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A Closed-Loop Design for Scalable High-Order Consensus

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***Abstract*— This paper studies the problem of coordinating a group of n^{th}-order integrator systems. As in the well-studied conventional consensus problem, we consider linear and distributed control with only local and relative measurements. We propose a closed-loop dynamic that we call *serial consensus* and prove it achieves n^{th}-order consensus regardless of model order and underlying network graph. This alleviates an important scalability limitation in conventional consensus dynamics of order $n \geq 2$, whereby they may lose stability if the underlying network grows. The distributed control law which achieves the desired closed loop dynamics is shown to be localized and obey the limitation to relative state measurements. Furthermore, through use of the small-gain theorem, the serial consensus system is shown to be robust to both model and feedback uncertainties. We illustrate the theoretical results through examples.**

I. Introduction

Properties of dynamical systems over networks have been a subject of significant research over the last two decades. A problem of interest is the coordination of agents in a network through localized feedback, leading to the prototypical distributed consensus dynamics studied early on by [1]–[3]. Over the years, it has become clear that the structural constraints imposed by the network topology in consensus problems often lead to fundamentally poor dynamic behaviors in large networks. This concerns controllability [4], performance [5], [6] and disturbance propagation [7], [8], but, as recently highlighted in [9], also stability. The poor stability properties characterized in earlier work [9] (which motivate the present work) apply to higher-order consensus. Here the local dynamics of each agent is modeled as an n^{th}-order integrator, with $n \geq 2$, and the control is a weighted average of neighbors' relative states. This is a theoretical generalization of first-order consensus [10], but is also relevant in practice. For example, a model where $n = 3$ and thus has consensus in position, velocity and acceleration, can capture flocking behaviors [11].

More specifically, [9] shows that conventional high-order consensus $(n \geq 3)$ is not *scalably stable* for many growing graph structures. When the network grows beyond a certain size, stability is lost. The same holds for second-order consensus ($n = 2$) in, for example, directed ring graphs, as also observed in [12]. To address this lack of scalable stability we propose an alternative generalization of the first-order consensus dynamics, which achieves consensus regardless of underlying network graph and model order n, thereby also ensuring scalable stability.

To illustrate our proposed controller, consider first the conventional second-order consensus system. Here the controller $u(t) = -L_1\dot{x}(t) - L_0x(t) + u_{\text{ref}}(t)$, with $L_{0,1}$ being weighted graph Laplacians, is used to achieve the closed loop

$$\ddot{x}(t) = -L_1\dot{x}(t) - L_0x + u_{\text{ref}}(t). \tag{1}$$

While for first-order consensus ($\dot{x} = Lx + u_{\text{ref}}$), a sufficient condition for convergence to consensus is that the graph underlying the graph Laplacian L contains a connected spanning tree [13]. However, this no longer suffices when $n \geq 2$ as in (1). Therefore, we instead propose the following controller $u(t) = -(L_2 + L_1)\dot{x}(t) - L_2L_1x(t) + u_{\text{ref}}(t)$. The reason for this choice of controller is best illustrated by considering the resulting closed loop in the Laplace domain:

$$(sI + L_2)(sI + L_1)X(s) = U_{\text{ref}}. \tag{2}$$

For this system, like for the first-order case, it is sufficient that the graphs underlying L_1 and L_2 each contain a connected spanning tree for the system to eventually coordinate in both x and its derivative $\dot{x}$ (regardless of network size!). This closed loop system, which we will call *serial consensus*, thus mimics one core property of the standard consensus protocol, and can also be generalized to any order n.

The main results of this paper elucidate key properties of the proposed n^{th}-order serial consensus. The controller is proven to remain localized (within an n-hop neighborhood) and implementable through relative measurements. We also prove that the closed loop will achieve consensus in all n states. Furthermore, we study the robustness of the proposed closed loop and show that the system will still coordinate when subject to unstructured uncertainty, whose permissible size is independent of the network size. The beneficial properties of the form (2) (generalized to any order n) are thus not contingent on an idealized implementation.

The remainder of this paper is organized as follows. We begin by introducing the n^{th}-order consensus model and defining our choice of control structure. Subsequently, we define and motivate the serial consensus system. In Sec. III and IV, we provide proofs for the stability and robustness of the serial consensus system, respectively. The main results are then illustrated through examples in Sec. V. Finally, Sec. VI offers our conclusions.

II. Problem Setup

We start by introducing some notation and graph theory before introducing the general n^{th}-order consensus problem

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for which we propose the new serial consensus setup. We discuss its properties and end with some useful definitions.

A. Network Model and Definitions

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ denote a graph of size $N = |\mathcal{V}|$. The set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the set of edges. The graph can be equivalently represented by the adjacency matrix $W \in \mathbb{R}^{N\times N}$ where $w_{i,j} > 0 \iff (j,i) \in \mathcal{E}$. The graph is called *undirected* if $W = W^T$. The graph contains a *connected spanning tree* if for some $i \in \mathcal{V}$ there is a path from i to any other vertex $j \in \mathcal{V}$.

Associated with a weighted graph we have the weighted graph Laplacian $L(\mathcal{G})$, defined as

$$[L(\mathcal{G})]_{i,j} = \begin{cases} -w_{i,j}, & \text{if } i \neq j \\ \sum_{k\neq i} w_{i,k}, & \text{if } i = j \end{cases}.$$

The graph dependence is omitted when clear from context. Under the condition that that the graph generating the graph Laplacian contains a connected spanning tree, L will have a simple and unique eigenvalue at 0 and the remaining eigenvalues will lie strictly in the right half plane (RHP).

We will also consider networks with a growing number of nodes. These are then described by a graph family $\{\mathcal{G}_N\}_{N\to\infty}$, where N is the size of the growing network.

The space of all proper, real rational, and stable transfer matrices will be denoted $\mathcal{RH}_\infty$. We will use $\|\cdot\|_{\mathcal{H}_\infty}$ for the $\mathcal{H}_\infty$ norm, following the notation in [14].

B. n^{th}-Order Consensus

Let the system be modeled as N agents with identical n^{th}-order integrator dynamics, i.e.

$$\frac{\mathrm{d}^n x_i(t)}{\mathrm{d}t^n} = u_i(t), \tag{3}$$

for all $i \in \mathcal{V}$. We will use the convention $x_i^{(0)}(t) = x_i(t)$ and $x_i^{(k)}(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} x_i(t)$ to denote time derivatives. When clear, we may omit the time argument for brevity.

In this paper, we consider the problem of synchronizing agents to achieve a state of consensus, formally defined as:

Definition 1 (n^{th}-order consensus): The multi-agent system (3) achieves n^{th}-order consensus if $\lim_{t\to\infty} |x_i^{(k)}(t) - x_j^{(k)}(t)| = 0$, for all $i, j \in \mathcal{V}$ and $k \in \{0, 1 \ldots, n-1\}$.

C. Control Structure

A linear state feedback controller of (3) can be written as

$$u(t) = u_{\text{ref}}(t) - \sum_{k=0}^{n-1} A_k x^{(k)}(t), \tag{4}$$

where $u_{\text{ref}}(t) \in \mathbb{R}^N$ is a feedforward term and $A_k \in \mathbb{R}^{N\times N}$ represents the feedback of the k^{th} derivative. We will restrict this class of controllers in three ways. The controllers

i) can only use *relative* feedback;
ii) have a limited gain;
iii) depend on the local neighborhood of each agent.

The constraint for relative feedback can be expressed as $A_k \mathbf{1}_N = 0$ for all k. Meanwhile, a limited gain can be represented by requiring that $\|A_k\|_\infty \leq c$. To capture the notion of locality, consider the adjacency matrix W representing the communication and measurement structure, which we here assume to be the same. That is, if $W_{i,j} = 1$, then agent i can directly receive or measure the relative distance to agent j. Next, consider the non-negative matrix W^q. This matrix has the property that $[W^q]_{i,j} \neq 0$ if and only if there is a path of length q from agent j to agent i. Thus, if we want the controller to only depend on information that is at most q steps away from each agent the following implication should hold: $\sum_{k=0}^q W_{i,j}^k = 0 \implies [A_k]_{i,j} = 0$. Putting all the conditions together gives us a family of controllers that we will consider in this paper:

Definition 2 (q-step implementable relative feedback): A relative feedback controller of the form (4) is q-step implementable with respect to the adjacency matrix W and gain $c > 0$ if $A_k \in \mathcal{A}^q(W, c)$ for all k, where

$$\mathcal{A}^q(W,c) = \left\{ A \;\middle|\; \begin{matrix} \left[\sum_{k=0}^q W^k\right]_{i,j} = 0 \implies A_{i,j} = 0, \\ A\mathbf{1}_N = 0, \; \|A\|_\infty \leq c \end{matrix} \right\}.$$

The conventional controller for achieving n^{th}-order consensus can be realized as (4) where each A_k is given by a graph Laplacian, e.g., $A_k = L_k \in \mathcal{A}^1(W, c)$.

D. A Novel Design: Serial Consensus

We propose the following controller of (3), expressed in the Laplace domain, to achieve n^{th}-order consensus

$$U(s) = U_{\text{ref}}(s) + \left(s^n I - \prod_{k=1}^n (sI + L_k) \right) X(s), \tag{5}$$

where L_k are graph Laplacians and U_{ref} is the transformed reference signal. In this case, it is more instructive to consider the closed-loop dynamics, which take the following form:

Definition 3 (n^{th}-order serial consensus system): For all $k \in \{1, 2 \ldots, n\}$, let L_k be a weighted and directed graph Laplacian. The n^{th}-order serial consensus system is then

$$\left(\prod_{k=1}^n (sI + L_k) \right) X(s) = U_{\text{ref}}(s). \tag{6}$$

We refer to this form as serial consensus because the same closed-loop dynamics can be realized through interconnecting n first-order consensus systems in series.

The closed-loop dynamics in (6) can also be transformed to state-space form by introducing the alternative variables Ξ_k with the corresponding states ξ_k. These relate to X through $\Xi_1 = X(s)$, $\Xi_k = (sI + L_{k-1})\Xi_{k-1}$ for $k \in \{2, \ldots, n-1\}$, and $s\Xi_n = -L_n \Xi_n + U_{\text{ref}}$. This leads to the following continuous-time state-space representation

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1 & I & & \\ & -L_2 & \ddots & \\ & & \ddots & I \\ & & & -L_n \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ u_{\text{ref}} \end{bmatrix}. \tag{7}$$

The serial consensus form has several advantages, which will be the focus of this paper. First, however, we show that it satisfies the controller constraints that we impose, as given by Definition 2. In other words, we will discuss how the closed-loop structure in (6) can be implemented on a network.

When analyzing the serial consensus controller of (5) we will utilize the following assumption on the graph structure.

Assumption 1: (Connected spanning tree) All graphs underlying the graph Laplacians L_k contain a connected spanning tree.

E. Implementing Serial Consensus

The following proposition ensures that the serial consensus system can be achieved by controlling the n^{th}-order integrator system (3) with an n-step implementable relative feedback controller as defined in Definition 2.

Proposition 1: Consider the n^{th}-order serial consensus as defined in (6). If each $L_k \in \mathcal{A}^1(W, c)$ for some constant c and adjacency matrix W, then the controller in (5) is an n-step implementable relative feedback controller with respect to W and a finite gain c'.

To prove this proposition, we first require the following two lemmas, whose proofs can be found in [15].

Lemma 1: If $A_1 \in \mathcal{A}^{q_1}(W, c_1)$ and $A_2 \in \mathcal{A}^{q_2}\ (W, c_2)$ then the sum satisfies $(A_1 + A_2) \in \mathcal{A}^{\max(q_1, q_2)}\ (W, c_1 + c_2)$.

Lemma 2: Let $A_1 \in \mathcal{A}^{q_1}(W, c_1)$ and $A_2 \in \mathcal{A}^{q_2}(W, c_2)$ then the product satisfies $(A_1 A_2) \in \mathcal{A}^{q_1+q_2}(W, c_1 c_2)$.

Now we can prove Proposition 1.

Proof: The serial consensus controller can be expanded to the matrix polynomial

$$\begin{aligned} U(s) &= U_{\text{ref}}(s) + \left(s^n I - \prod_{k=1}^{n} (sI + L_k) \right) X(s) \\ &= U_{\text{ref}}(s) + \left((s^n - s^n) I - \sum_{k=0}^{n-1} s^k A_k \right) X(s), \end{aligned}$$

for some matrices A_k. To show the proposition, we need to show that $A_k \in \mathcal{A}^q(W, c')$ for all $k = 0, \ldots, n-1$, with $q \leq n$ and $c' < \infty$. Let

$$\begin{aligned} \mathcal{I}_k = \{ \alpha \mid |\alpha| = n - k,\ \alpha \subset \{1, 2, \ldots, n\}, \\ i < j \implies \alpha(i) < \alpha(j) \} \end{aligned}$$

denote all the ordered subsets of the range $[1, n]$ with size $n - k$. Then

$$A_k = \sum_{\alpha \in \mathcal{I}_k} \prod_{j \in \alpha} L_j, \text{ for all } k \in [0, n-1].$$

Since all $\alpha \in \mathcal{I}_k$ has $n - k$ elements we can show that $\prod_{j \in \alpha} L_j = B_\alpha \in \mathcal{A}^{n-k}\ (W, c^{n-k})$ by applying Lemma 2 recursively. Now we have a sum

$$A_k = \sum_{\alpha \in \mathcal{I}_k} B_\alpha.$$

The number of ordered subsets of the range $[1, n]$ with size $n - k$ is given by the binomial coefficients and therefore the size of $|\mathcal{I}_k| = \binom{n}{n-k}$. Applying Lemma 1 recursively shows that $A_k \in \mathcal{A}^{n-k}(W, \binom{n}{n-k} c^{n-k})$. Clearly, we have that $n - k \leq n$ and $\binom{n}{n-k} c^{n-k} \leq \binom{n}{\lceil n/2 \rceil} \max(c, c^n) < \infty$ for all k. Let $c' = \binom{n}{\lceil n/2 \rceil} \max(c, c^n)$ and then $A_k \in \mathcal{A}^n(W, c')$ holds true for all k. ■

Example 1: For clarity, let us consider the controller for the case $n = 3$. Then the controller is

$$\begin{aligned} U(s) &= U_{\text{ref}}(s) + \left(s^3 I - \prod_{k=1}^{3} (sI + L_k) \right) X(s) \\ &= U_{\text{ref}}(s) - \big(s^2 (L_3 + L_2 + L_1) + \\ &\quad s(L_3 L_2 + L_3 L_1 + L_2 L_1) + L_3 L_2 L_1 \big) X(s). \end{aligned}$$

Here, $A_0 = L_3 L_2 L_1$, $A_1 = L_3 L_2 + L_3 L_1 + L_2 L_1$, and $A_2 = L_3 + L_2 + L_1$. The proposition asserts that if L_1, L_2, and L_3 share a sparsity pattern and have bounded gains, then the resulting controller gains A_0, A_1, and A_2 will be sparse and have bounded gains.

Remark 1: Proposition 1 may be conservative. For instance, if W represents the complete graph, then any relative feedback controller would trivially be 1-step implementable.

III. Stability of Serial Consensus

In this section, we prove the stability of the serial consensus. Using this result, we further demonstrate that the serial consensus satisfies a notion of scalable stability.

A. Stability

Theorem 1: Consider the n^{th}-order serial consensus system as defined in Definition 3 under Assumption 1 and with $U_{\text{ref}} \in \mathcal{RH}_\infty$. The closed-loop dynamics have the following properties:

(i) The poles of (6) are given by the union of the eigenvalues of $-L_k$.
(ii) The solution achieves n^{th}-order consensus.

Proof: (i) Any square matrix can be unitarily transformed to upper-triangular form by the Schur traingularization theorem. Let $U_k L_k U_k^H = T_k$ be upper triangular. Then the block diagonal matrix $U = \text{diag}(U_1, U_2, \ldots U_n)$ is a unitary matrix that upper triangularizes A in (7). For any triangular matrix the eigenvalues lie on the diagonal and these will be the eigenvalues of each $-L_k$.

(ii) First, consider the closed loop dynamics of (6) which will be

$$X(s) = \left(\prod_{k=n}^{1} (sI + L_k)^{-1} \right) U_{\text{ref}}(s).$$

Since U_{ref} is stable, we know that the limit $\lim_{s \to 0} U_{\text{ref}}(s) = U_{\text{ref}}(0)$ exists. To prove that the system achieves n^{th}-order consensus we want to show that

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} C(s) X(s) = 0$$

for some transfer matrix $C(s)$, which encodes the consensus states. But since the reference dependence is only related to $U_{\text{ref}}(0)$, we can simplify the problem to only consider impulse responses. But the impulse response has the same transfer function as the initial value response where $\xi_n(0) = U_{\text{ref}}(0)$. Therefore, WLOG, assume that $U_{\text{ref}}(s) = 0$ and an arbitrary initial condition

$$\xi(0) = [\xi_1^T(0), \, \xi_2^T(0), \, \ldots, \, \xi_n^T(0)]^T.$$

The solution of (7) is given by $\exp(At)\xi(0) = S\exp(J(A)t)S^{-1}\xi(0)$ where $J(A)$ is the Jordan normal form of A and S is an invertible matrix. From (i) and the diagonal dominance of the graph Laplacians we know that all eigenvalues of A lie in the left half plane. By Assumption 1 it follows that the zero eigenvalue for each L_k is simple. Now we prove that these n zero eigenvalues form a Jordan block of size n. Let e_k denote the k^{th} 1-block vector, e.g. $\mathbf{e}_1 = [\mathbf{1}_N^T, \, 0_N, \, \ldots, \, 0_N]^T$ and $e_2 = [0_N, \, \mathbf{1}_N^T, \, \ldots, \, 0_N]^T$. Then e_1 is an eigenvector, since $A\mathbf{e}_1 = 0$. For $k \in \{2, 3 \ldots, n\}$ we have $Ae_k = e_{k-1}$ which implies that $A^k e_k = 0$. This shows that there is a Jordan block of size n with an invariant subspace spanned by the vectors e_k. Since all other eigenvectors make up an asymptotically stable invariant subspace, it follows that $\xi(t)$ will converge towards a solution in $\text{span}(e_1, e_2 \ldots, e_n)$ and thus $\lim_{t\to\infty} \xi_k(t) = \alpha_k(t)\mathbf{1}_N$. Now, since $x(t) = \xi_1(t)$, it follows that $\lim_{t\to\infty} x(t) = \alpha_1(t)\mathbf{1}_N$, and furthermore, since

$$\dot{\xi}_k = -L_k\xi_k + \xi_{k+1} \to \xi_{k+1} \text{ as } t \to \infty,$$

for $k \in \{1, \ldots, n-1\}$, it follows that $\lim_{t\to\infty} x^{(k)}(t) = \alpha_{k+1}(t)1_N$, proving convergence to n^{th}-order consensus. ■

The proposition shows that the stability analysis of the n^{th}-order serial consensus can be reduced to verifying that the n first-order consensus systems $\dot{x} = -L_k x$ achieve consensus. This is equivalent to determining whether the graphs underlying each L_k, all contain a connected spanning tree. Consequently, in conjunction with Proposition 1, this result demonstrates that n^{th}-order consensus is achievable using a local relative feedback controller with finite gain. Notably, this achievement is independent of the number of agents, thus ensuring scalability, which we will discuss next.

B. Scalable Stability

Coordinating a multi-agent system is inherently a decentralized problem where the goal for each agent is to coordinate with its nearest neighbors. However, when the controllers only depend on local measurements there is a possibility that controllers that manage to coordinate N agents stop doing so as the number of agents increases. More specifically, consider the growing graph family $\{\mathcal{G}_N\}$ and corresponding graph Laplacians $L(\mathcal{G}_N)$. Then we can consider the following notion of stability.

Definition 4 (Scalable stability [9, Def. 2.1]): A consensus control design is scalably stable if the resulting closed-loop system achieves consensus over any graph in the family $\{\mathcal{G}_N\}$.

In [9] it was shown that for the 3rd and higher-order consensus problem with controller $A_k = a_k L(\mathcal{G}_N)$ in (4), the closed loop system will become unstable if the algebraic connectivity $\lambda_2(L(\mathcal{G}_N)) \to 0$ as $N \to \infty$. The serial consensus alleviates this scalability issue. Theorem 1 shows that the serial consensus will be stable, regardless of the network size, as long as the underlying graphs are sufficiently connected. The result is summarized in the following corollary.

Corollary 1: For any n, the controller (5) is scalably stable over any graph family $\{\mathcal{G}_N\}$ that underlies $L_k(\mathcal{G}_N)$, provided each $\mathcal{G}_N$ satisfies Assumption 1.

Remark 2: Note that, by Theorem 1, scalable stability is also achieved when the graph families underlying each L_k are different. This can even be achieved with $||L_k||_\infty$ being arbitrarily small.

IV. Robustness of Serial Consensus

The controller proposed in (5) is a relative state-feedback controller, specifically designed to ensure that the closed loop system achieves n^{th}-order consensus as guaranteed through Theorem 1. However, the n^{th}-order integrator system may be an idealization of the system. Implementing the relative state feedback may require observers to be fully realized, and there can be unmodeled dynamics. These potential sources of errors call for a robust controller. We will now present two theorems, which prove that the serial consensus is robust towards two different types of uncertainties.

A. Additive Perturbation

The following theorem asserts that the n^{th}-order serial consensus controller can handle additive perturbations.

Theorem 2: Consider the n^{th}-order serial consensus system as defined in Definition 3, under Assumption 1, with $L_k = L$ for all k, and $L = L^T$. Then the perturbed system

$$(sI + L)^n X = U_{\text{ref}} + \left(\sum_{k=0}^{n} \Delta_k s^k L^{n-k} \right) X,$$

where $U_{\text{ref}}, \Delta_k \in \mathcal{RH}_\infty$, achieves n^{th}-order consensus if

$$\|\Delta_0\|_{\mathcal{H}_\infty} + \|\Delta_n\|_{\mathcal{H}_\infty} + \sum_{k=1}^{n-1} \|\Delta_k\|_{\mathcal{H}_\infty} \sqrt{\frac{k^k}{n^n}(n-k)^{n-k}} < 1.$$

Proof: First, note that the closed-loop system can be represented by the block diagram in Fig. 1, which in turn can be simplified to Fig. 2. Since U_{ref} is stable we can apply the small-gain theorem which asserts that $U(s)$ (as defined in the figures) will be stable if

$$\left\| \sum_{k=0}^{n} \Delta_k s^k L^{n-k} (sI + L)^{-n} \right\|_{\mathcal{H}_\infty} < 1.$$

Applying the triangle inequality and submultiplicativity on the left-hand side (LH) yields

$$LH \leq \sum_{k=0}^{n} \|\Delta_k\|_{\mathcal{H}_\infty} \left\| s^k L^{n-k} (sI + L)^{-n} \right\|_{\mathcal{H}_\infty}. \quad (8)$$

Since L is symmetric, it is possible to unitarily diagonalize it. Let $U = U^H$ denote one such unitary matrix. Then $L = U\Lambda U^H$ where Λ is a non-negative real diagonal matrix.

$$\|s^k L^{n-k}(sI + L)^{-n}\|_{\mathcal{H}_\infty} = \|s^k \Lambda^{n-k}(sI + \Lambda)^{-n}\|_{\mathcal{H}_\infty}.$$

For a diagonal matrix the singular values are given by the absolute value of the diagonal. Let, $\lambda > 0$ be an arbitrary

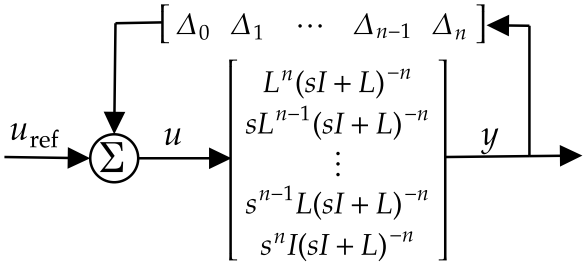


Fig. 1: Block diagram illustrating the perturbation model in proof of Theorem 2.

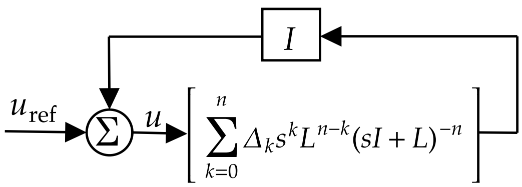


Fig. 2: Block diagram illustrating the perturbation model in proof of Theorem 2.

positive constant. The maximum gain for each diagonal can then be calculated through

$$\max_{\omega}\left|\frac{\omega^k\lambda^{n-k}}{(j\omega+\lambda)^n}\right| = \sqrt{\max_{\omega}\frac{\omega^{2k}\lambda^{2n-2k}}{(\omega^2+\lambda^2)^n}}.$$

The latter optimization problem is given by a continuous function and thus the derivative must be 0 at the maximum. Simple calculus shows that the optimum is found at $\omega^2 = \lambda^2 k/(n-k)$ for $k = 0, 1, \ldots n-1$ and at $\omega = \infty$ for $k = n$. Inserting yields

$$\max_{\omega}\left|\frac{\omega^k\lambda^{n-k}}{(j\omega+\lambda)^n}\right| = \begin{cases}\sqrt{\frac{k^k}{n^n}(n-k)^{n-k}} & \text{if } 0 < k < n\\ 1 & \text{else}\end{cases}.$$

In the case where $\lambda = 0$. Then we have for $k = 0, \ldots, n-1$

$$\max_{\omega}\left|\frac{\omega^k 0^{n-k}}{(j\omega+0)^n}\right| = 0,$$

and for $k = n$

$$\max_{\omega}\left|\frac{\omega^n}{(j\omega+0)^n}\right| = 1.$$

This is less restrictive than for $\lambda > 0$ and thus we can use the result for $\lambda > 0$. Plugging this into the upper bound of the LH (8) results in the sought inequality.

Finally, we must ensure that stability of the closed loop in Fig. 2 implies n^{th}-order consensus. Since the transfer matrix from u to y in Fig. 1 is stable it follows that $Y(s)$ will be stable. This means that we have shown the following $\lim_{t\to\infty} L^{n-k}x^{(k)}(t) = 0$. By Assumption 1 the 0 eigenvalue of L is unique and therefore 0 is a unique eigenvalue of L^{n-k} too. Subsequently, $\lim_{t\to\infty} x^{(k)}(t) \in \ker(L^{n-k})$. Since $L^{n-k}\mathbf{1}_N = 0$ it follows that $\lim_{t\to\infty} x^{(k)}(t) \in \mathrm{span}(\mathbf{1}_N)$ and that the agents will reach consensus in all the $n-1$ first time derivatives and thus achieve n^{th}-order consensus. ∎

It is worth noting that the norm bound on the uncertainty blocks Δ_k is independent of the number of agents in the system. Therefore, the serial consensus implementation can be considered *scalably robust* in the sense that it allows equally sized perturbations, regardless of network size. This is not the case for localized conventional consensus, following the results in [9].

B. Multiplicative Perturbation

It is also possible to see the closed-loop serial consensus system as a series of interconnected first-order systems. Therefore it is also interesting to consider the robustness with respect to the each factor. The following theorem gives a sufficient condition for the closed-loop system to achieve n^{th}-order consensus.

Theorem 3: Consider the n^{th}-order serial consensus system as defined in Definition 3, under Assumption 1, with $L_k = L_k^T$ for all k. Then the perturbed system

$$(sI+s\Delta_0+(I+\Delta_n)L_n)\prod_{k=1}^{n-1}(sI+(I+\Delta_k)L_k)\,X = U_{\mathrm{ref}},$$

where $U_{\mathrm{ref}}, \Delta_k \in \mathcal{RH}_\infty$, achieves n^{th}-order consensus if

$$\|\Delta_k\|_{\mathcal{H}_\infty} < 1, \text{ for all } k$$

and

$$\|\Delta_0\|_{\mathcal{H}_\infty} + \|\Delta_n\|_{\mathcal{H}_\infty} < 1.$$

Proof: First, note that we can construct $X(s) = \Xi_1(s)$, $s\Xi_k = -(I+\Delta_k)L_k\Xi_k + \Xi_{k+1}$ for $k = 1, \ldots, n-1$, and $s(I+\Delta_0)\Xi_n = -(I+\Delta_n)L_n\Xi_n + U_{\mathrm{ref}}$. For Ξ_n we have exactly the first-order case of Theorem 2 and thus $\lim_{t\to\infty}\xi_n(t) = \alpha_n(t)\mathbf{1}_N$ if $\|\Delta_0\|_{\mathcal{H}_\infty} + \|\Delta_n\|_{\mathcal{H}_\infty} < 1$. Consider the following induction hypothesis: if $\Xi_{k+1}(s) = \mathbf{1}_N G_{k+1}(s) + H_{k+1}(s)$ where $H_{k+1}(s) \in \mathcal{RH}_\infty$, then $\Xi_k = \mathbf{1}_N G_k(s) + H_k(s)$ for some $H_k(s) \in \mathcal{RH}_\infty$. We then have

$$s\Xi_k = -(I+\Delta_k)L_k\Xi_k + \Xi_{k+1}$$

which can be represented by the block diagram Fig. 3. Here, note that

$$L_k(sI+L_k)^{-1}\Xi_{k+1} = (sI+L_k)^{-1}L_k(H_{k+1}(s))$$

and the potentially unstable term of Ξ_{k+1} can be ignored. Reusing a result from the previous proof we have $\|L_k(sI+L_k)^{-1}\|_{\mathcal{H}_\infty} = 1$ and therefore $L_k\Xi_k \in \mathcal{RH}_\infty$ if $\|\Delta_k\|_{\mathcal{H}_\infty} < 1$. Since the 0 eigenvalue of L_k is unique, it follows that $\Xi_k(s) = \mathbf{1}_N G_k(s) + H_k(s)$ with $H_k \in \mathcal{RH}_\infty$ which proves the induction hypothesis since we have already shown the base case $\Xi_n(s) = \mathbf{1}_N G_n(s) + H_n(s)$. It is left to prove that the system will reach n^{th}-order consensus. Note that $L_1X(s) = L_1\Xi_1(s)$ is stable and therefore we get through the final value theorem

$$\lim_{t\to\infty} L_1x(t) = \lim_{s\to 0} sL_1\Xi_1(s) = 0.$$

Furthermore, we have for all k: $\lim_{s\to 0} sL_k\Xi_k(s) = 0$. This, combined with $s^2\Xi_k(s) = -(I+\Delta_k)sL_k\Xi_k(s) + s\Xi_{k+1}(s)$

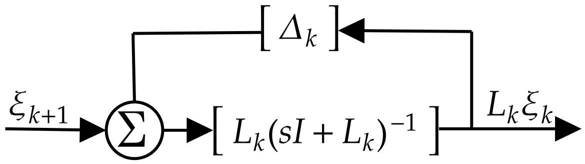


Fig. 3: Block diagram illustrating the perturbation model of a general first-order consensus block which is used in the proof of Theorem 3.

shows that

$$\lim_{t\to\infty} L_{k+1}x^{(k)}(t) = \lim_{s\to 0} s(s^k L_{k+1}X(s))$$
$$= \lim_{s\to 0} sL_{k+1}\Xi_{k+1}(s) = 0.$$

Finally, since each L_k has a unique 0 eigenvalue with the corresponding eigenvector $\mathbf{1}_N$, we see that n^{th}-order consensus will be achieved ■

This theorem shows that the n^{th}-order serial consensus is robust in its construction.

V. Examples

A. 2^{nd}-Order Consensus on Circular Graph

Consider the directed cycle graph, which can be represented by the adjacency matrix

$$[W_{\text{cycle}}]_{i,j} = 1 \text{ iff } i - j = 1 \mod N.$$

The corresponding graph Laplacian L_{cycle} is a circulant matrix and therefore, the eigenvalues are known analytically. In particular, the eigenvalue with the second smallest real part is $\lambda_2(L_{\text{cycle}}) = 1 - \exp(2\pi\mathbf{i}/N) = 1 - \cos(2\pi/N) - \mathbf{i}\sin(2\pi/N)$. For large N, this eigenvalue can be approximated with a first-order Taylor approximation, which yields $\lambda_2(L_c) \approx -\mathbf{i}2\pi/N$. This eigenvalue will cause problems when designing a controller using the conventional consensus. To see this, consider the closed loop dynamics

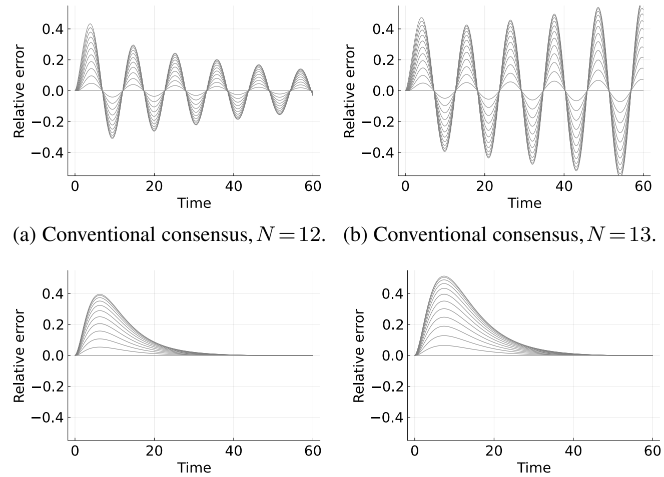
$$s^2 I + 2p_1 sL_{\text{cycle}} + p_0 L_{\text{cycle}} = U_{\text{ref}}.$$

The system can be diagonalized and, in particular, two of the poles are given by the equation $s^2 + 2p_1\lambda_2(L_{\text{cycle}}) + p_0\lambda_2(L_{\text{cycle}}) = 0$. In the case when p_0 and p_1 are designed independently of the network size N, then for sufficiently large N the roots can be approximated as

$$s_p = -p_1\lambda_2 \pm \sqrt{p_1^2\lambda_2^2 - p_0\lambda_2} \approx \pm(1+\mathbf{i})\sqrt{\frac{\pi p_0}{N}}.$$

Since one of these poles lies in the RHP, it follows that the closed loop system will become unstable when N is sufficiently large, for any (fixed) choice of p_0 and p_1.

For the serial consensus it is sufficient to check that all eigenvalues but the unique 0 eigenvalue of L_{cycle} lie in the RHP, or equivalently, if $0 < \text{Re}(\lambda_2(L_{\text{cycle}})) = 1 - \cos(2\pi/N)$ which is clearly true for any finite N. Alternatively, it is also sufficient to check that the underlying graph contains a connected spanning tree.



(a) Conventional consensus, $N=12$. (b) Conventional consensus, $N=13$.

(c) Serial consensus, $N=12$. (d) Serial consensus, $N=13$.

Fig. 4: 3[rd]-order consensus in a chain of vehicles is considered. The plots show the inter-vehicle relative errors over time when the lead vehicle moves at constant acceleration. Panels (a) and (b) show that the addition of one agent destabilizes the closed loop for the conventional consensus. Panels (c) and (d) illustrate the fact that the serial consensus will remain stable under such agent additions.

B. 3^{rd}-Order Consensus

It has been shown that the conventional consensus $x^{(n)} = -\sum_{k=0}^{n-1} L(\mathcal{G}_N)x^{(k)}$ cannot achieve scalable stability for any graph family $\{\mathcal{G}_N\}$ such that the corresponding graph Laplacian has an eigenvalue that decreases towards zero as the graph is growing, i.e. if $\lim_{N\to\infty}\text{Re}(\lambda_2(L(\mathcal{G}_N))) = 0$. At least, this is not possible with the conventional consensus control. However, for the serial consensus this is no longer a problem. The controller

$$U(s) = U_{\text{ref}} + \left(s^3 I - \prod_{k=1}^{3}(sI + L(\mathcal{G}_N))\right)X(s)$$

will achieve consensus as long as each of the underlying graphs $\{\mathcal{G}_N\}$ contains a connected spanning tree. To illustrate this, consider the graph defined by the adjacency matrix $W_{\text{path}} \in \mathbb{R}^{N\times N}$, defined as

$$[W_{\text{path}}]_{i,j} = \begin{cases} 1 & \text{if } |i-j| = 1 \text{ and } i \neq 1 \\ 0 & \text{else} \end{cases}.$$

This corresponds to a bidirectional path graph with a leader (Agent 1). Let L_{path} be the associated graph Laplacian. It is true that $\lim_{N\to\infty}\lambda_2(L_{\text{path}}) = 0$ and thus any conventional control design with L_{path} will eventually lead to an unstable closed loop. For this example, let the conventional control law be $u(t) = u_{\text{ref}}(t) - 6L_{\text{path}}\ddot{x} - 4L_{\text{path}}\dot{x} - 2L_{\text{path}}x$ and the serial consensus controller (5) be defined with the same graph Laplacians $L_k = 2kL_{\text{path}}$. The response to a constant acceleration of the leader is shown in Fig. 4. Here we see that the addition of a 13^{th} agent to the system destabilizes the closed loop for the conventional consensus while the serial consensus only loses some performance.

C. Robustness of the 2^{nd}-Order Serial Consensus.

Theorems 2 and 3 show that the serial consensus can be perturbed and still achieve n^{th}-order consensus. Now we want to illustrate what the block Δ_k can be. Consider the perturbed 2^{nd}-order consensus system in Theorem 2. Writing out all terms we get

$$s^2(I+\Delta_2)X = U_{\text{ref}} - (s(2I+\Delta_1)LX + (I+\Delta_0)L^2X).$$

In this form, the Δ_2 block can represent potential model errors. While we might control a system modeled as N identical double-integrator systems, the reality may differ. This is obviously the case for vehicle platoons, which are often modeled as chains of identical double integrators. Through our theorem we can for instance allow Δ_2 to be a diagonal transfer matrix with elements $[\Delta_2]_{i,i} = \frac{k_i}{T_i s+1}$ where $|k_i| < 1$ and $T_i > 0$ for all i. In this scenario, the closed loop system would remain stable despite the heterogeneous agents. The blocks Δ_1 and Δ_0 are also important. For instance, the signals $L^2x(t)$ and $L\dot{x}(t)$ may not be directly measured, but instead estimated through linear filters. This could be thought of as unmodeled dynamics, which these blocks can capture.

If we focus on Theorem 3, then the perturbed model is

$$(s(\Delta_0+I)+(\Delta_2+I)L_2)(sI+(\Delta_1+I)L_1)X = U_{\text{ref}}.$$

The theorem only asserts robustness for symmetrical graph Laplacians L_k. However, since each Δ_k can be a constant matrix, it is possible to construct new (asymmetric) graph Laplacians $L'_k = (I+\Delta_k)L_k$ by designing the Δ_k blocks.

VI. Conclusion

This work has introduced the n^{th}-order serial consensus system, which serves as a natural generalization of the well-known consensus protocols. The stability of this system can be analyzed by considering n regular first-order consensus protocols. The controller proposed for achieving n^{th}-order serial consensus has been shown to be implementable using relative measurements confined to a local neighborhood of each agent and can therefore be considered a distributed control scheme. Robustness of the proposed system has also been analyzed. This has been addressed in terms of two different types of model perturbations. The analysis showed that the size, measured in the $\mathcal{H}_\infty$ norm, of the allowable uncertainties were independent of the number of agents.

Future and ongoing work will explore the performance of the serial consensus and its relation to string stability. It would also be interesting to consider an implementation where each agent employs an observer to compute their control action.

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