2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Parameter Adaptation for General Regulator Problems With Compensation of Internal Model

Gian Paolo Incremona, Patrizio Colaneri, Leonid Mirkin

***Abstract*— This paper investigates the general regulator problem with an internal model capable of adapting to the unknown parameters of persistent disturbances and/or reference signals affecting the measured output. Specifically, the proposed architecture based on stable compensators of internal model (CIM) allows to reduce the stabilization problem for an augmented system (the plant plus internal model) to that of a process without the internal model and with the complexity of the plant. Then, it is shown that, under certain conditions on the plant model, the proposed scheme makes the parameters of the internal model affect the closed-loop dynamics affinely. This, in turn, facilitates the incorporation of simple adaptation mechanisms with a global convergence.**

***Index Terms*— Parameter adaptation, internal model principle, regulator problem.**

I. Introduction

Traditionally, for a continuous-time linear time-invariant (LTI) plant, namely $P : u \mapsto y$, with input $u(t) \in \mathbb{R}^m$ and measured output $y(t) \in \mathbb{R}^p$, addressing the regulator problem consists of designing a stabilizing feedback controller $R : y \mapsto u$ capable of asymptotically rejecting persistent disturbances and/or reference signals of a known class on the so-called *regulated signal*. Introducing a suitable matrix $E \in \mathbb{R}^{p_e \times p}$ assumed to be normalized, i.e., such that $EE' = I$, such a regulated signal can be indicated as $e(t) = Ey(t) \in \mathbb{R}^{p_e}$.

In the literature, Internal Model Principle [1] is a valid solution to the regulator problem, whose idea is that of including a $p_e \times p_e$ model of persistent exogenous signals, namely M, into the controller, see e.g., [2, § 4.4]. Specifically, condition for robust regulation is that each pole of M has the geometric multiplicity p_e, see [2], [3], and the stabilizer, provided that no unstable cancellations occur, is commonly designed for an augmented plant

$$P_{\text{aug}} := (E'ME + I - E'E)P, \tag{1}$$

including the internal model. This procedure presents however two significant drawbacks. On the one hand, the dimension of the augmented plant increases when the complexity of the internal model increases (as for instance for repetitive control [4] in case of periodic signals, whose model is infinite dimensional). On the other hand, a complex dependence of the parameters of the stabilizer on those of the internal model can occur.

Recently, alternative solutions to the mentioned issues have been proposed. Inspired by the delay compensation in repetitive control [5], a general internal model in the state-feedback case with compensation has been introduced, for instance, in [6]. Enhanced results have been presented then in [7], where an original approach based on *compensators of internal model* (CIM) is proposed. The main advantage of such elements is that of reducing the stabilization of P_{aug} to that of a system having the same complexity as the non-augmented plant P. One of those elements is in parallel to the "central controller" in the regulation channel, while the other one connects two measurement channels.

However, in [7], the parameters of the persistent disturbances and/or reference signals affecting the measured output are assumed constant and available, which is not always true in practice. In this paper, we overcome this assumption by extending the contribution to the case of unknown parameters. Differently from [7], where no adaptive internal model control elements are presented, here we show that, under certain conditions for the plant, that should be minimum-phase with a relative degree at most 2, the control architecture proposed in [7] allows the reduction of the stabilization problem to that of a *simpler* version of P, and the parameters of the internal model affects the closed-loop dynamics affinely. This, in turn, facilitates the incorporation of simple adaptation mechanisms with a global convergence. This is an advantage and it is worth to notice that having parameters of the internal model which affect the closed-loop dynamics affinely is more suitable for real-world control problems, thus paving the way for the application of the proposal to different domains, such as electro-mechanical systems, as the case study illustrated in this work. Moreover, while the number of parameters in early studies of adaptive internal model control, see the book [8] and the references therein, always equals the state dimension of the internal model, in the proposed method the number of parameters has the dimension of the number of the uncertain frequencies in the internal model (sometimes it is even lower), thus making the proposed parametrization more economic.

Notation: The closed right half of the complex plane is denoted by $\bar{\mathbb{C}}_0$, while the sets of real and natural numbers are indicated as $\mathbb{R}$ and $\mathbb{N}$, respectively. The complex-conjugate transpose of a matrix A is denoted by A'. The notation $\text{spec}(A)$ stands for the matrix spectrum when A is a square matrix or for the set of poles if A is an LTI system. By $\mathcal{H}_\infty$

Supported by the Israel Science Foundation (grant no. 3177/21), by the Italian Ministry for Research (PRIN 2022 grant no. 2022LP77J4) and, in part, by Sakranut Graydah at Politecnico di Milano and Technion.

G. P. Incremona and P. Colaneri are with Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, 20133 Milan, Italy (e-mails: *gianpaolo.incremona@polimi.it* and *patrizio.colaneri@polimi.it*). L. Mirkin is with the Faculty of Mechanical Engineering, Technion—IIT, Haifa 3200003, Israel (e-mail: *mirkin@technion.ac.il*).

we denote the set of holomorphic and bounded functions in the open right-half plane, whereas $\mathcal{L}_2$ is the set of square-integrable functions. Linear systems in the time domain are denoted by capital letters with no argument, like G, and $G(s)$ stands for the corresponding transfer function. The compact notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B$$

is used for transfer functions in terms of their state-space realizations.

II. Preliminaries on Internal Model Compensation

In this section we recall some preliminaries about the instrumental results presented in [7].

The considered control problem is that of designing a stabilizer, namely R_{s}, inside the controller defined as

$$R = R_{\mathrm{s}}(E'ME + I - E'E), \tag{2}$$

with internal model M for a plant P with proper transfer function $P(s)$. According to [7], the following assumptions hold:

$\boldsymbol{\mathcal{A}_1}$: $\mathrm{spec}(M) \in \bar{\mathbb{C}}_0$, $M^{-1} \in \mathcal{H}_\infty$, and $M(\infty) = I$,

$\boldsymbol{\mathcal{A}_2}$: $p_e = m$ and $EP(s)$ has full rank,

$\boldsymbol{\mathcal{A}_3}$: $\mathrm{spec}((EP)^{-1}) \cap \mathrm{spec}(M) = \varnothing$.

This means that all poles of M are unstable, that the regulated channel is neither underactuated nor has redundancies and, finally, no unstable cancellations occur in P_{aug}.

It is known that R_{s} internally stabilizes P_{aug} if the system (the gang of four)

$$T_{4,\mathrm{aug}} := \begin{bmatrix} I \\ -R_{\mathrm{s}} \end{bmatrix} (I - P_{\mathrm{aug}}R_{\mathrm{s}})^{-1} \begin{bmatrix} I & P_{\mathrm{aug}} \end{bmatrix} \tag{3}$$

is stable. Now, apart from [5, Thm. 1] and [6, Thm. 1], it is convenient to recall more in detail [7, Thm. 1] to introduce the concept of *internal model compensator*. Consider the matrix $E_\perp \in \mathbb{R}^{(p-p_e)\times p}$ as any matrix satisfying

$$E_\perp' E_\perp = I - E'E,$$

so that, if $p > p_e$, then every such $E_\perp$ has full row rank and satisfies $E_\perp \begin{bmatrix} E' & E_\perp' \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$. [7, Thm. 1] says that the problem of stabilizing P_{aug} can be recast into that of stabilizing an equivalent plant, namely

$$\bar{P} := (I + E_\perp' \Upsilon_2 E) P_{\mathrm{aug}} (I + \Upsilon_1 E P_{\mathrm{aug}})^{-1}, \tag{4}$$

by including two CIM terms, namely Υ_1 and Υ_2, into the controller given by

$$R_{\mathrm{s}} = \bar{R}(I + E_\perp' \Upsilon_2 E) - \Upsilon_1 E. \tag{5}$$

Looking at (5), differently from [5], [6], a cascade block $I + E_\perp' \Upsilon_2 E$ is present together with the parallel element $-\Upsilon_1 E$. The former connects the regulated signal e with its complement in y. Note that, if $e = y$, then $E_\perp$ is no more present, so is Υ_2, and (5) coincides with the controller structure in [5], [6].

III. Structure of $\bar{P}$

To better highlight the advantages of the considered control architecture, the simpler plant $\bar{P}$ is now analyzed.

From the result recalled in the previous section, one of the main advantages is that the regulator problem could be simplified avoiding the stabilization of a high-dimensional augmented plant P_{aug}. Specifically, $\bar{P}$ can always be chosen to have the same complexity as the original plant P without the dynamics of the internal model M. Qualitative arguments supporting this claim, as well as a more useful relation between $\bar{P}$ and Υ_1 and Υ_2, are hereafter reported.

In fact, by exploiting (4), using (1) and relations

$$\begin{bmatrix} E \\ E_\perp \end{bmatrix} \bar{P} = \begin{bmatrix} MEP \\ E_\perp P + \Upsilon_2 MEP \end{bmatrix} (I + \Upsilon_1 MEP)^{-1},$$

one obtains

$$\begin{bmatrix} E\bar{P} \\ E_\perp \bar{P} \end{bmatrix} (I + \Upsilon_1 MEP) = \begin{bmatrix} MEP \\ E_\perp P + \Upsilon_2 MEP \end{bmatrix}. \tag{6}$$

Therefore, since EP and M are invertible by $\boldsymbol{\mathcal{A}_1}$ and $\boldsymbol{\mathcal{A}_2}$, post-multiplying (6) by $(EP)^{-1}M^{-1}$one has

$$\begin{bmatrix} E\bar{P} \\ E_\perp \bar{P} \end{bmatrix} ((EP)^{-1}M^{-1} + \Upsilon_1) = \begin{bmatrix} I \\ E_\perp P (EP)^{-1} M^{-1} + \Upsilon_2 \end{bmatrix}.$$

Taking $EP(s)$ strictly proper, it always holds that $I + \Upsilon_1 MEP$ and the first row of the expression above imply that $E\bar{P}$ is invertible as well, so $\bar{P}$ has to be found such that

$$\begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} = \begin{bmatrix} I \\ E_\perp \bar{P} \end{bmatrix} (E\bar{P})^{-1} - \begin{bmatrix} I \\ E_\perp P \end{bmatrix} (EP)^{-1} M^{-1}. \tag{7}$$

Looking at (7), we can report some considerations which confirm the claim that the complexity of $\bar{P}$ shall not exceed that of P. The first observation is that the terms $(E\bar{P})^{-1}$ and $E_\perp \bar{P}(E\bar{P})^{-1}$ should match unstable, non-proper, parts of $(EP)^{-1}M^{-1}$ and $E_\perp P(EP)^{-1}M^{-1}$, respectively. Then, since M^{-1} is stable by virtue of $\boldsymbol{\mathcal{A}_1}$, so instabilities above are associated only with the plant P, without the internal model. Note that, if EP is stably invertible, then one could select $\bar{P} = P$, which is not always true in general.

Now, before analyzing the parameter in the internal model, it is convenient to define the state-space expressions for the system, as in [7]. More precisely, let the state-space realizations of P and M be

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{and} \quad M(s) = \left[\begin{array}{c|c} A_{\mathrm{m}} & B_{\mathrm{m}} \\ \hline C_{\mathrm{m}} & I \end{array}\right],$$

whose state dimensions are n and n_{m}, respectively, and where we assume that $ED = 0$, and zeros of $M(s)$ and $EP(s)$ are disjoint. Letting matrices $B^{\#} \in \mathbb{R}^{m\times n}$ and $B^{\perp} \in \mathbb{R}^{(n-m)\times n}$ so that

$$\begin{bmatrix} B^\perp \\ B^{\#} \end{bmatrix} B = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} B^\perp \\ B^{\#} \end{bmatrix} \neq 0,$$

one can write

$$\begin{bmatrix} \Upsilon_1(s) \\ \Upsilon_2(s) \end{bmatrix} = \left[\begin{array}{c|c} A_{\mathrm{m}} - B_{\mathrm{m}}C_{\mathrm{m}} & B_{\mathrm{m}} \\ \hline C_0 & 0 \\ E_\perp CX + E_\perp DC_0 & 0 \end{array}\right], \tag{8}$$

where $C_0 := B^{\#}X(A_{\rm m} - B_{\rm m}C_{\rm m}) - B^{\#}AX$, and

$$\bar{P}(s) = \left[\begin{array}{c|c} A + XB_{\rm m}EC & B \\ \hline C & D \end{array}\right] \tag{9}$$

satisfying (7), with $X \in \mathbb{R}^{n \times n_{\rm m}}$ being the unique bounded solution of the generalized Sylvester equation

$$\begin{bmatrix} B^{\perp} \\ 0 \end{bmatrix} X(A_{\rm m} - B_{\rm m}C_{\rm m}) - \begin{bmatrix} B^{\perp}A \\ -EC \end{bmatrix} X = \begin{bmatrix} 0 \\ C_{\rm m} \end{bmatrix}. \tag{10}$$

Moreover, since we are interested in the effects of the CIM elements on closed-loop system functions, in particular we introduce the closed-loop sensitivity, S, and disturbance sensitivity, $T_{\rm d}$, given by

$$\begin{bmatrix} S & T_{\rm d} \end{bmatrix} := (I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix}, \tag{11}$$

whose counterparts associated with $\bar{P}$ and $\bar{R}$ in (4) and (5) are $\begin{bmatrix} \bar{S} & \bar{T}_{\rm d} \end{bmatrix} := (I - \bar{P}\bar{R})^{-1} \begin{bmatrix} I & \bar{P} \end{bmatrix}$.

In the following, we will show that, under certain conditions for EP, the considered control structure makes the parameters of the internal model affect the closed-loop dynamics affinely.

IV. Parameter in the Internal Model

In this section we will explicitly consider the dependence of the internal model on some unknown parameter to design an adaptation mechanism to estimate it.

A. Structure of M

First, assume that

$$M^{-1}(s) = M_0(s) + M_1(s)\Theta, \tag{12}$$

for some $M_0, M_1 \in \mathcal{H}_\infty$ and the static unknown parameter $\Theta \in \mathbb{R}^{\nu \times m}$, with $\nu \in \mathbb{N}$. Specifically, assume $M_0(s)$ be bi-proper and $M_1(s)$ be strictly proper, with relative degree 2. For example, consider an exogenous signal having a DC component and two harmonics at $\omega_1 > 0$ and $\omega_2 > 0$. If only ω_1 is uncertain (i.e., $\nu = 1$), one can select $M(s) = \phi(s)/(s^2 + \omega_1^2)$ for some monic second-order Hurwitz $\phi(s)$, so that

$$M_0(s) = \frac{s^2}{\phi(s)}, \quad M_1(s) = \frac{1}{\phi(s)}, \quad \text{and} \quad \Theta = \omega_1^2.$$

If instead both the harmonics are unknown (i.e., $\nu = 2$), and $M(s) = \phi(s)/(s(s^2 + \omega_1^2)(s^2 + \omega_2^2))$ for some monic 5th-order Hurwitz $\phi(s)$, then

$$M_0(s) = \frac{s^5}{\phi(s)}, \quad M_1(s) = \frac{1}{\phi(s)} \begin{bmatrix} s^3 & s \end{bmatrix},$$

and

$$\Theta = \begin{bmatrix} \omega_1^2 + \omega_2^2 \\ \omega_1^2\omega_2^2 \end{bmatrix}.$$

Therefore, equation (7) reads

$$\begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} = \begin{bmatrix} I \\ E_\perp \bar{P} \end{bmatrix} (E_1\bar{P})^{-1} - \begin{bmatrix} I \\ E_\perp P \end{bmatrix} (EP)^{-1}M_0 - \begin{bmatrix} I \\ E_\perp P \end{bmatrix} (EP)^{-1}M_1\Theta. \tag{13}$$

The logic in the choice of $\bar{P}$ is to cancel all instabilities in the last two terms in the right-hand side of (13). If we also want it to be independent of Θ, then we need to *assume* that both $(EP)^{-1}M_1$ and $E_\perp P(EP)^{-1}M_1$ are stable. To this end, the following sufficient assumption is introduced.

$\boldsymbol{\mathcal{A}_4}$: $EP(s)$ is minimum-phase with a relative degree of at most 2, and $E_\perp P(s)$ has no unstable poles that are not poles of $EP(s)$.

If this is indeed the case, then $\bar{P}$ is as in (9) modulo the replacement

$$M^{-1}(s) = \left[\begin{array}{c|c} A_0 & B_0 + B_1\Theta \\ \hline C_0 & I \end{array}\right] \to M_0(s) = \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & I \end{array}\right]$$

in its derivation (so that $B_{\rm m}$ there is B_0, rather than $B_0 + B_1\Theta$). Also, Υ_1 and Υ_2 from (7) shall then be complemented with the last term in (13), which might increase their dimensions. Hence, under assumption $\boldsymbol{\mathcal{A}_4}$, we could always choose $\bar{P}$ independent of Θ.

B. Internal model compensation

Let now the model M be such that in (12) $M^{-1} = M_0 + M_1\hat{\Theta}$, where $\hat{\Theta} \in \mathbb{R}^{\nu \times m}$ is the estimate of the unknown parameter Θ, and

$$\begin{bmatrix} M_0(s) & M_1(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_0 & B_0 & B_1 \\ \hline C_0 & I & 0 \end{array}\right]. \tag{14}$$

One can define $\bar{P}$ as a system independent of $\hat{\Theta}$, such that it satisfies the following relation achieved from (7) and (12):

$$\begin{bmatrix} \hat{\Upsilon}_1 \\ \hat{\Upsilon}_2 \end{bmatrix} := \begin{bmatrix} I \\ E_\perp \bar{P} \end{bmatrix} (E\bar{P})^{-1} - \begin{bmatrix} I \\ E_\perp P \end{bmatrix} (EP)^{-1}M_0 \in \mathcal{H}_\infty, \tag{15}$$

where $\hat{\Upsilon}_1$ and $\hat{\Upsilon}_2$ are the CIM elements such that no dependence on $\hat{\Theta}$ occurs. To compute these compensators we only need to replace $B_{\rm m}$ in (8) with B_0, as already mentioned at the end of § IV-A.

Our goal becomes therefore to derive a state-space realization of the system

$$\Psi := \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix},$$

where

$$\begin{aligned} \Psi_{11} &:= \bar{R} + (I - \bar{R}P)(EP)^{-1}E - (E\bar{T}_{\rm d})^{-1}M_0^{-1}E, \\ \Psi_{12} &:= (E\bar{T}_{\rm d})^{-1}M_0^{-1}M_1, \\ \Psi_{21} &:= M_0^{-1}E, \\ \Psi_{22} &:= -M_0^{-1}M_1. \end{aligned}$$

Taking into account that $(E\bar{T}_{\rm d})^{-1} = (I - \bar{R}\bar{P})(E\bar{P})^{-1}$ and $\bar{P}(E\bar{P})^{-1} = E' + E'_\perp E_\perp \bar{P}(E\bar{P})^{-1}$, it can be shown that

$$\Psi = \left[\begin{array}{cc:c} I & \bar{R} & 0 \\ \hdashline 0 & 0 & I \end{array}\right] \Psi_0,$$

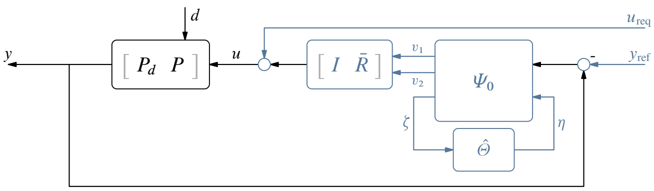


Fig. 1. Proposed control scheme.

where

$$\Psi_0 := \left[\begin{array}{c:c} -\hat{\Upsilon}_1 & (E\bar{P})^{-1} \\ E' + E'_\perp \hat{\Upsilon}_1 & -E' - E'_\perp E_\perp \bar{P}(E\bar{P})^{-1} \\ \hdashline I & -I \end{array}\right] \times \left[\begin{array}{c:c} M_0^{-1}E & 0 \\ \hdashline 0 & M_0^{-1}M_1 \end{array}\right] + \left[\begin{array}{c:c} 0 & 0 \\ E'_\perp E_\perp & 0 \\ \hdashline 0 & 0 \end{array}\right]. \quad (16)$$

Therefore, it results that the parameters of the internal model affinely affect the closed-loop dynamics such that, letting v_1, v_2, and ζ the outputs of system Ψ_0, with e and η being the inputs (see Fig. 1), one has

$$\eta = \hat{\Theta}\zeta, \quad (17)$$

with $\zeta = Me$, and both ζ and e directly measurable. The following assumption needs to be introduced.

$\boldsymbol{\mathcal{A}_5}$: $\exists \Theta = \bar{\Theta}$ in (12) such that $e = \bar{e} \in \mathcal{L}_2$.

In other words, the previous assumption implies that there is a value $\bar{\Theta}$ coherent with the actual persistent components of exogenous signals, such that the internal model with (12) ensures asymptotic regulation. This condition implies that the expression of the error becomes

$$e = M_1\tilde{\Theta}\zeta + \bar{e}, \quad (18)$$

with $\tilde{\Theta} := \hat{\Theta} - \bar{\Theta}$ and $\bar{e}$ decaying to zero. Hence, this relation is the instrumental base to design an adaptation scheme, as detailed in the next section.

V. Adaption of the Internal Model

We are now in a position to describe the adaptation method included in the proposed scheme. In the previous section, although relation (18) is instrumental to this end, a clear dependence on the unknown parameters still appears, thus preventing the application of adaptation mechanisms.

Without loss of generality, take now $m = 1$ such that the regulation error satisfies (18) for a measurable $\zeta : \mathbb{R}_+ \to \mathbb{R}$ and an unknown $\Theta \in \mathbb{R}^\nu$, and in (14) A_0 is Hurwitz and (A_0, B_1) controllable.

Then, the state-space relation (18) can be written as

$$\begin{bmatrix} \dot{x}_{M_1} \\ \dot{x}_\theta \end{bmatrix} = \begin{bmatrix} A_0 & -B_1\zeta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{M_1} \\ x_\theta \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \hat{\Theta}\zeta,$$

$$e(t) = \begin{bmatrix} C_0 & 0 \end{bmatrix} \begin{bmatrix} x_{M_1} \\ x_\theta \end{bmatrix},$$

with x_{M_1} and x_θ being the internal model and parameter states, respectively, while $\left[\begin{smallmatrix} x_{M_1}(0) \\ x_\theta(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ \bar{\Theta} \end{smallmatrix}\right]$.

Now, introduce the function $z : \mathbb{R}_+ \to \mathbb{R}^{n_0 \times \nu}$, where n_0 is the dimension of x_{M_1} and ν is the dimension of Θ, satisfying

$$\dot{z}(t) = A_0 z(t) + B_1\zeta(t), \quad z(0) = 0,$$

and the auxiliary variable $x_1 := x_{M_1} + zx_\theta$. In this case, applying the similarity transformation with $\left[\begin{smallmatrix} I & z(t) \\ 0 & I \end{smallmatrix}\right]$, the state equation reads

$$\begin{bmatrix} \dot{x}_1 \\ x_\theta \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_\theta \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \hat{\Theta}\zeta,$$

where $\left[\begin{smallmatrix} x_1(0) \\ x_\theta(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ \bar{\Theta} \end{smallmatrix}\right]$, and the output equation reads

$$\begin{aligned} e(t) &= C_0 x_1(t) - C_0 Z(t) x_\theta(t) \\ &= C_0 x_1(t) - C_0 Z(t)\hat{\Theta}(t) + C_0 Z(t)\tilde{\Theta}(t). \end{aligned}$$

Hence, we have the static relation

$$e_1 = C_0 z\tilde{\Theta} \quad \text{or} \quad e_1' = \tilde{\Theta}'(C_0 z)', \quad (19)$$

where

$$e_1 := e + C_0 z\hat{\Theta} - M_1\hat{\Theta}\zeta$$

can be measured. The latter can now be adopted to apply a suitable (in whatever appropriate sense) adaptation law.

As an example, consider the strategy in [9, § 3.A], where a *normalized least square estimation algorithm* is designed as

$$\dot{\hat{\Theta}} = \frac{\Pi(C_0 z)' e_1}{N} + Q, \quad (20)$$

where e_1 is defined in (19), $Q \geq 0$ (with $Q = 0$ corresponding to the pure least square approach) , and Π such that $\mathrm{trace}(\Pi) < \infty$ solving

$$\dot{\Pi} = -\frac{\Pi(C_0 z)'(C_0 z)\Pi}{N} + 2\alpha\frac{\Pi}{N}, \quad \Pi(0) = k_0 I > 0.$$

with $\alpha > 0$, $k_0 > 0$, and the scalars

$$N = c + (C_0 z)(C_0 z)' + (C_0 z)\Pi(C_0 z)', \quad c > 0.$$

The analysis of the algorithm follows from the selection of the non-negative function $V = \tilde{\Theta}'\Pi^{-1}\tilde{\Theta}$, thus ensuring that $\dot{\hat{\Theta}}$ is bounded and $\dot{\hat{\Theta}} \in \mathcal{L}_2$, see [9, § 3.A].

VI. Numerical Example

In this section the proposed approach is assessed relying on an armature-controlled DC motor connected to a rigid mechanical load, see [10, § 6.5]. Let the shaft angle θ_{sh} and its angular velocity ω_{sh} measurable, and the output $y = \left[\begin{smallmatrix} \theta_{\mathrm{sh}} \\ \omega_{\mathrm{sh}} \end{smallmatrix}\right]$, while the control input is u representing the armature voltage. The controlled plant is instead captured by

$$P(s) = \begin{bmatrix} P_\theta(s) \\ P_\omega(s) \end{bmatrix} = \begin{bmatrix} 1/s \\ 1 \end{bmatrix} \frac{K_{\mathrm{m}}}{(Js + f)R_{\mathrm{a}} + K_{\mathrm{m}}^2},$$

with K_{m} being the motor torque coefficient, R_{a} the armature resistance (the inductance is neglected), and J and f being the moment of inertia and viscous friction coefficient of the rigid load, respectively. The load disturbance is given by an external torque, namely τ_{e}, and is equal to $k_\tau\tau_{\mathrm{e}}$, with $k_\tau :=$

$R_\mathrm{a}/K_\mathrm{m}$. The regulated signal is the shaft angle θ_sh, such that $E = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The motor numerical data are reported in Table I.

TABLE I

MOTOR AND LOAD PARAMETERS

K_m [N m/A]	R_a [Ω]	J [kg m^2]	f [N m s/rad]	τ_max [N m]
0.126	2.08	0.008	0.005	0.235

Considering the scheme in Fig. 1, let y_ref and u_req represent the nominal command following requirements, such that, given a feedback controller R, it holds

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y_\mathrm{ref} \\ u_\mathrm{req} \end{bmatrix} + \begin{bmatrix} S \\ RS \end{bmatrix} \left(P k_\tau \tau_\mathrm{e} - y_\mathrm{ref} + P u_\mathrm{req} \right),$$

where S is the sensitivity function in (11), and in nominal conditions (i.e., without uncertainties and disturbances) one has $y_\mathrm{ref} = P u_\mathrm{req}$, $\tau_\mathrm{e} = 0$, implying $y = y_\mathrm{ref}$. Since instead the disturbance affects the closed-loop system, a suitable design of R is required so that, if at some ω_i

$$ES(\mathrm{j}\omega_i) = 0 \quad \text{and} \quad ET_\mathrm{d}(\mathrm{j}\omega_i) = 0, \tag{21}$$

then, the corresponding harmonic of τ_e, y_ref, and u_req do not affect the regulated signal. For the regulator design we rely on the same procedure described in [7, § IV.A], with form (2) and an internal model $M(s)$ having poles at each $s = \mathrm{j}\omega_i$. We will focus now on the design of the internal model and the discussion on the adaptation algorithm.

A. Choice of $\bar{P}$, Υ_i and adaptation design

Consider that (21) has to be fulfilled for three frequencies,

$$\omega_0 = 0, \quad \omega_1 = \frac{1}{2}\pi, \quad \text{and} \quad \omega_2 = \frac{8}{3}\pi.$$

so that the internal model can be chosen as

$$M(s) = \frac{(s + a_\mathrm{m})^5}{s(s^2 + \omega_1^2)(s^2 + \omega_2^2)},$$

with $a_\mathrm{m} > 0$, satisfying $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{A}}_3$. Then, making reference to (8), $\bar{P}$, Υ_1, and Υ_2 are such that, given

$$P(s) = \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{7.5672}{s(s + 1.578)} \quad \text{and} \quad k_\tau = 16.5187,$$

and $a_\mathrm{m} = 4$, one has

$$\bar{P}(s) = \begin{bmatrix} 1 \\ s - 20 \end{bmatrix} \frac{7.5672}{s^2 - 18.42s + 281.1},$$

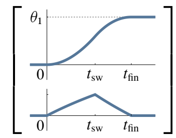
and the internal model compensators are

$$\Upsilon_1(s) = \frac{422.84(s^2 + 4.88s + 6.28)(s^2 + 5.84s + 14.33)}{(s + 4)^5},$$

$$\Upsilon_2(s) = -\frac{312.65(s^2 + 3.56s + 3.83)(s^2 + 4.63s + 17.09)}{(s + 4)^5}.$$

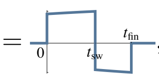
Pose now the closed-loop poles at $s = -2$ by selecting in (2) the stabilizer $\bar{R}(s) = -\begin{bmatrix} 22.6436 & 2.9631 \end{bmatrix}$. Finally, in the following, the reference y_ref is chosen to be the time-optimal one (also known as bang-bang) to achieve θ_1 under a limited torque $\tau_\mathrm{max}/4$, with τ_max as indicated in Table I. The optimal torque is instead designed for the load dynamics $J\ddot{\theta}_\mathrm{sh} + f\dot{\theta}_\mathrm{sh} = \tau$ under $\theta_\mathrm{sh}(0) = \dot{\theta}_\mathrm{sh}(0) = 0$ [11, Ch. 7]. Therefore, letting t_sw and t_fin be the switching and final times, respectively, one obtains

$$y_\mathrm{ref}(t) = \begin{bmatrix} \theta_\mathrm{opt}(t) \\ \omega_\mathrm{opt}(t) \end{bmatrix} =$$



where $\omega_\mathrm{opt} = \dot{\theta}_\mathrm{opt}$, whereas the required voltage $u_\mathrm{req} = (1/P_\theta)\theta_\mathrm{opt}$ is of the form

$$u_\mathrm{req}(t) = \frac{R_\mathrm{a}}{K_\mathrm{m}} \tau_\mathrm{opt}(t) + K_\mathrm{m}\, \omega_\mathrm{opt}(t) =$$



and τ_opt is bang-bang in the range $[-\tau_\mathrm{max}/3, \tau_\mathrm{max}/3]$.

As for the adaptation of the internal model, starting from (19), letting $\bar{\omega} \in \mathbb{R}^\nu$ be the nominal frequency and $\omega \in \mathbb{R}^\nu$ be the actual frequency, the parameter Θ is defined as

$$\Theta = \omega^2 - \bar{\omega}^2$$

if $\nu = 1$, and as

$$\Theta = \begin{bmatrix} \omega_1^2 + \omega_2^2 - \bar{\omega}_1^2 - \bar{\omega}_2^2 \\ \omega_1^2\omega_2^2 - \bar{\omega}_1^2\bar{\omega}_2^2 \end{bmatrix}$$

if $\nu = 2$. Then, according to [9, § 3.A], the normalized least square estimation algorithm in (20) is adopted, selecting $Q = 0$, $\alpha = k_0 = 1$ and $c = 0.1$.

B. Simulations

In the following, we consider two different scenarios to assess the proposed scheme with adaptation of the internal model.

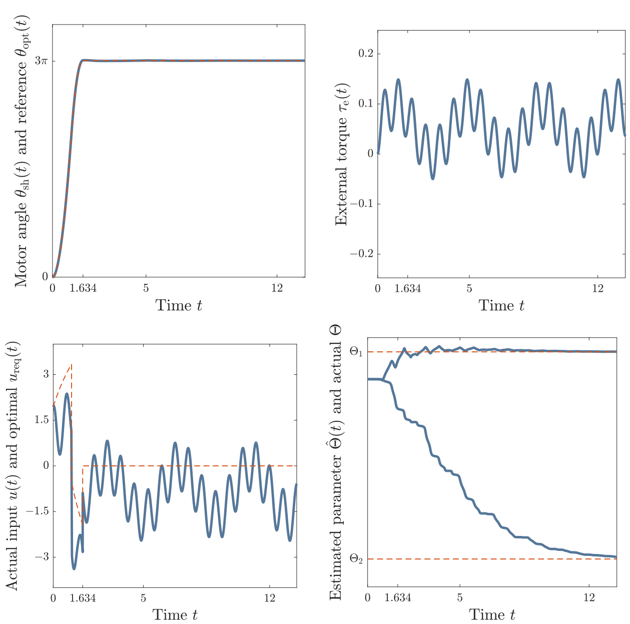


Fig. 2. Simulations in the Scenario 1.

Scenario 1: In this scenario we assume that the external (disturbance) torque applied to the motor shaft contains the three frequencies ω_0, ω_1 and ω_2 previously introduced, while the nominal ones are

$$\bar{\omega}_0 = 0, \quad \bar{\omega}_1 = \frac{3\pi}{2}, \quad \bar{\omega}_2 = \pi,$$

and it has the following form:

$$\tau_\mathrm{e}(t) = 0.05[1 + \sin(\omega_1 t) - \cos(\omega_2 t)].$$

Considering a fifth-order internal model, the simulated responses of the controller are illustrated in Fig. 2. Specifically, the trajectory of the output is illustrated (top left), where the optimal reference (dashed line) is perfectly tracked despite the presence of the external disturbance (top right). The latter is compensated by the control action (bottom left) which replicates the opposite shape of the external torque overlapped to the optimal input u_req (dashed line), by virtue of the proposed stabilizer with adaptive internal model. The two components of $\hat{\Theta}$ (bottom right) indeed adapt to the actual values by virtue of the adaptation mechanism introduced in (20).

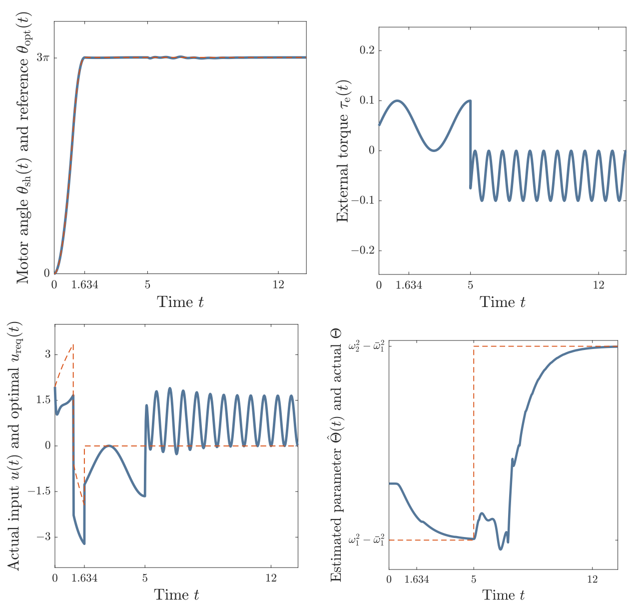


Fig. 3. Simulations in the Scenario 2.

Scenario 2: In this scenario we assume that the disturbance torque applied to the motor shaft contains always three frequencies, but, to assess the robustness of the proposed method in a more realistic and complex setting, the nominal ones are only the following:

$$\bar{\omega}_0 = 0, \quad \bar{\omega}_1 = \frac{3\pi}{2},$$

whereas the shape of the external torque is

$$\tau_\mathrm{e}(t) = 0.05 \begin{cases} \sin(\omega_1 t) + 1 & \text{if } 0 < t < 5 \\ \cos(\omega_2 t) - 1 & \text{if } t > 5 \end{cases}.$$

Considering a third-order internal model, the simulated responses of the controller are reported in Fig. 3. Analogously to the previous case, although the internal model contains only two out of the three frequencies of the disturbance, the parameter (bottom right) perfectly adapts to the variation of the load, again assessing the validity of the proposed scheme. Hence, again the trajectory of the output (top left) perfectly tracks the optimal reference (dashed line) despite the presence of the external disturbance (top right), compensated by the control action (bottom left).

Note that, while in early studies of adaptive internal model control (see, e.g., [8]), the number of parameters is equal to the state dimension n_m of M, the parameter Θ in (12) has the dimension of the number of uncertain frequencies in the internal model. The latter is always upperbounded by $\lfloor n_\mathrm{m}/2 \rfloor$, as in the examples above. In other words, our parametrization is more economic.

VII. Conclusions

The paper has proposed a novel procedure of designing adaptive internal model controllers. First, a control scheme is defined showing its capability of recasting the stabilization problem of high-dimensional augmented systems into that of simpler plants. Then, under the assumption of minimum phase plant and relative degree at most 2, it can be shown that the static parameter of the internal model affects the closed-loop dynamics affinely, paving the way to incorporate adaptation mechanisms, whose purpose is to tune the model to uncertain or changing exosystems.

The future work is to extend the proposed adaptive internal model controller architecture to a wider class of plants. In particular, we plan to relax the conditions adopted in this work about the plant zeros and relative degree, while keeping the static parameter of the internal model affinely affecting the closed-loop dynamics.

References

[1] B. A. Francis and W. M. Wonham, “The internal model principle of control theory,” *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.
[2] A. Isidori, *Lectures in Feedback Design for Multivariable Systems*. Cham, CH: Springer-Verlag, 2017.
[3] A. Saberi, A. A. Stoorvogel, and P. Sannuti, *Control of Linear Systems with Regulation and Input Constraints*. London, UK: Springer-Verlag, 2000.
[4] S. Hara, Y. Yamamoto, T. Omata, and M. Nakano, “Repetitive control system: A new type servo system for periodic exogenous signals,” *IEEE Trans. Automat. Control*, vol. 33, pp. 659–668, 1988.
[5] L. Mirkin, “On dead-time compensation in repetitive control,” *IEEE Control Syst. Lett.*, vol. 4, no. 4, pp. 791–796, 2020.
[6] G. P. Incremona, L. Mirkin, and P. Colaneri, “Integral sliding-mode control with internal model: A separation,” *IEEE Control Syst. Lett.*, vol. 6, pp. 446–451, 2022.
[7] P. Colaneri, G. P. Incremona, and L. Mirkin, “On Internal Model Compensation for General Regulator Problems,” in *62nd IEEE Conf. on Dec. and Cont. (CDC)*, pp. 2657–2662, Singapore, 2023.
[8] V. Nikiforov, and D. Gerasimov, “Adaptive Regulation: Reference Tracking and Disturbance Rejection”, ser. Lecture Notes in Control and Inform. Sci. Cham, CH: Springer-Verlag, 2022, vol. 491.
[9] G.C. Goodwin, D.Q. Mayne, “A Parameter Estimation Perspective of Continuous Time Model Reference Adaptive Control,” *Automatica*, vol. 23, pp. 57–70, 1987.
[10] K. Ogata, *System Dynamics*, 4th ed. Upper Saddle River, NJ: Prentice-Hall, 2004.
[11] M. Athans and P. L. Falb, *Optimal Control: An Introduction to the Theory and Its Applications*. New York, NY: McGraw-Hill, 1966.