri-linear form we get

$$\begin{aligned}
|\widehat{\Sigma}_3| &= \Delta t|c(\alpha_u^n, \mathbf{e}_\zeta^{n-1}, P_r^\zeta\zeta_h^{n-1}) + c(\mathbf{u}_h^n, \mathbf{e}_\zeta^{n-1}, \alpha_\zeta^{n-1})| \\
&\leq C\Delta t(\|\nabla P_r^\zeta\zeta_h^{n-1}\|_\infty + \|P_r^\zeta\zeta_h^{n-1}\|_\infty)\|\nabla\alpha_u^n\|\|\nabla\mathbf{e}_\zeta^{n-1}\| \\
&+ C\Delta t\|\mathbf{u}_h^n\|_\infty\|\nabla\mathbf{e}_\zeta^{n-1}\|\|\nabla\alpha_\zeta^{n-1}\| \\
&\leq \tfrac{\nu\Delta t}{7}\|\nabla\mathbf{e}_\zeta^{n-1}\|^2 + K_4\Delta t\|\nabla\alpha_u^n\|^2 + K_5\Delta t\|\nabla\alpha_\zeta^{n-1}\|^2.
\end{aligned} \tag{25}$$

For the fourth term, we obtain

$$\begin{aligned}
|\widehat{\Sigma}_4| &= \Delta t|c(\mathbf{e}_\zeta^{n-1}, \mathbf{e}_u^n, P_r^\zeta\zeta_h^{n-1}) + c(\mathbf{e}_\zeta^{n-1}, \mathbf{u}_h^n, \alpha_\zeta^{n-1})| \\
&\leq C\Delta t(\|\nabla P_r^\zeta\zeta_h^{n-1}\|_\infty + \|P_r^\zeta\zeta_h^{n-1}\|_\infty)\|\mathbf{e}_\zeta^{n-1}\|\|\nabla\mathbf{e}_u^n\| \\
&+ C\Delta t(\|\mathbf{u}_h^n\|_\infty + \|\nabla\mathbf{u}_h^n\|_\infty)\|\mathbf{e}_\zeta^{n-1}\|\|\nabla\alpha_\zeta^{n-1}\| \\
&\leq \tfrac{\nu\Delta t}{7}\|\nabla\mathbf{e}_\zeta^{n-1}\|^2 + K_6\Delta t\|\nabla\mathbf{e}_u^n\|^2 + K_7\Delta t\|\alpha_\zeta^{n-1}\|^2.
\end{aligned} \tag{26}$$

For the fifth term, we obtain

$$|\widehat{\Sigma}_5| \quad \leq \quad K_8\Delta t\|\nabla \mathbf{e}_u^{n-1}\|^2 + K_9\Delta t\|\alpha_\zeta^{n-1}\|^2 . \tag{27}$$

For the sixth term, we obtain

$$|\widehat{\Sigma}_6| \quad \leq \quad \tfrac{\nu\Delta t}{7}\|\nabla \mathbf{e}_\zeta^{n-1}\|^2 + K_{10}\Delta t\|\beta_\sigma^{n-1}\|^2 . \tag{28}$$

For the seventh term, we obtain

$$|\widehat{\Sigma}_7| \quad \leq \quad \tfrac{\varepsilon\Delta t}{2}\|e_\sigma^{n-1}\|^2 + \tfrac{\hat{\eta}^2\Delta t}{2\varepsilon}\|\nabla \alpha_\zeta^n\|^2 . \tag{29}$$

For the eighth and ninth terms, we obtain

$$|\widehat{\Sigma}_8| \quad \leq \quad \tfrac{\nu\Delta t}{7}\|\nabla \mathbf{e}_\zeta^{n-1}\|^2 + K_{11}\Delta t\|\nabla \alpha_u^{n-1}\|^2 . \tag{30}$$

and

$$|\widehat{\Sigma}_9| \quad \leq \quad \tfrac{\nu\Delta t}{7}\|\nabla \mathbf{e}_\zeta^{n-1}\|^2 + K_{12}\Delta t\|\nabla \mathbf{e}_u^{n-1}\|^2 . \tag{31}$$

Finally, we estimate the tenth term using trace inequality as follows

$$|\widehat{\Sigma}_{10}| \leq \alpha\Delta t\|\mathbf{e}_\zeta^{n-1}\cdot\tau\|_{0,\Gamma}^2 + C\Delta t\|\nabla \alpha_\zeta^{n-1}\|^2 . \tag{32}$$

Combining all inequalities, dropping unneeded terms on the left-hand side and summing from $n = m+1$ to N yield

$$\begin{aligned}
&(\|\mathbf{e}_\zeta^m\|^2 + \varepsilon\|e_\sigma^m\|^2) + 2\Delta t\sum_{i=m}^{N-1}\nu\|\nabla \mathbf{e}_\zeta^i\|^2 \\
&\leq C_1\Delta t\sum_{i=m}^{N-1}(\|\mathbf{e}_\zeta^m\|^2 + \varepsilon\|e_\sigma^m\|^2) \\
&+ C_2\Delta t\Bigg[\sum_{i=R+1}^{d_u}\lambda_i^u + \sum_{i=M+1}^{d_p}\lambda_i^p + (1+\frac{\hat{\eta}^2}{\varepsilon})\sum_{i=R+1}^{d_\zeta}\lambda_i^\zeta \\
&+ \sum_{i=M+1}^{d_\sigma}\lambda_i^\sigma + \sum_{i=1}^{N}\|\mathbf{g}_h^i - \mathbf{g}_r^i\|_{0,\Gamma}^2 .\Bigg]
\end{aligned} \tag{33}$$

Finally desired result follows by applying discrete Gronwall inequality. ■

Notice that from (9), we easily have

$$\sum_{i=1}^{N}\Delta t\|\mathbf{g}_h^i - \mathbf{g}_r^i\|_{0,\Gamma}^2 \leq C\sum_{i=1}^{N}\Delta t\|\zeta_h^i - \zeta_r^i\|^2 .$$

Therefore combining Lemma 4.1 and Lemma 4.2, we have the following theorem.

Theorem 4.3. Let $(\mathbf{u}_h^n, p_h^n, \zeta_h^n, \sigma_h^n)$ be the finite element solutions of optimality system associated with (2) and $(\mathbf{u}_r^n, p_r^n, \zeta_r^n, \sigma_r^n)$ be the corresponding ROM solutions, for $n = 1, 2, \ldots, N$. Then, we have

$$\begin{aligned}
&(\|\mathbf{u}_h - \mathbf{u}_r\|_{l^\infty(0,T;L^2(\Omega))}^2 + \varepsilon\|p_h - p_r\|_{l^\infty(0,T;L^2(\Omega))}^2) \\
&+ (\|\zeta_h - \zeta_r\|_{l^\infty(0,T;L^2(\Omega))}^2 + \varepsilon\|\sigma_h - \sigma_r\|_{l^\infty(0,T;L^2(\Omega))}^2) \\
&+ \|\mathbf{g}_h - \mathbf{g}_r\|_{l^2(0,T;L^2(\Gamma))}^2 \leq C\Delta t\Bigg[\sum_{i=R+1}^{d_u}\lambda_i^u + \sum_{i=M+1}^{d_p}\lambda_i^p \\
&+ (1+\frac{\hat{\eta}^2}{\varepsilon})\sum_{i=R+1}^{d_\zeta}\lambda_i^\zeta + \sum_{i=M+1}^{d_\sigma}\lambda_i^\sigma\Bigg] .
\end{aligned} \tag{34}$$

where C is independent of h and Δt.

V. Numerical Experiments

In this section, we perform a numerical investigation of the pressure stabilized ROM algorithm in the simulation of control of flow separation over a forward-facing step channel. The step of unit height is located at a distance of two units from the entrance. The width of the channel is three units and the length is sixteen units. At the channel entrance the flow is prescribed to be fully developed parabolic flow: $u(x=0, 0 \leq y \leq 3) = 4y(3-y)/9$, $v(x=0, 0 \leq y \leq 3) = 0$ and at the outflow, a psuedo stress-free condition is applied. The computational grid was nonuniform in both the stream-wise and cross-flow coordinate directions and a fine grid was used in regions where sharp variations in velocities were expected. All the computations were done with $51x51$ grid and a time step size $t = 1/20$ for the Reynolds' number 2000. For this configuration, separation and re-attachment occur at two places. One on the lower wall in front of the step and another behind the step. But the latter leads to significantly larger wake spread. After the re-attachment on the lower wall behind the step, the flow slowly recovers towards a fully developed Poiseuille flow.

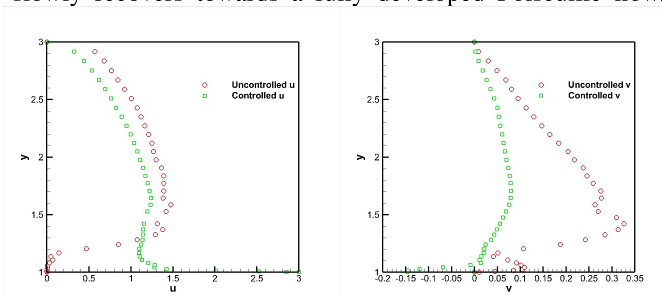

Figure 5.1. *Controlled and uncontrolled u-velocity (left) and v-velocity (right) profiles at* $x = 2.5$

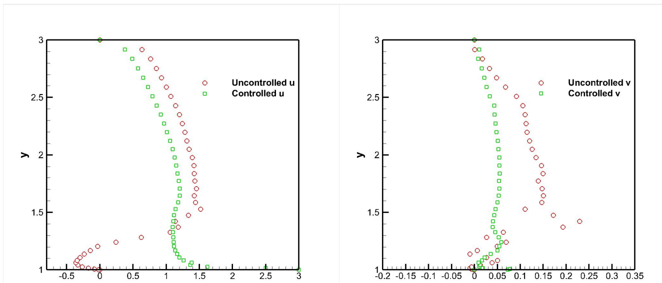

Figure 5.2. *Controlled and uncontrolled u-velocity (left) and v-velocity (right) profiles at* $x = 3$

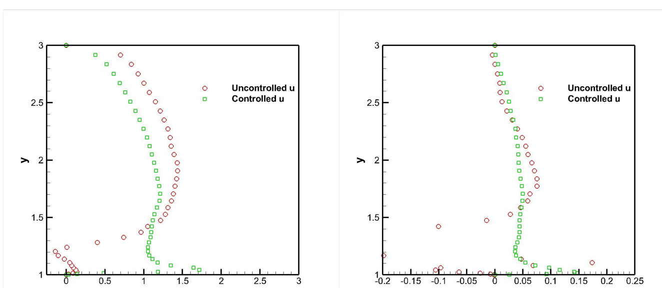

Figure 5.3. *Controlled and uncontrolled u-velocity (left) and v-velocity (right) profiles at* $x = 3.5$

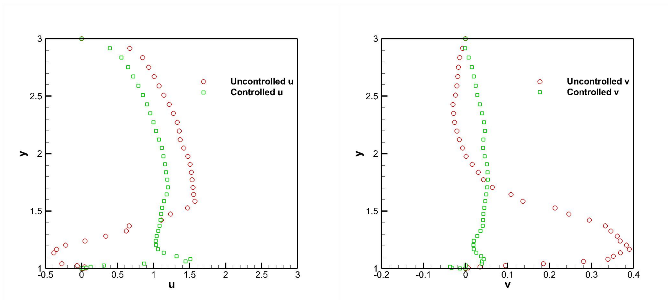

Figure 5.4. *Controlled and uncontrolled u-velocity (left) and v-velocity (right) profiles at* $x = 4$

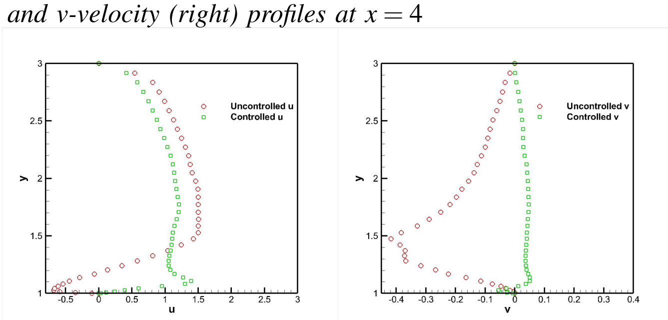

Figure 5.5. *Controlled and uncontrolled u-velocity (left) and v-velocity (right) profiles at* $x = 4.5$

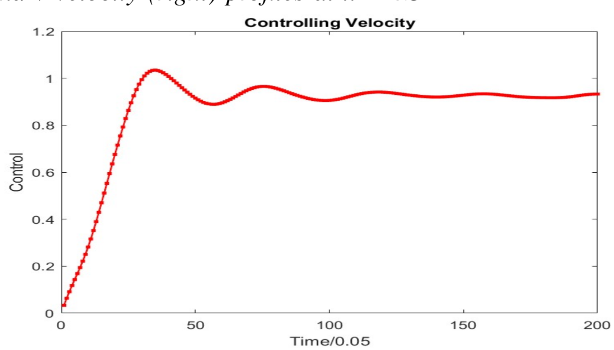

Figure 5.6. *The computed optimal control as function of non-dimensional time*

In the numerical investigation of the new pressure stabilized-ROM, we first showed that the new ROM yields accurate velocity and pressure approximations that are close to the direct numerical simulation results. Next, we employed this ROM to compute a reduced-order controller. The purpose of the controller is to alleviate flow separation and reduce wake spread in the channel. Therefore, we formulate an optimal control problem that minimizes the enstrophy of the flow. The control action (actuation) is effected through tangential blowing through a single slot on a part of the boundary: $2 \leq x \leq 3$ and $y = 1$. Ten POD spatial basis functions (equal number of pressure and velocity modes) are determined by the POD at this Reynolds number $Re = 2000$. A Galerkin projection is then employed to obtain suitable reduced-order dynamic models. Optimal controller is computed using a sequential quadratic programming (SQP) method. For the ROM construction for control, we employed $g(t) = t/10$ in $[0,T] = [0,10]$ in the control action to generate the snapshots. Computed controls at various actuator positions are shown in Fig. 5.6. The resulting horizontal and velocity velocity profiles with and without control at various stations in the channel are shown in Fig. 5.1-5.4. They clearly indicate that flow separation is mitigated by the control action. Substantial reduction in the wake spread is also seen. The re-attachment length has been reduced by more than 80% compared to the uncontrolled case.

VI. Concluding Remarks

In this paper, we proposed a new pressure-stabilized proper orthogonal decomposition reduced order model (POD-ROM) for the control of viscous incompressible flows. The new pressure stabilized ROM is a velocity-pressure ROM and it does not require the fulfillment of the inf-sup condition, which can be prohibitively expensive with current ROM approaches [1]. Moreover, the present method does not require weakly divergence free snapshots. We proposed and rigorously analyzed a decoupled time-stepping scheme that uncouples the computation of velocity and pressure greatly improving computational efficiency. In the numerical investigations, we provided a numerical comparison of the new pressure-stabilized ROM in the simulation of control of flow past on a forward-facing step channel. Our results showed the feasibility of proposed scheme and yields an efficient control approximations.

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